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A SUBSOLUTION-SUPERSOLUTION METHOD FOR QUASILINEAR SYSTEMS

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Abstract. Assuming that a system of quasilinear equations of gradient type admits a strict supersolution and a strict subsolution, we show that it also admits a positive solution.

1. Introduction

Consider the quasilinear elliptic system

$$\begin{gathered} -\Delta_p u = H_u(x,u,v) \quad \text{in } \Omega \\ -\Delta_q v = H_v(x,u,v) \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with boundary of class C^2, Δ_p and Δ_q are the $p-$ and $q-$Laplace operators with $1 < p, q < N$, and $H : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function.

The solvability of system (1.1) has been extensively studied by various methods, fibering [3], bifurcation [4], via the mountain pass theorem [2] etc. We use the super- and sub- solution method by assuming that (1.1) admits a strict supersolution and a strict subsolution and construct two sequences of approximate solutions whose limit is shown to be a solution of the system. The same approach can also be applied to nonvariational and Hamiltonian systems. It is worth mentioning that, as far as (1.1) is concerned, the classical super- and sub- solution method is not directly applicable because the "restriction" of the function $H(x,.,.)$ between the super- and sub- solution is not necessarily differentiable.

We make the following assumptions:

- (H1) $s \mapsto H_u(x,s,t)$ and $s \mapsto H_v(x,s,t)$ are nondecreasing for a.e. $x \in \Omega$ and every $t > 0$.
- (H2) $t \mapsto H_u(x,s,t)$ and $t \mapsto H_v(x,s,t)$ are nondecreasing for a.e. $x \in \Omega$ and every $s > 0$.
- (H3) $H_u(x,0,t) = H_v(x,s,0) = 0$ for a.e. $x \in \Omega$ and every $s, t > 0$.

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(H4) There exists $C > 0$ such that $|H_u(x,s,t)| \le C(1+|s|^{p^*-1}+|t|^{\frac{q^*(p^*-1)}{p^*}})$ and $|H_v(x,s,t)| \le C(1+|s|^{\frac{p^*(q^*-1)}{q^*}}+|t|^{q^*-1})$ a.e. in Ω, where $p^* := \frac{Np}{N-p}$ and $q^* := \frac{Nq}{N-q}$ are the critical Sobolev exponents.

Note that if (H1)–(H4) are satisfied then

$$|H(x,s,t)| \le c(1+|s|^{p^*}+|t|^{q^*}) \text{ a.e.in } \Omega,$$

for some $c > 0$.

Let $E = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. The energy functional $\Phi : E \to \mathbb{R}$ associated to (1.1) is

$$\Phi(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q - \int_\Omega H(x,u(x),v(x))dx.$$

It is clear that if (H1)–(H4) are satisfied, then Φ is a C^1-functional whose critical points are solutions to (1.1).

Definition. A pair of nonnegative functions $(\overline{u},\overline{v}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ is said to be a strict supersolution for (1.1) if $-\Delta_p\overline{u} > H_u(x,\overline{u},\overline{v})$ and $-\Delta_q\overline{v} > H_v(x,\overline{u},\overline{v})$ in Ω. A pair of nonnegative functions $(\underline{u},\underline{v})$ is said to be a strict subsolution if $-\Delta_p\underline{u} < H_u(x,\underline{u},\underline{v})$ and $-\Delta_q\underline{v} < H_v(x,\underline{u},\underline{v})$ a.e. in Ω.

Theorem 1.1. *Assume that hypotheses (H1)–(H4) hold and* (1.1) *admits a strict supersolution* $(\overline{u},\overline{v})$ *and a strict subsolution* $(\underline{u},\underline{v})$ *with* $\underline{u} < \overline{u}$ *and* $\underline{v} < \overline{v}$ *in* Ω. *Then* (1.1) *has a solution* (u_0,v_0) *with* $u_0, v_0 > 0$ *in* Ω.

Proof. For a function $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we define

$$\widehat{F}(x,s,t) = \begin{cases} F(x,s,t) & \text{if } \underline{u}(x) \le s \le \overline{u}(x),\ \underline{v}(x) \le t \le \overline{v}(x), \\ F(x,\underline{u}(x),t) & \text{if } s < \underline{u}(x),\ \underline{v}(x) \le t \le \overline{v}(x), \\ F(x,s,\underline{v}(x)) & \text{if } \underline{u}(x) \le s \le \overline{u}(x),\ t < \underline{v}(x), \\ F(x,\underline{u}(x),\underline{v}(x)) & \text{if } s < \underline{u}(x),\ t < \underline{v}(x), \\ F(x,\overline{u}(x),t) & \text{if } \overline{u}(x) < s,\ \underline{v}(x) \le t \le \overline{v}(x), \\ F(x,\overline{u}(x),\overline{v}(x)) & \text{if } \overline{u}(x) < s,\ \overline{v}(x) < t, \\ F(x,\underline{u}(x),\overline{v}(x)) & \text{if } s < \underline{u}(x),\ \overline{v}(x) < t, \\ F(x,\overline{u}(x),\underline{v}(x)) & \text{if } \overline{u}(x) < s,\ t < \underline{v}(x), \\ F(x,s,\overline{v}(x)) & \text{if } \underline{u}(x) \le s \le \overline{u}(x),\ \overline{v}(x) < t. \end{cases}$$

We will construct two sequences $u_n \in W_0^{1,p}(\Omega)$ and $v_n \in W_0^{1,q}(\Omega)$, $n \in \mathbb{N}$, as follows: consider the problem

$$\begin{gathered} -\Delta_p u = \widehat{H}_u(x,u,\overline{v}) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

The Euler-Lagrange functional associated with the above system is

$$\widehat{\Phi}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega\int_0^u \widehat{H}_u(x,s,\overline{v})ds\,dx$$

which is bounded from below, weakly lower semicontinuous and coercive in $W_0^{1,p}(\Omega)$. Therefore, the infimum of $\widehat{\Phi}(.)$ is achieved at some point $u_1 \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ which is a solution of (1.2). We claim that $\underline{u}(x) \le u_1(x) \le \overline{u}(x)$ for every $x \in \Omega$. Indeed, assume that the set

$$\widetilde{\Omega} := \{x \in \Omega : u_1(x) < \underline{u}(x)\}$$

is nonempty. Since it is open, it must have positive measure and

$$-\Delta_p u_1 = H_u(x, \underline{u}, \overline{v}) \quad \text{in } \widetilde{\Omega}, \tag{1.3}$$

while,

$$-\Delta_p \underline{u} < H_u(x, \underline{u}, \overline{v}) \quad \text{in } \widetilde{\Omega}. \tag{1.4}$$

Multiplying (1.3) and (1.4) with $\underline{u} - u_1$ and integrating over $\widetilde{\Omega}$, we get

$$\int_{\Omega} |\nabla u_0|^{p-2} \nabla u_1 \nabla(\underline{u} - u_1) = \int_{\Omega} H_u(x, \underline{u}, \overline{v})(\underline{u} - u_1),$$

and

$$\int_{\Omega} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla(\underline{u} - u_1) = \int_{\Omega} H_u(x, \underline{u}, \overline{v})(\underline{u} - u_1),$$

which combined yield

$$\int_{\Omega} \Big[|\nabla \underline{u}|^{p-2} \nabla \underline{u} - |\nabla u_1|^{p-2} \nabla u_1 \Big] \nabla(\underline{u} - u_1) < 0,$$

contradicting the strong monotonicity of the $-\Delta_p$ operator. Thus $\widetilde{\Omega}$ is empty. Similarly, $u_1(x) \le \overline{u}(x)$ for every $x \in \Omega$.

Consider the problem

$$\begin{gathered} -\Delta_q v = \widehat{H}_v(x, u_1, v) \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

Working as in (1.2) we can show that it admits a solution $v_1 \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ with $\underline{v}(x) \le v_1(x) \le \overline{v}(x)$. Assuming now that $u_n \in W_0^{1,p}(\Omega)$ and $v_n \in W_0^{1,q}(\Omega)$, $n = 1, \dots k-1$, have been defined, we let $u_k \in W_0^{1,p}(\Omega)$ be a solution of (1.2) with v_{k-1} in the place of $\overline{v}$ and $v_k \in W_0^{1,p}(\Omega)$ be a solution of (1.5) with u_k in the place of u_1. Since $\widehat{H}_u(x, s, t)$ and $\widehat{H}_v(x, s, t)$ are bounded, the sequences $u_n \in W_0^{1,p}(\Omega)$ and $v_n \in W_0^{1,q}(\Omega)$, $n \in \mathbb{N}$, are also bounded, so $u_n \to u_0$ weakly in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$ weakly in $W_0^{1,q}(\Omega)$. Exploiting the continuity of $H_u(x, ., .)$ and $H_v(x, ., .)$ and the Sobolev embedding we easily deduce that (u_0, v_0) is a solution of the system

$$\begin{gathered} -\Delta_p u = \widehat{H}_u(x, u, v) \quad \text{in } \Omega \\ -\Delta_q v = \widehat{H}_v(x, u, v) \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

while $\underline{u}(x) \le u_0(x) \le \overline{u}(x)$, $\underline{v}(x) \le v_0(x) \le \overline{v}(x)$ for every $x \in \Omega$. Thus

$$\widehat{H}_u(x, u_0, v_0) = H_u(x, u_0, v_0), \quad \widehat{H}_v(x, u_0, v_0) = H_v(x, u_0, v_0).$$

Consequently, (u_0, v_0) is a critical point of $\Phi(., .)$ and therefore a solution of (1.1). On account of (H1)(i), we have

$$-\Delta_p \underline{u} < H_u(x, \underline{u}, \underline{v}) \le H_u(x, u_0, v_0) = -\Delta_p u_0 \quad \text{in } \Omega,$$

and so, by the strong comparison principle in [5, Proposition 2.2] , we deduce that

$$0 \le \underline{u} < u_0 \text{ in } \Omega.$$

Similarly, $v_0 > 0$ in Ω. □

Remark 1.2. In the case of a single equation, the existence of a solution is established by minimizing (locally) the energy functional. By making use of the fact that this solution is a minimizer, an application of the mountain pass principle provides a second solution [1, 3]. However, in our case it is not clear that the solution (u_0, v_0) provided by the previous Theorem is a (local) minimizer of $\Phi(.,.)$.

Let λ_1 denote the principal eigenvalue of the $p-$Laplace operator and μ_1 the principal eigenvalue of the $q-$Laplace operator in Ω.

Corollary 1.3. *Assume that hypotheses (H1)–(H4) hold. Then* (1.1) *admits a strict supersolution* $(\overline{u}, \overline{v})$ *and*

$$\lim_{s\to 0^+} \frac{H_u(x,s,t)}{s^{p-1}} > \lambda_1, \quad \lim_{t\to 0^+} \frac{H_v(x,s,t)}{t^{q-1}} > \mu_1 \tag{1.6}$$

for a.e. $x \in \Omega$ *and* $s, t > 0$. *Then* (1.1) *has a solution* (u_0, v_0) *with* $u_0, v_0 > 0$ *in* Ω.

Proof. Let $\varphi_1 > 0$ be an eigenfunction corresponding to λ_1 and $\psi_1 > 0$ an eigenfunction corresponding to μ_1. In view of (1.6) there exists $\varepsilon > 0$ such that $(\varepsilon\varphi_1, \varepsilon\psi_1)$ is a strict subsolution of (1.1). Furthermore, as a consequence of the maximum principle [6], by taking ε sufficiently small we have that $\varepsilon\varphi_1 < \overline{u}$ and $\varepsilon\psi_1 < \overline{v}$ in Ω. Theorem 1.1 implies that (1.1) has a solution (u_0, v_0) with $u_0, v_0 > 0$ in Ω. □

We now present a simple (academic) example. Assume that $H(.,.,.)$ is a C^1 function satisfying (H1)–(H3) and

$$H_u(x, \xi s, \xi t) = \xi^\alpha H_u(x, s, t), \quad H_v(x, \xi s, \xi t) = \xi^\alpha H_v(x, s, t)$$

for some $\alpha \in [1, \min\{p-1, q-1\}]$ and every $s, t, \xi > 0$. Then H satisfies (H4) since

$$\begin{aligned} H_u(s,t) &= H_u\big(\sqrt{s^2+t^2}\frac{s}{\sqrt{s^2+t^2}}, \sqrt{s^2+t^2}\frac{t}{\sqrt{s^2+t^2}}\big) \\ &= (s^2+t^2)^{\frac{\alpha}{2}} H_u\big(\frac{s}{\sqrt{s^2+t^2}}, \frac{t}{\sqrt{s^2+t^2}}\big) \le M(s^2+t^2)^{\frac{\alpha}{2}} \\ &\le C_1(1+s^\alpha+t^\alpha), \end{aligned}$$

for some $C_1 > 0$, where $M = \sup\{H_u(s,t) : s^2+t^2 = 1\}$. Similarly, $H_u(s,t) \le C_2(1+s^\alpha+t^\alpha)$ for some $C_2 > 0$.

If $\widehat{u}$, $\widehat{v}$ are the solutions of

$$\begin{gathered} -\Delta_p u = 1 \quad \text{in } \Omega \\ -\Delta_q v = 1 \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

then there exists $\zeta > 0$ such that $(\overline{u}, \overline{v}) := (\zeta\widehat{u}, \zeta\widehat{v})$ is a strict supersolution of (1.1). Indeed, if

$$M = \sup_{x\in\Omega} \big\{H_u(x, \widehat{u}(x), \widehat{v}(x)), H_v(x, \widehat{u}(x), \widehat{v}(x))\big\},$$

then for $\zeta > \max\{M^{1/(1-p-\alpha)}, M^{1/(1-q-\alpha)}\}$ we have

$$-\Delta_p(\zeta\widehat{u}) = \zeta^{p-1} > M\zeta^\alpha \ge \zeta^\alpha H_u(x, \widehat{u}, \widehat{v}) = H_u(x, \zeta\widehat{u}, \zeta\widehat{v}).$$

Similarly, $-\Delta_q(\zeta\widehat{v}) > H_v(x, \zeta\widehat{u}, \zeta\widehat{v})$. On the other hand, (1.6) is satisfied because $\alpha < \min\{p-1, q-1\}$. By Corollary 1.3, (1.1) admits a positive solution.

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