# NONEXISTENCE OF SOLUTIONS TO KPP-TYPE EQUATIONS OF DIMENSION GREATER THAN OR EQUAL TO ONE

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ABSTRACT. In this article, we consider a semilinear elliptic equations of the form $\Delta u + f(u) = 0$, where $f$ is a concave function. We prove for arbitrary dimensions that there is no solution bounded in $(0, 1)$. The significance of this result in probability theory is also discussed.

## 1. INTRODUCTION AND STATEMENT OF MAIN RESULT

In this article, we study semilinear elliptic equations of the form $\Delta u + f(u) = 0$. On the nonlinear term $f : [0, 1] \to \mathbb{R}$ we assume that

  (i) $f$ is continuous,
 (ii) $f$ is positive on $(0, 1)$,
(iii) the mapping $z \mapsto f(z)/z$ is strictly decreasing.

Under these three conditions, we consider the Kolmogorov Petrovskii Piscunov-type (KPP-type) equation

$$\Delta u + f(u) = 0 \tag{1.1}$$

$$0 < u < 1, \quad \text{in } \mathbb{R}^d. \tag{1.2}$$

Our main result is as follows.

**Theorem 1.1.** *Problem* (1.1)-(1.2) *has no solution for dimension* $d \geq 1$.

Semilinear elliptic equations of the form (1.1) have been widely studied. We mention here only two reviews [14, 15], where the exact number of positive solutions with different nonlinearities are studied. In [14] the differential equation is considered on a bounded domain, in [15] the equation is studied in the whole space $\mathbb{R}$, however, it is subject to the boundary condition $u \to 0$ as $|x| \to \infty$. The case of concave $f$ has also been studied by several authors. In [1] the assumption on $f$ is similar to ours, however, the problem is given on a bounded domain with Dirichlet boundary condition. In that paper the existence and uniqueness of the positive solution is proved. Castro et al. studied the case of concave nonlinearities in a series of papers, see e.g. [2, 3]. In these works the problem is given on a bounded domain with Dirichlet boundary condition. A generalized logistic equation, with

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$f(u) = mu - qu^p$ is studied in [8] on a bounded domain with Dirichlet boundary condition again.

Summarizing, we can say that our equation (1.1) has been widely studied, however, in the papers where it is considered in the whole space $\mathbb{R}$, it is always subject to the boundary condition $u \to 0$ as $|x| \to \infty$. In these publications the aim is to determine the exact number of the so-called *fast and slow decay* solutions. Hence, according to the authors knowledge, there is no result available concerning problem (1.1)-(1.2) under the assumptions given on $f$.

**Remark 1.2** (Low dimensions)**.** Our theorem can be proved very easily for $d \leq 2$. To see this, recall that $\Delta$ is a so-called *critical* operator in $\mathbb{R}^d$ when $d = 1, 2$. Second order elliptic operators $L$ with no zeroth order term are classified as being *subcritical* or critical according to whether the operator possesses or does not possess a minimal positive Green's function. In probabilistic terms criticality/subcriticality is captured by the *recurrence/transience* of the corresponding diffusion process (see [12, Chapter 4]).

Another equivalent condition for $L$ to be critical is that all positive functions $h$ that are superharmonic (i.e. $Lh \leq 0$) are in fact harmonic (i.e. $Lh \equiv 0$). (See again [12, Chapter 4])

Now, observe that (1.1)-(1.2) and the positivity of $f$ on $(0, 1)$ implies

$$\Delta u = -f(u) < 0 \quad \text{in } \mathbb{R}^d. \tag{1.3}$$

By the above criterion for critical operators, this is impossible in dimension one or two.

The most important model case is the classical KPP equation, when

$$f(u) := \beta u (1 - u) \tag{1.4}$$

with $\beta > 0$. (In fact this particular nonlinearity is intimately related to the distribution of a *branching Brownian motion*; see more on the subject in the next paragraph.) We will present a proof for our result that works basically for concave functions; in fact, (iii) of Assumption 1 is related to the concaveness of the function.

The connection between the KPP equation and branching Brownian motion has already been discovered by H. P. McKean — it first appeared in the classic work [10, 11].

Let $Z = (Z(t))_{t \geq 0}$ be the $d$-dimensional binary branching Brownian motion with a spatially and temporally constant branching rate $\beta > 0$. The informal description of this process is as follows. A single particle starts at the origin, performs a Brownian motion on $\mathbb{R}^d$, after a mean–$1/\beta$ exponential time dies and produces two offspring, the two offspring perform independent Brownian motions from their birth location, die and produce two offspring after independent mean–$1/\beta$ exponential times, etc. Think of $Z(t)$ as the subset of $\mathbb{R}^d$ indicating the locations of the particles $z_1^t, \ldots, z_t^{N_t}$ alive at time $t$ (where $N_t$ denote the number of particles at $t$). Write $P_x$ to denote the law of $Z$ when the initial particle starts at $x$. The natural filtration is denoted by $\{\mathcal{F}_t,\ t \geq 0\}$.

Then, as is well known (see e.g. [4, Chapter 1]), the law of the process can be described via its Laplace functional as follows. If $f$ is a positive measurable function, then

$$E_x \exp\left( -\sum_{i=1}^{N_t} f(z_i^t) \right) = 1 - u(x, t), \tag{1.5}$$

where $u$ solves the initial value problem

$$\dot{u} = \frac{1}{2}\Delta u + f(u) \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+$$
$$u(\cdot, 0) = 1 - e^{-f(\cdot)} \quad \text{in } \mathbb{R}^d \tag{1.6}$$
$$0 \leq u \leq 1 \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+,$$

with $f$ of the form (1.4).

Equation (1.1)-(1.2) appears when one studies certain 'natural' martingales associated with branching Brownian motion (see e.g. [5]). To understand this, let $\widehat{\mathcal{F}}_t := \sigma(\bigcup_{s \geq t} \mathcal{F}_s)$ and consider the tail $\sigma$-algebra $\widehat{\mathcal{F}}_\infty := \bigcap_{t \geq 0} \widehat{\mathcal{F}}_s$. Choosing appropriate (sequences of) $f$'s one can then express the probabilities of various events $A_t \in \widehat{\mathcal{F}}_t$, for $t > 0$, in terms of the function $u$ in (1.6). Letting $t \to \infty$ then leads to the conclusion that if $A \in \widehat{\mathcal{F}}_\infty$ denotes a certain tail event (e.g. having strictly positive limit for a certain nonnegative 'natural' martingale, or local/global extinction) then the function $u(x) := P_x(A)$ is either constant ($= 0$ or $= 1$), or it must solve (1.1)-(1.2). Hence, it immediately follows from our main theorem that *the tail $\sigma$-algebra is trivial*, that is, all those events $A$ satisfy $P_\cdot(A) \equiv 0$ or $P_\cdot(A) \equiv 1$.

Note that if $\beta > 0$ is replaced by a smooth nonnegative function $\beta(\cdot)$ that does not vanish everywhere, then this corresponds to having *spatially dependent* branching rate for the branching Brownian motion. It would be desirable therefore to investigate whether our main theorem can be generalized for such $\beta$'s.

## 2. Proof of the theorem

The proof is based on two ideas: The application of the semilinear elliptic maximum principle, which is generalized here fore concave functions, and a comparison between the semilinear and the linear problems. Using these two ideas we will show that the *minimal positive solution* of (1.1) is $u_{\min} \equiv 1$, hence (1.1) has no solution satisfying (1.2).

First we state and prove a semilinear maximum principle. The results in this form is a generalization of [6, Proposition 7.1] for the particular case when the elliptic operator is $L = \Delta$.

**Lemma 2.1** (Semilinear elliptic maximum principle). *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function, for which the mapping $z \mapsto f(z)/z$ is strictly decreasing. Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. If $v_i \in C^2(D) \cap C(\bar{D})$ satisfy $v_i > 0$ in $D$, $\Delta v_i + f(v_i) = 0$, in $D$ for $i = 1, 2$, and $v_1 \geq v_2$ on $\partial D$, then $v_1 \geq v_2$ in $\bar{D}$.*

*Proof.* Note that the function $w := v_1 - v_2$ satisfies

$$\Delta w + f(v_1) - f(v_2) = 0. \tag{2.1}$$

We show that $w \geq 0$ in $D$. Suppose to the contrary that there exists a point $y \in D$ where $w$ is negative. Let $\Omega_0 := \{x \in D \mid w(x) < 0\}$. Let $\Omega$ be the connected component of $\Omega_0$ containing $y$. Since $w \geq 0$ on $\partial D$, one has $\Omega \subset\subset D$ and

$$w < 0 \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Let us multiply the equation $\Delta v_1 + f(v_1) = 0$ by $w$ and equation (2.1) by $v_1$, then subtract the second equation from the first, and integrate on $\Omega$. Using that

$w = v_1 - v_2$ one obtains

$$I + II := \int_\Omega (w\Delta v_1 - v_1 \Delta w) + \int_\Omega (v_1 f(v_2) - v_2 f(v_1)) = 0. \qquad (2.3)$$

Using Green's second identity and that $w = 0$ on $\partial\Omega$, along with the fact that $\partial_\nu w \geq 0$ on $\partial\Omega$, we obtain

$$I = -\int_{\partial\Omega} v_1 \partial_\nu w \leq 0,$$

where $\nu$ denotes the unit outward normal to $\partial\Omega$. Furthermore, since $v_1 < v_2$ in $\Omega$, using (iii) of Assumption 1, we have that also $II < 0$:

$$v_1 f(v_2) - v_2 f(v_1) = v_1 v_2 \left[ \frac{f(v_2)}{v_2} - \frac{f(v_1)}{v_1} \right] < 0.$$

It follows that the left hand side of (2.3) is negative, while its right hand side is zero. This contradiction proves that in fact $w \geq 0$ in $D$. $\qquad\square$

**Remark 2.2** (Spatially dependent $f$'s)**.** One can similarly prove the analogous more general result for the case, when $f : D \times [0, \infty) \to \mathbb{R}$ is continuous in $u$ and bounded in $x$, and $u \mapsto f(x, u)/u$ is strictly decreasing.

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which is positive in $(0, 1)$. Based on ideas in [9] and using the comparison between the linear and the semilinear equations, we prove the following lemma.

**Lemma 2.3** (Radially symmetric solutions)**.** *Assume in addition that $f$ satisfies* $\liminf_{z\downarrow 0} \frac{f(z)}{z} > 0$ *(this is automatically satisfied under assumption that the mapping $z \mapsto f(z)/z$ is strictly decreasing). Then for any $y \in \mathbb{R}^d$ and $p \in (0, 1)$ there exists a ball $\Omega := B_R(y)$ (with some $R > 0$) and a radially symmetric $C^2$ function $v : \Omega \to \mathbb{R}$ such that*

$$\Delta v + f(v) = 0$$
$$v > 0 \quad in\ \Omega$$
$$v = 0 \quad on\ \partial\Omega \quad v(y) = p\,.$$

*Proof.* We show the existence of a radially symmetric solution of the form $v(x) = V(|x - y|)$. Let $V \in C^2([0, \infty))$ be the solution of the initial value problem

$$(r^{d-1}V'(r))' + r^{d-1}f(V(r)) = 0$$
$$V(0) = p,\ V'(0) = 0. \qquad (2.4)$$

Writing $\Delta$ in polar coordinates, one sees that it is sufficient to prove that there exists an $R > 0$ such that $V(R) = 0$ and $V(r) > 0$ for all $r \in [0, R)$. To this end, consider the *linear* initial value problem

$$(r^{d-1}W'(r))' + r^{d-1}mW(r) = 0$$
$$W(0) = p,\ W'(0) = 0, \qquad (2.5)$$

where $m > 0$ is chosen so that $f(u) > mu$ holds for all $u \in (0, p)$. (Our assumptions on $f$ guarantee the existence of such an $m$.) It is known that $W$ has a first root, which we denote by $\rho$. Note that in this case $-m$ is the first eigenvalue of the Laplacian on the ball $B_\rho$. We now show that $V$ has a root in $(0, \rho]$. In order to do

so let us multiply (2.5) by $V$ and (2.4) by $W$, then subtract one equation from the other, and finally, integrate on $[0, \rho]$. We obtain

$$
\begin{aligned}
I + II := &\int_0^\rho [(r^{d-1} W'(r))' V(r) - (r^{d-1} V'(r))' W(r)] \, \mathrm{d}r \\
&+ \int_0^\rho r^{d-1} [mW(r)V(r) - W(r)f(V(r))] \, \mathrm{d}r = 0 \, .
\end{aligned}
\tag{2.6}
$$

Suppose now that $V$ has no root in $(0, \rho]$. Then, integrating by parts, $I = \rho^{d-1} W'(\rho) V(\rho) < 0$.

Next, observe that by integrating (2.4), one gets $V'(r) < 0$ (i.e. $V$ is decreasing). Hence $V(r) < p$, yielding $mV(r) - f(V(r)) < 0$. Therefore $II$, and thus the whole left hand side of (2.6) are negative; contradiction. This contradiction proves that $V$ in fact has a root in $(0, \rho]$. $\qquad\square$

**Remark 2.4** (Spatially dependent f's)**.** When $f$ depends also on $x$, our method breaks down as it is no longer possible to use ordinary differential equations to show the existence of a solution attaining a value close to one at a given point.

There is one easy case though: it is immediately seen that if there exists a $g(u)$, with $f(x, u) \geq g(u)$ and $g(u)$ satisfies the conditions of Theorem 1.1, then Theorem 1.1 remains valid for $f(x, u)$ as well.

Indeed, we know that $u_{min} \geq 1$, where $u_{min}$ is the minimal positive solution for the semilinear equation with $g$. Recall (see e.g. [6, 7]) that one way of constructing the minimal positive solution is as follows. One takes large balls $B_R(0)$, and positive solutions with zero boundary condition on these balls (in our case we know from [9] that there exist such positive solutions for arbitrarily large $R$'s), and finally, lets $R \to \infty$; using the monotonicity in $R$ that follows from the semilinear elliptic maximum principle (Lemma 2.1), the limiting function exists and positive. It is standard to prove that it solves the equation on the whole space, and by Lemma 2.1 again it must be the *minimal* such solution.

Now suppose that $0 < v$ solves the semilinear equation with $f(x, u)$. Then $v$ is a *supersolution*: $0 \geq \Delta v + g(v)$; hence by the above construction of $u_{min}$ and by an obvious modification of the proof of Lemma 2.1, $v \geq u_{min} \geq 1$.

The general case is harder. For example, when $f(x, u) := \beta(x)(u - u^2)$ and $\beta$ is a smooth nonnegative bounded function, the mere existence of positive solutions on large balls is no problem as long as the generalized principal eigenvalue of $\Delta + \beta$ on $\mathbb{R}^d$ is positive. (The method in [13], pp. 26-27 goes through for $f(x, u) := \beta(x)(u - u^2)$ even though $\beta$ is constant in [13].) The problematic part is to show that the solution is large at the center of the ball.

*Proof of Theorem 1.1.* Suppose that problem (1.1)-(1.2) has a solution. Choose an arbitrary point $y \in \mathbb{R}^d$ and an arbitrary number $p \in (0, 1)$. Note that by Assumption 1, $f$ satisfies the conditions of Lemma 2.3 and consider the ball $B_R(y)$ and the radially symmetric function $v$ on it, which are guaranteed by Lemma 2.3. We can apply Lemma 2.1 with $D = B_R(y)$, $v_1 = u$ and $v_2 = v$ and obtain that $u \geq v$. In particular then, $u(y) \geq v(y) = p$. Since $y$ and $p$ were arbitrary, we obtain that $u \geq 1$, in contradiction with (1.2). Consequently, (1.1)-(1.2) has no solution. $\qquad\square$

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