Electronic Journal of Differential Equations, Vol. 2006(2006), No. 107, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

PERIODIC SOLUTIONS FOR A KIND OF RAYLEIGH EQUATION WITH TWO DEVIATING ARGUMENTS

YUANHENG WU, BING XIAO, HONG ZHANG

Abstract. In this paper, we use the coincidence degree theory to establish new results on the existence of T-periodic solutions for the Rayleigh equation with two deviating arguments of the form

$$x'' + f(x(t), x'(t)) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) = p(t).$$

1. Introduction

Consider the Rayleigh equation with two deviating arguments

$$x'' + f(x(t), x'(t)) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) = p(t), \tag{1.1}$$

where $\tau_1,\ \tau_2,\ p : \mathbb{R} \to \mathbb{R}$ and $f,\ g_1,\ g_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions, $f(x,0) = 0$, τ_1, τ_2 and p are T-periodic, g_1 and g_2 are T-periodic in the first argument, and $T > 0$. In recent years, the problem of the existence of periodic solutions of (1.1) has been extensively studied in the literature. We refer the reader to [2, 4, 5, 6, 3] and the references cited therein. Moreover, in the above-mentioned literature, we find the following assumptions:

(H0) $g_1(t,x) + g_2(t,x) = g(x)$, $g(x) \in C(\mathbb{R},\mathbb{R})$ and there exist constants $k_1 \ge 0$ and $k_2 \ge 0$ such that one of the following conditions holds:
(1) $xg(x) > 0$, for all $|x| > k_1$, and $g(x) \ge -k_2$, for all $x \le -k_1$,
(2) $xg(x) > 0$, for all $|x| > k_1$, and $g(x) \le k_2$, for all $x \ge k_1$;

(H1) $g_1(t,x)+g_2(t,x) = g(x)$, $g(x) \in C^1(\mathbb{R},\mathbb{R})$ and there exists a constant $K \ge 0$ such that
$$|g'(x)| \le K, \forall x \in \mathbb{R};$$

(H2) $f(x,y) = f(y)$, and there exist constants $r \ge 0$ and $K > 0$ such that
$$|f(y)| \le r|y| + K, \forall y \in \mathbb{R};$$

(H3) $f(x,y) = f(y)$, and there exists constants $n \ge 1$ and $\sigma > 0$ such that
$$yf(y) \ge \sigma|y|^{n+1}, \quad \forall y \in \mathbb{R} \quad \text{or} \quad yf(y) \le -\sigma|y|^{n+1}, \quad \forall y \in \mathbb{R}.$$

2000 *Mathematics Subject Classification.* 34C25, 34D40.
Key words and phrases. Rayleigh equation; deviating argument; periodic solution; coincidence degree.

Submitted January 27, 2006. Published September 8, 2006.
Supported by the NNSF of China and by project 05JJ40009 from the Hunan Provincial Natural Science Foundation of China.

These conditions have been considered for the existence of periodic solutions of (1.1). However, to the best of our knowledge, few authors have considered (1.1) without the assumptions (H0)–(H3). Thus, it is worth while to continue to investigate the existence of periodic solutions of (1.1) in this case.

The main purpose of this paper is to establish sufficient conditions for the existence of T-periodic solutions of (1.1). The results of this paper are new and they complement previously known results. In particular, we do not use assumptions (H0)–(H3), and we illustrate our results with examples in Section 4.

For ease of exposition, throughout this paper we will adopt the following notation:

$$|x|_k = \Big(\int_0^T |x(t)|^k dt\Big)^{1/k}, \quad |x|_\infty = \max_{t\in[0,T]} |x(t)|.$$

Let

$$X = \{x|x \in C^1(\mathbb{R},\mathbb{R}) : x(t+T) = x(t), \text{ for all } t \in \mathbb{R}\},$$
$$Y = \{x|x \in C(\mathbb{R},\mathbb{R}), x(t+T) = x(t), \text{ for all } t \in \mathbb{R}\}$$

be two Banach spaces with the norms

$$\|x\|_X = \max\{|x|_\infty, |x'|_\infty\}, \quad \text{and} \quad \|x\|_Y = |x|_\infty.$$

Define a linear operator $L : D(L) \subset X \to Y$, with $D(L) = \{x|x \in X : x'' \in C(\mathbb{R},\mathbb{R})\}$ and for $x \in D(L)$,

$$Lx = x''. \tag{1.2}$$

We also define the nonlinear operator $N : X \to Y$ by

$$Nx = -f(x(t), x'(t)) - g_1(t, x(t-\tau_1(t))) - g_2(t, x(t-\tau_2(t))) + p(t). \tag{1.3}$$

It is easy to see that

$$\ker L = \mathbb{R}, \quad \text{and} \quad \operatorname{Im} L = \{x : x \in Y, \int_0^T x(s)ds = 0\}.$$

Thus, the operator L is a Fredholm operator with index zero. Define the continuous projectors $P : X \to \ker L$ and $Q : Y \to Y$ by setting

$$Px(t) = x(0) = x(T), \quad Qx(t) = \frac{1}{T}\int_0^T x(s)ds.$$

and let

$$L_P = L|_{D(L)\cap \ker P} : D(L) \cap \ker P \to \operatorname{Im} L$$

Then, according to [4], we have that L_P has continuous inverse L_P^{-1} on $\operatorname{Im} L$ defined by

$$L_P^{-1}y(t) = -\frac{t}{T}\int_0^T (t-s)y(s)ds + \int_0^t (t-s)y(s)ds.$$

2. Preliminary Results

In view of (1.2) and (1.3), the operator equation $Lx = \lambda Nx$ is equivalent to the equation

$$x'' + \lambda[f(x(t), x'(t)) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t)))] = \lambda p(t), \tag{2.1}$$

where $\lambda \in (0, 1)$.

For convenience of use, we introduce the Continuation Theorem; see [1].

Lemma 2.1. *Let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;*
(2) *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \ker L$;*
(3) *The Brouwer degree, $\deg\{QN, \Omega \cap \ker L, 0\}$, is not equal to zero.*

Then the equation $Lx = Nx$ has at least one T-periodic solution in $\overline{\Omega}$.

The following lemma will be useful for proving our main results in Section 3.

Lemma 2.2. *Assume that the following conditions are satisfied:*

(A1) *One of the following conditions holds:*
  (1) *$(g_i(t, u_1) - g_i(t, u_2))(u_1 - u_2) > 0$, for $i = 1, 2, u_i \in \mathbb{R}$, for all $t \in \mathbb{R}$ and $u_1 \neq u_2$,*
  (2) *$(g_i(t, u_1) - g_i(t, u_2))(u_1 - u_2) < 0$, for $i = 1, 2, u_i \in \mathbb{R}$, for all $t \in \mathbb{R}$ and $u_1 \neq u_2$;*

(A2) *There exists a constant $d > 0$ such that one of the following conditions holds:*
  (1) *$x(g_1(t, x) + g_2(t, x) - p(t)) > 0$, for all $t \in \mathbb{R}$, $|x| \geq d$,*
  (2) *$x(g_1(t, x) + g_2(t, x) - p(t)) < 0$, for all $t \in \mathbb{R}$, $|x| \geq d$.*

If $x(t)$ is a T-periodic solution of (2.1), then

$$|x|_\infty \leq d + \sqrt{T}|x'|_2. \tag{2.2}$$

Proof. Let $x(t)$ be a T-periodic solution of (2.1). Set

$$x(t_{\max}) = \max_{t \in \mathbb{R}} x(t), \quad x(t_{\min}) = \min_{t \in \mathbb{R}} x(t),$$

where $t_{\max}, t_{\min} \in \mathbb{R}$. Then

$$x'(t_{\max}) = 0, \quad x''(t_{\max}) \leq 0, \quad \text{and} \quad x'(t_{\min}) = 0, \quad x''(t_{\min}) \geq 0. \tag{2.3}$$

In view of $f(x, 0) = 0$ and (2.1), Equation (2.3) implies

$$\begin{aligned} &g_1(t_{\max}, x(t_{\max} - \tau_1(t_{\max}))) + g_2(t_{\max}, x(t_{\max} - \tau_2(t_{\max}))) - p(t_{\max}) \\ &= -\frac{x''(t_{\max})}{\lambda} \geq 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &g_1(t_{\min}, x(t_{\min} - \tau_1(t_{\min}))) + g_2(t_{\min}, x(t_{\min} - \tau_2(t_{\min}))) - p(t_{\min}) \\ &= -\frac{x''(t_{\min})}{\lambda} \leq 0. \end{aligned} \tag{2.5}$$

Since $g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - p(t)$ is a continuous function on $\mathbb{R}$, it follows from (2.4) and (2.5) that there exists a constant $t_1 \in \mathbb{R}$ such that

$$g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) = 0. \tag{2.6}$$

Next we show that the following claim is true.

Claim: If $x(t)$ is a T-periodic solution of (2.1), then there exists a constant $t_2 \in \mathbb{R}$ such that

$$|x(t_2)| \leq d. \tag{2.7}$$

Proof. Assume, by way of contradiction, that (2.7) does not hold. Then

$$|x(t)| > d, \quad \text{for all } t \in \mathbb{R}, \tag{2.8}$$

which, together with (A2) and (2.6), implies that one of the following relations holds:

$$x(t_1 - \tau_1(t_1)) > x(t_1 - \tau_2(t_1)) > d; \tag{2.9}$$

$$x(t_1 - \tau_2(t_1)) > x(t_1 - \tau_1(t_1)) > d; \tag{2.10}$$

$$x(t_1 - \tau_1(t_1)) < x(t_1 - \tau_2(t_1)) < -d; \tag{2.11}$$

$$x(t_1 - \tau_2(t_1)) < x(t_1 - \tau_1(t_1)) < -d. \tag{2.12}$$

Suppose that (2.9) holds, in view of (A1)(1), (A1)(2), (A2)(1) and (A2)(2), we consider following four cases:
Case (i). If (A2)(1) and (A1)(1) hold, according to (2.9), we obtain

$$\begin{aligned} 0 &< g_1(t_1, x(t_1 - \tau_2(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) \\ &< g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \end{aligned}$$

which contradicts (2.6). This contradiction implies (2.7).
Case (ii). If (A2)(1) and (A1)(2) hold, according to (2.9), we obtain

$$\begin{aligned} 0 &< g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_1(t_1))) - p(t_1) \\ &< g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \end{aligned}$$

which contradicts (2.6). This contradiction implies (2.7) .
Case (iii). If (A2)(2) and (A1)(1) hold, according to (2.9), we obtain

$$\begin{aligned} 0 &> g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_1(t_1))) - p(t_1) \\ &> g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \end{aligned}$$

which contradicts (2.6). This contradiction implies (2.7).
Case (iv). If (A2)(2) and (A1)(2) hold, according to (2.9), we obtain

$$\begin{aligned} 0 &> g_1(t_1, x(t_1 - \tau_2(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) \\ &> g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \end{aligned}$$

which contradicts (2.6). This contradiction implies (2.7).

Suppose that (2.10) (or (2.11), or (2.12)) holds, using the methods similarly to those used in Cases (i)–(iv), we can show that (2.7) is true. This completes the proof of the above claim. □

Let $t_2 = mT + t_0$, where $t_0 \in [0, T]$ and m is an integer. Then, using the Schwarz inequality and the relation

$$|x(t)| = |x(t_0) + \int_{t_0}^{t} x'(s)ds| \le d + \int_0^T |x'(s)|ds, t \in [0, T],$$

we obtain

$$|x|_\infty = \max_{t\in[0,T]} |x(t)| \le d + \sqrt{T}|x'|_2.$$

This completes the proof. □

3. Main Results

Theorem 3.1. *Suppose that (A1)(1) and (A2)(1) hold, and there exist nonnegative constants* m_1, m_2 , m_3 *and* m_4 *such that* $2m_1+4m_3 < \frac{1}{2T^2}$, *and one of the following conditions holds:*

(1) $f(x,y) \leq 0$ *for all* $x \in \mathbb{R}$, $y \in \mathbb{R}$, $|g_2(t,x)| \leq m_3|x| + m_4$ *for all* $t \in \mathbb{R}$, $x \in \mathbb{R}$, *and*
$$g_1(t,x) + g_2(t,x) - p(t) \leq m_1 x + m_2, \quad \forall t \in \mathbb{R},\ x \geq d;$$
(2) $f(x,y) \geq 0$ *for all* $x \in \mathbb{R}, y \in \mathbb{R}$, $|g_2(t,x)| \leq m_3|x| + m_4$ *for all* $t \in \mathbb{R}$, $x \in \mathbb{R}$, *and*
$$g_1(t,x) + g_2(t,x) - p(t) \geq m_1 x - m_2, \quad \forall t \in \mathbb{R},\ x \leq -d.$$

Then (1.1) *has at least one* T*-periodic solution.*

Proof. We shall seek to apply Lemma 2.1. To do this, it suffices to prove that the set of all possible T-periodic solutions of (2.1) are bounded. Let $x(t)$ be a T-periodic solution of (2.1). Integrating (2.1) from 0 to T, we have
$$\int_0^T f(x(t),x'(t))dt + \int_0^T [g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_2(t))) - p(t)]dt = 0. \tag{3.1}$$
Set
$$\begin{gathered}
[x(t-\tau_1(t)) < -d] = \{t | t \in [0,T], x(t-\tau_1(t)) < -d\},\\
[x(t-\tau_1(t)) \geq -d] = \{t | t \in [0,T], x(t-\tau_1(t)) \geq -d\},\\
[x(t-\tau_1(t)) > d] = \{t | t \in [0,T], x(t-\tau_1(t)) > d\},\\
[x(t-\tau_1(t)) \leq d] = \{t | t \in [0,T], x(t-\tau_1(t)) \leq d\}.
\end{gathered}$$
Then, in view of (A2)(1), (3.1) implies
$$\begin{aligned}
&\int_{[x(t-\tau_1(t))<-d]} |g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_1(t))) - p(t)|dt\\
&= -\int_{[x(t-\tau_1(t))<-d]} [g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_1(t))) - p(t)]dt\\
&= \int_{[x(t-\tau_1(t))\geq -d]} [g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_1(t))) - p(t)]dt\\
&\quad - \int_0^T [g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_1(t))) - p(t)]dt\\
&= \int_{[x(t-\tau_1(t))\geq -d]} [g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_1(t))) - p(t)]dt\\
&\quad - \int_0^T [g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_2(t))) - p(t)]dt\\
&\quad - \int_0^T g_2(t,x(t-\tau_1(t)))dt + \int_0^T g_2(t,x(t-\tau_2(t)))dt\\
&= \int_{[x(t-\tau_1(t))\geq -d]} [g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_1(t))) - p(t)]dt\\
&\quad - \int_0^T g_2(t,x(t-\tau_1(t)))dt + \int_0^T g_2(t,x(t-\tau_2(t)))dt + \int_0^T f(x(t),x'(t))dt,
\end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
&\int_{[x(t-\tau_1(t))>d]} |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt\\
&=\int_{[x(t-\tau_1(t))>d]} [g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)]dt\\
&=-\int_{[x(t-\tau_1(t))\le d]} [g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)]dt\\
&\quad+\int_0^T [g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)]dt\\
&=-\int_{[x(t-\tau_1(t))\le d]} [g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)]dt\\
&\quad+\int_0^T [g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-p(t)]dt\\
&\quad+\int_0^T g_2(t,x(t-\tau_1(t)))dt-\int_0^T g_2(t,x(t-\tau_2(t)))dt\\
&=-\int_{[x(t-\tau_1(t))\le d]} [g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)]dt\\
&\quad+\int_0^T g_2(t,x(t-\tau_1(t)))dt-\int_0^T g_2(t,x(t-\tau_2(t)))dt-\int_0^T f(x(t),x'(t))dt.
\end{aligned}
\tag{3.3}
$$

Now suppose that (1) (or (2)) holds. We shall consider two cases as follows.
Case 1: If (1) holds, it follows from (2.2) and (3.2) that

$$
\begin{aligned}
&\int_{[x(t-\tau_1(t))<-d]} |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt\\
&\le\int_{[x(t-\tau_1(t))\ge -d]} |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt\\
&\quad+\int_0^T |g_2(t,x(t-\tau_1(t)))|dt+\int_0^T |g_2(t,x(t-\tau_2(t)))|dt\\
&\le\int_{\{t|t\in[0,T],|x(t-\tau_1(t))|\le d\}} |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt\\
&\quad+\int_{[x(t-\tau_1(t))>d]} |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt \qquad (3.4)\\
&\quad+\int_0^T (m_3|x(t-\tau_1(t))|+m_4)dt+\int_0^T (m_3|x(t-\tau_2(t))|+m_4)dt\\
&\le T(\max\{|g_1(t,x)+g_2(t,x)-p(t)|:t\in\mathbb{R},|x|\le d\})\\
&\quad+\int_0^T (m_1|x(t-\tau_1(t))|+m_2)dt+2T(m_3|x|_\infty+m_4)\\
&\le T(\max\{|g_1(t,x)+g_2(t,x)-p(t)|:t\in\mathbb{R},|x|\le d\}+m_2+2m_4)\\
&\quad+T(m_1+2m_3)|x|_\infty\\
&\le T(\theta_1+m_2+2m_4)+T(m_1+2m_3)(\sqrt{T}|x'|_2+d),
\end{aligned}
$$

where $\theta_1 = \max\{|g_1(t,x) + g_2(t,x) - p(t)| : t \in \mathbb{R}, |x| \le d\}$. Then, (3.4) implies

$$\begin{aligned} &\int_0^T |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)|dt \\ &= \int_{[x(t-\tau_1(t))<-d]} |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)|dt \\ &\quad + \int_{[x(t-\tau_1(t))\ge -d]} |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)|dt \\ &\le 2T(\theta_1 + m_2 + m_4) + 2T(m_1 + m_3)(\sqrt{T}|x'|_2 + d), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \int_0^T |f(x(t), x'(t))|dt &= -\int_0^T f(x(t), x'(t))dt \\ &= \int_0^T [g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) - p(t)]dt \\ &= \int_0^T [g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)]dt \\ &\quad - \int_0^T g_2(t, x(t-\tau_1(t)))dt + \int_0^T g_2(t, x(t-\tau_2(t)))dt \\ &\le \int_0^T |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)|dt \\ &\quad + \int_0^T |g_2(t, x(t-\tau_1(t)))|dt + \int_0^T |g_2(t, x(t-\tau_2(t)))|dt \\ &\le 2T(\theta_1 + m_2 + 2m_4) + 2T(m_1 + 2m_3)(\sqrt{T}|x'|_2 + d). \end{aligned} \tag{3.6}$$

Case 2: If (2) holds, it follows from (2.2) and (3.3) that

$$\begin{aligned} &\int_{[x(t-\tau_1(t))>d]} |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)|dt \\ &\le \int_{[x(t-\tau_1(t))\le d]} |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)|dt \\ &\quad + \int_0^T |g_2(t, x(t-\tau_1(t)))|dt + \int_0^T |g_2(t, x(t-\tau_2(t)))|dt \\ &\le \int_{\{t|t\in[0,T],\ |x(t-\tau_1(t))|\le d\}} |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)|dt \\ &\quad + \int_{[x(t-\tau_1(t))<-d]} |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)|dt \\ &\quad + \int_0^T (m_3|x(t-\tau_1(t))| + m_4)dt + \int_0^T (m_3|x(t-\tau_2(t))| + m_4)dt \\ &\le T(\max\{|g_1(t,x) + g_2(t,x) - p(t)| : t \in \mathbb{R}, |x| \le d\}) \\ &\quad + \int_0^T (m_1|x(t-\tau_1(t))| + m_2)dt + 2T(m_3|x|_\infty + m_4) \\ &\le T(\theta_1 + m_2 + 2m_4) + T(m_1 + 2m_3)(\sqrt{T}|x'|_2 + d), \end{aligned} \tag{3.7}$$

which implies

$$\begin{aligned}
&\int_0^T |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt\\
&=\int_{[x(t-\tau_1(t))>d]}|g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt\\
&\quad+\int_{[x(t-\tau_1(t))\le d]}|g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt\\
&\le 2T(\theta_1+m_2+m_4)+2T(m_1+m_3)(\sqrt{T}|x'|_2+d),
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
&\int_0^T |f(x(t),x'(t))|dt\\
&=\int_0^T f(x(t),x'(t))dt\\
&=-\int_0^T [g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-p(t)]dt\\
&\le\int_0^T |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt\\
&\quad+\int_0^T |g_2(t,x(t-\tau_1(t)))|dt+\int_0^T |g_2(t,x(t-\tau_2(t)))|dt\\
&\le 2T(\theta_1+m_2+2m_4)+2T(m_1+2m_3)(\sqrt{T}|x'|_2+d).
\end{aligned}\tag{3.9}$$

Multiplying (2.1) by $x(t)$ and then integrating from 0 to T, by (2.3), (3.5) , (3.6), (3.8) and (3.9), we have

$$\begin{aligned}
&|x'|_2^2\\
&=\lambda\int_0^T \{f(x(t),x'(t))+[g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-p(t)]\}x(t)dt\\
&=\lambda\int_0^T \{f(x(t),x'(t))+[g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)\\
&\quad-g_2(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))]\}x(t)dt\\
&\le\int_0^T |f(x(t),x'(t))||x(t)|dt\\
&\quad+\int_0^T |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)||x(t)|dt\\
&\quad+\int_0^T |g_2(t,x(t-\tau_1(t)))||x(t)|dt+\int_0^T |g_2(t,x(t-\tau_2(t)))||x(t)|dt\\
&\le |x|_\infty\{\int_0^T |f(x(t),x'(t))|dt\\
&\quad+\int_0^T |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_1(t)))-p(t)|dt+2T(m_3|x|_\infty+m_4)\}\\
&\le 2T[(2\theta_1+2m_2+4m_4)+(2m_1+4m_3)(\sqrt{T}|x'|_2+d)](\sqrt{T}|x'|_2+d)\\
&=2(2m_1+4m_3)T^2|x'|_2^2+2T[(2\theta_1+2m_2+4m_4)+2(2m_1+4m_3)d]\sqrt{T}|x'|_2
\end{aligned}\tag{3.10}$$

$$+ 2Td[(2\theta_1 + 2m_2 + 4m_4) + (2m_1 + 4m_3)d].$$

Since $0 \le 2m_1 + 4m_3 < \frac{1}{2T^2}$, (3.10) implies that there exists a positive constant D_1 such that

$$|x'|_2 \le D_1 \quad \text{and} \quad |x|_\infty \le \sqrt{T}|x'|_2 + d \le D_1. \tag{3.11}$$

In view of (3.5), (3.6), (3.8) and (3.9), it follows from (2.1) that

$$\begin{aligned}
&\int_0^T |x''(t)|dt \\
&\le \int_0^T |f(x(t), x'(t))|dt + \int_0^T |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t) \\
&\quad - g_2(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t)))|dt \\
&\le \int_0^T |f(x(t), x'(t))|dt + \int_0^T |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_1(t))) - p(t)|dt \\
&\quad + \int_0^T |g_2(t, x(t-\tau_1(t)))|dt + \int_0^T |g_2(t, x(t-\tau_2(t)))|dt \\
&\le 2T[(2\theta_1 + 2m_2 + 4m_4) + (2m_1 + 4m_3)(\sqrt{T}|x'|_2 + d)] \\
&\le 2T[(2\theta_1 + 2m_2 + 4m_4) + (2m_1 + 4m_3)(\sqrt{T}D_1 + d)] := D_2.
\end{aligned} \tag{3.12}$$

Since $x(0) = x(T)$, it follows that there exists a constant $\zeta \in [0, T]$ such that $x'(\zeta) = 0$ and

$$|x'(t)| = |x'(\zeta) + \int_\zeta^t x''(s)ds| \le \int_0^T |x''(t)|dt \le D_2, \quad \forall t \in [0, T],$$

which, together with (3.11), implies

$$\|x\|_X \le |x|_\infty + |x'|_\infty < D_1 + D_2 + 1 := M_1.$$

If $x \in \Omega_1 = \{x | x \in \ker L \cap X \text{ and } Nx \in \operatorname{Im} L\}$, then there exists a constant M_2 such that

$$x(t) \equiv M_2 \quad \text{and} \quad \int_0^T [g_1(t, M_2) + g_2(t, M_2) - p(t)]dt = 0.$$

Thus,

$$|x(t)| \equiv |M_2| < d, \quad \text{for all } x(t) \in \Omega_1. \tag{3.13}$$

Let $M = M_1 + d + 1$. Set

$$\Omega = \{x | x \in X, |x|_\infty < M, \ |x'|_\infty < M\}.$$

It is easy to see from (1.3) and (1) that N is L-compact on $\overline{\Omega}$. We have from (3), (3.13) and the fact $M > \max\{M_1, d\}$ that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, we define a continuous function $H(x, \mu)$ by setting

$$H(x, \mu) = -(1-\mu)x - \mu \cdot \frac{1}{T}\int_0^T [g_1(t, x) + g_2(t, x) - p(t)]dt; \quad \mu \in [0, 1].$$

In view of (A2)(1), we have

$$xH(x, \mu) \ne 0 \quad \text{for all } x \in \partial\Omega \cap \ker L.$$

Hence, using the homotopy invariance theorem, we obtain

$$\begin{aligned}\deg\{QN, \Omega \cap \ker L, 0\} &= \deg\{-\frac{1}{T}\int_0^T [g_1(t,x) + g_2(t,x) - p(t)]dt, \Omega \cap \ker L, 0\} \\ &= deg\{-x, \Omega \cap \ker L, 0\} \neq 0.\end{aligned}$$

In view of the discussions above, from Lemma 2.1 we complete the proof of Theorem 3.1. □

A similar argument leads to the following result.

Theorem 3.2. *Suppose that (A1)(2) and (A2)(2) holds, and there exist nonnegative constants m_1, m_2 , m_3 and m_4 such that $2m_1 + 4m_3 < \frac{1}{2T^2}$, and one of the following two conditions holds:*

1. *$f(x,y) \geq 0$ for all $x \in \mathbb{R}, y \in \mathbb{R}$, $|g_2(t,x)| \leq m_3|x| + m_4$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}$, and $g_1(t,x) + g_2(t,x) - p(t) \geq -m_1x - m_2$, for all $t \in \mathbb{R}$, $x \geq d$;*
2. *$f(x,y) \leq 0$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, $|g_2(t,x)| \leq m_3|x| + m_4$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}$, and*

$$g_1(t,x) + g_2(t,x) - p(t) \leq -m_1x + m_2, \quad \textit{for all } t \in \mathbb{R},\ x \leq -d.$$

Then (1.1) *has at least one T-periodic solution.*

4. Examples and Remarks

Example 4.1. Let $g(t,x) = x^{13} + \frac{1}{72\pi^2}x$ for $t \in \mathbb{R}$, $x \leq 0$, and $g(t,x) = \frac{1}{36\pi^2}x$ for $t \in \mathbb{R}$, $x > 0$. Then the Rayleigh equation

$$x'' - (x')^4 + g(t, x(t - \sin(t))) = e^{\cos^2 t}, \tag{4.1}$$

has at least one 2π-periodic solution.

Proof. Let $g_2(t,x) = \frac{1}{72\pi^2}x$ for $t \in \mathbb{R}$, $x \in \mathbb{R}$, $g_1(t,x) = x^{13}$ for $t \in \mathbb{R}$, $x \leq 0$, and $g_1(t,x) = \frac{1}{72\pi^2}x$ for $t \in \mathbb{R}$, $x > 0$. Then (4.1) is equivalent to the equation

$$x'' - (x')^4 + g_1(t, x(t - \sin(t))) + g_2(t, x(t - \sin(t))) = e^{\cos^2 t}. \tag{4.2}$$

From (4.2), we have $f(x,y) = -y^4 \leq 0$, $\tau_1(t) = \tau_2(t) = \sin t$, $p(t) = e^{\cos^2 t}$ and $g_1(t,x) + g_2(t,x) - p(t) = \frac{1}{36\pi^2}x - e^{\cos^2 t} \leq \frac{1}{36\pi^2}x + e$, for all $t \in \mathbb{R}$, $x > 0$. It is straightforward to check that all the conditions needed in Theorem 3.1 are satisfied. Therefore, (4.2) has at least one 2π-periodic solution. This implies that (4.1) has at least one 2π-periodic solution. □

Remark 4.2. Equation (4.1) is a very simple version of Rayleigh equation. Obviously, the conditions (H0)–(H3) are not satisfied. Therefore, the results in [2, 4, 5, 6, 3] and the references cited therein cannot be applied to (4.1). This implies that the results of this paper are essentially new.

Example 4.3. Let $g_1(t,x) = -\frac{1}{72\pi^2}x$ for $t \in \mathbb{R}$, $x \in \mathbb{R}$, $g_2(t,x) = -x^{13}$ for $t \in \mathbb{R}$, $x \leq 0$, and $g_2(t,x) = -\frac{1}{72\pi^2}x$ for $t \in \mathbb{R}$, $x > 0$. Then, the Rayleigh equation

$$x'' + x^4(x')^6 + g_1(t, x(t - \cos(t))) + g_2(t, x(t - \sin(t))) = \frac{1}{4}\cos^2 t. \tag{4.3}$$

has at least one 2π-periodic solution.

Proof. From (4.3), we can obtain $f(x,y) = x^4y^6$, $\tau_1(t) = \cos(t)$, $\tau_2(t) = \sin(t)$, $p(t) = \frac{1}{4}cos^2 t$ and $g_1(t,x) + g_2(t,x) - p(t) = -\frac{1}{36\pi^2}x - \frac{1}{4}cos^2 t \geq -\frac{1}{36\pi^2}x - \frac{1}{4}$, for $t \in \mathbb{R}$, $x > 0$. It is obvious that all the conditions needed in Theorem 3.2 are satisfied. Hence, by Theorem 3.2, equation (4.3) has at least one 2π-periodic solution. □

Remark 4.4. In view of (4.3), it is clear that (H0)–(H3), do not hold for (4.3), and so the results obtained in [2, 4, 5, 6, 3] and the references cited therein cannot be applied to (4.3).

Remark 4.5. Using the methods similarly to those used for (1.1), we can study the Rayleigh equation with multiple deviating arguments

$$x'' + f(x(t), x'(t)) + \sum_{i=1}^{n} g_i(t, x(t - \tau_i(t))) = p(t), \tag{4.4}$$

where $\tau_i (i = 1, 2, \dots, n)$, $p : \mathbb{R} \to \mathbb{R}$ and f, $g_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $f(x, 0) = 0$, τ_i and p are T-periodic, g_i are T-periodic in the first argument, and $T > 0$ $(i = 1, 2, \dots, n)$. One may also establish the results similarly to those in Theorems 3.1 and 3.2 under some minor additional assumptions on $g_i(t, x)$ $(i = 1, 2, \dots, n)$.

References

[1] R. E. Gaines, J. Mawhin; *Coincide degree and nonlinear differential equations*, Lecture Notes in Math.,N0. 568, Spring-Verlag, 1977.

[2] Genqiang Wang; *A priori bounds for periodic solutions of a delay Rayleigh equation*, Appl. Math. Lett. 12(1999), 41-44.

[3] Xiankai Huang and Z. G. Xiang; *On existence of 2π-periodic solutions for delay Duffing equation $x'' + g(t, x(t - \tau(t))) = p(t)$*, Chinese Science Bulletin, 39(1994), 201-203.

[4] Shiping Lu, Weigao Ge; *Some new results on the existence of periodic solutions to a kind of Rayleigh equation with a deviating argument*, Nonlinear Analysis, 56(2004), 501-514.

[5] Shiping Lu, Weigao Ge, Zuxiou Zheng; *Periodic solutions for neutral differential equation with deviating arguments*, Applied Mathematics and Computation. 152(2004), 17-27.

[6] Shiping Lu, Weigao Ge, Zuxiou Zheng; *A new result on the existence of periodic solutions for a kind of Rayleigh equation with a deviating argument* (in Chinese), Acta Mathematica Sinica, 47(2004), 299-304 .

Yuanheng Wu
College of Continuing Education, Guangdong University of Foreign Studies, Guangzhou 510420, China
E-mail address: wyhcd2006@yahoo.com.cn

Bing Xiao
Department of Mathematics, Hunan University of Arts and Science, Changde, Hunan 415000, China
E-mail address: changde1218@yahoo.com.cn

Hong Zhang
Department of Mathematics, Hunan University of Arts and Science, Changde, Hunan 415000, China
E-mail address: hongzhang320@yahoo.com.cn