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INSTABILITY RESULTS FOR CERTAIN THIRD ORDER NONLINEAR VECTOR DIFFERENTIAL EQUATIONS

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Abstract. Our goal in this paper is to obtain sufficient conditions for instability of the zero solution to the non-linear vector differential equation

$$\dddot{X} + F(X, \dot{X})\ddot{X} + G(\dot{X}) + H(X) = 0.$$

An example illustrates the results obtained.

1. Introduction

It is well-known that, since Lyapunov [17] proposed his famous second (or direct) method on the stability of motion, the problems related to the investigation of instability of solutions of certain second-, third-, fourth-, fifth-, sixth-, seventh and eighth-order linear and nonlinear differential equations have been given great attention in the past decade due to the importance of the subject. During this period, instability of solutions for various higher order linear and nonlinear differential equations have been extensively studied and many results have been obtained in the literature (see, e.g., Bereketoğlu [2], Bereketoğlu and Kart [3], Ezeilo [4, 5, 6, 7, 8], Kipnis [9], Krasovskii [10], Liao and Lu [11], Li and Yu [12], Li and Duan [13], Losprime [14], Lu and Liao [15], Lu [16], Reissig et al [18], Sadek [19, 20], Skrapek [21, 22], Tejumola [23], Tiryaki [24, 25, 26], C. Tunç [27, 28, 29, 30, 31, 32, 33, 34, 35], C.Tunç and E. Tunç [36, 37, 38], C. Tunç and Şevli [39], E. Tunç [40] and the references cited in that works). Among which, the results performed on instability properties of linear and nonlinear scalar and vector differential equations of third order can briefly be summarized as follows: First, in 1966, Losprime [14] took into consideration the third-order scalar linear differential equation with periodic coefficients

$$\dddot{x} + \ddot{x} + S(t)\dot{x} + T(t)x = 0.$$

Losprime [14] found the regions of stability and instability of this differential equation by means of some expansions and the Lyapunov's second (or direct) method (see, Lyapunov [17]). Then, in 1974, Kipnis [9] discussed the instability of the scalar linear differential equation

$$\dddot{x} + p(t)x = 0.$$

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The author presented that if the function p is continuous, ω-periodic, non-positive, and satisfies an inequality involving ω, then the above equation is unstable. Later, in 1980, by using Lyapunov's second (or direct) method, Skrapek [22] established sufficient conditions which guarantee the instability of the trivial solution of the scalar non-linear differential equation as

$$\dddot{x} + f_1(\ddot{x}) + f_2(\dot{x}) + f_3(x) + f_4(x, \dot{x}, \ddot{x}) = 0.$$

In 1995, Lu [16] discussed a similar problem for the third order nonlinear scalar differential equation

$$\dddot{x} + f(x, \dot{x})\ddot{x} + g(x) = 0.$$

In a similar manner, in 1996, Bereketoğlu and Kart [3] also studied instability of the trivial solution of scalar differential equation

$$\dddot{x} + f(\dot{x})\ddot{x} + g(x)\dot{x} + h(x, \dot{x}, \ddot{x}) = 0.$$

Together the above works, by using Lyapunov function approach, more recently the authors in [35, 40] also established some instability results for the zero solution of the non-linear vector differential equations of third order

$$\dddot{X} + F(\dot{X})\ddot{X} + G(\dot{X}) + H(X) = 0,$$

and

$$\dddot{X} + F(\dot{X})\ddot{X} + G(X)\dot{X} + H(X, \dot{X}, \ddot{X}) = 0$$

respectively. Furthermore, to the best of our knowledge in the relevant literature, no author except that mentioned above has investigated the instability of solutions of third order nonlinear vector differential equations of the form

$$\dddot{X} + A_1\ddot{X} + A_2\dot{X} + A_3X = 0$$

in which $X \in \mathbb{R}^n, A_1, A_2$ and A_3 are not necessarily $n \times n$-constant matrices.

In the present paper, we concern with the instability of the trivial solution $X = 0$ of nonlinear vector differential equation

$$\dddot{X} + F(X, \dot{X})\ddot{X} + G(\dot{X}) + H(X) = 0, \tag{1.1}$$

in which $X \in \mathbb{R}^n$; F is a continuous $n \times n$ -symmetric matrix; $G : \mathbb{R}^n \to \mathbb{R}^n, H : \mathbb{R}^n \to \mathbb{R}^n$ and $G(0) = H(0) = 0$. It will be supposed that the functions G and H are continuous. Throughout this paper, we use the following differential system

$$\begin{gathered} \dot{X} = Y, \dot{Y} = Z, \\ \dot{Z} = -F(X, Y)Z - G(Y) - H(X), \end{gathered} \tag{1.2}$$

which was obtained as usual by setting $\dot{X} = Y, \ddot{X} = Z$ in (1.1).

Let $J\left(F(X, Y)Y \mid X\right), J\left(F(X, Y)Y \mid Y\right), J_G(Y)$ and $J_H(X)$ denote the Jacobian matrices corresponding to $F(X, \dot{X}), G(Y)$ and $H(X)$, respectively:

$$J(F(X, Y)Y \mid X) = \Big(\frac{\partial}{\partial x_j}\sum_{k=1}^{n} f_{ik}y_k\Big) = \Big(\sum_{k=1}^{n}\frac{\partial f_{ik}}{\partial x_j}y_k\Big),$$

$$J(F(X, Y)Y \mid Y) = \Big(\frac{\partial}{\partial y_j}\sum_{k=1}^{n} f_{ik}y_k\Big) = F(X, Y) + \Big(\sum_{k=1}^{n}\frac{\partial f_{ik}}{\partial y_j}y_k\Big),$$

$$J_G(Y) = \Big(\frac{\partial g_i}{\partial y_j}\Big), \quad J_H(X) = \Big(\frac{\partial h_i}{\partial x_j}\Big) \quad (i, j = 1, 2, \dots, n),$$

where $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n)$, $(z_1, z_2, \dots, z_n)$, (f_{ik}), $(g_1, g_2, \dots, g_n)$ and $(h_1, h_2, \dots, h_n)$ are the components of X, Y, Z, F, G and H, respectively. It will also be assumed as basic throughout the paper that the Jacobian matrices, $J(F(X,Y)Y \mid X)$, $J(F(X,Y)Y \mid Y)$, $J_G(Y)$ and $J_H(X)$ exist, and are symmetric and continuous. The symbol $\langle X, Y\rangle$ will be used to denote the usual scalar product in $\mathbb{R}^n$ for given any X, Y in $\mathbb{R}^n$, that is, $\langle X, Y\rangle = \sum_{i=1}^n x_i y_i$; thus $\langle X, X\rangle = \|X\|^2$. It is well-known that the real symmetric matrix $A = (a_{ij})$, $(i, j = 1, 2, \dots, n)$ is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \mathbb{R}^n$ and X^T denotes the transpose of X.

The reason for investigation equation (1.1) has been inspired basically by the papers mentioned above. It is worth mentioning that the papers performed on the instability of solutions of third order nonlinear differential equation (see, e.g., [9, 14, 16, 22, 35, 40]) have been published without an example. But, this paper includes an explanatory example on the subject. It should be noted that the Lyapunov's second (or direct) method is used to verify the results established here.

2. Main results

Now, above all, we state the following algebraic results, lemmas, which are needed in the proofs of the main results.

Lemma 2.1. *Let A be a real symmetric $n \times n$-matrix and*
$$a' \ge \lambda_i(A) \ge a > 0 \quad (i = 1, 2, \dots, n),$$
where a', a are constants. Then
$$a'\langle X, X\rangle \ge \langle AX, X\rangle \ge a\langle X, X\rangle,$$
$$a'^2\langle X, X\rangle \ge \langle AX, AX\rangle \ge a^2\langle X, X\rangle.$$

For a proof of the above lemma, see Bellman [1].

Lemma 2.2. *Let Q, D be any two real $n \times n$ commuting symmetric matrices. Then*
(i) The eigenvalues $\lambda_i(QD)$, $(i = 1, 2, \dots, n)$, of the product matrix QD are real and satisfy
$$\max_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D) \ge \lambda_i(QD) \ge \min_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D).$$
(ii) The eigenvalues $\lambda_i(Q + D)$, $(i = 1, 2, \dots, n)$, of the sum of matrices Q and D are real and satisfy
$$\big\{ \max_{1\le j\le n} \lambda_j(Q) + \max_{1\le k\le n} \lambda_k(D)\big\} \ge \lambda_i(Q + D) \ge \big\{ \min_{1\le j\le n} \lambda_j(Q) + \min_{1\le k\le n} \lambda_k(D)\big\},$$
where $\lambda_j(Q)$ and $\lambda_k(D)$ are, respectively, the eigenvalues of Q and D.

For a proof of the above lemma, see Bellman [1]. We can now state our first main result.

Theorem 2.3. *In addition to the fundamental assumptions imposed on F, G and H appeared in (1.2), suppose that there are constants $a_1, a_2, \overline{a}_2$ and a_3 such that one of the following conditions is satisfied:*

- (i) *$\lambda_i(F(X,Y)) \le a_1$, $-\overline{a}_2 \le \lambda_i(J_G(Y)) \le -a_2 < 0$ and $\lambda_i(J_H(X)) \ge a_3 > 0$, $(i = 1, 2, \dots, n)$, for all $X, Y \in \mathbb{R}^n$, and $J(F(X,Y)Y \mid X)$ is positive-definite for all $X, Y \in \mathbb{R}^n$*

(i') $\lambda_i(F(X,Y)) \le a_1$, $-\bar{a}_2 \le \lambda_i(J_G(Y)) \le -a_2 < 0$ *and* $\lambda_i(J_H(X)) \le -a_3 < 0$, $(i = 1, 2, \dots, n)$, *for all* $X, Y \in \mathbb{R}^n$, *and* $J(F(X,Y)Y \mid X)$ *is positive-definite for all* $X, Y \in \mathbb{R}^n$.

Then the trivial solution $X = 0$ *of the system* (1.2) *is unstable.*

Proof. In order to prove the theorem it will suffice (see Krasovskii [10]) to show that there exists a continuous function $V_0 = V_0(X,Y,Z)$ which has the following Krasovskii properties:

(K1) In every neighborhood of $(0,0,0)$ there exists a point (ξ,η,ζ) such that $V_0(\xi,\eta,\zeta) > 0$.

(K2) The time derivative $\dot{V}_0 = \frac{d}{dt}V_0(X,Y,Z)$ along solution paths of the system (1.2) is positive-semi definite.

(K3) The only solution $(X,Y,Z) = (X(t),Y(t),Z(t))$ of the system (1.2) which satisfies $\dot{V}_0 = 0$ $(t \ge 0)$ is the trivial solution $(0,0,0)$.

We claim that the function $V_0 = V_0(X,Y,Z)$ defined by

$$\begin{aligned} 2V_0 = 2\alpha \int_0^1 \langle H(\sigma X), X\rangle d\sigma + 2\alpha\langle Y, Z\rangle + \alpha \int_0^1 \sigma \langle F(X,\sigma Y)Y, Y\rangle d\sigma \\ + \langle Y, Y\rangle - 2\langle X, Z\rangle, \end{aligned} \tag{2.1}$$

has all the three properties, where α is a positive constant. Indeed, it is clear from (2.1) that $V_0(0,0,0) = 0$. Since

$$H(0) = 0, \frac{\partial}{\partial\sigma}H(\sigma X) = J_H(\sigma X)X,$$

then

$$H(X) = \int_0^1 J_H(\sigma X)X d\sigma. \tag{2.2}$$

Hence, in view of assumption (i) of Theorem 2.3 and (2.2), we obtain

$$\begin{aligned} \int_0^1 \langle H(\sigma X), X\rangle d\sigma &= \int_0^1 \int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1 \\ &\ge \int_0^1 \int_0^1 \langle \sigma_1 a_3 X, X\rangle d\sigma_2 d\sigma_1 \\ &= \frac{a_3}{2}\langle X, X\rangle = \frac{a_3}{2}\|X\|^2. \end{aligned} \tag{2.3}$$

Obviously, it follows from assumption (i) of Theorem 2.3, (2.1) and (2.3) that

$$V_0(\varepsilon,0,0) \ge \frac{a_3}{2}\langle\varepsilon,\varepsilon\rangle = \frac{a_3}{2}\|\varepsilon\|^2 > 0$$

for all arbitrary $\varepsilon \in \mathbb{R}^n$, $\varepsilon \ne 0$. Thus, in every neighborhood of $(0,0,0)$ there exists a point (ξ,η,ζ) such that $V_0(\xi,\eta,\zeta) > 0$ for all ξ,η and ζ in $\mathbb{R}^n$. Next, let $(X,Y,Z) = (X(t),Y(t),Z(t))$ be an arbitrary solution of the system (1.2). Then, the total derivative of the function V_0 with respect to t along this solution path is

$$\begin{aligned} \dot{V}_0 &= \frac{d}{dt}V_0(X,Y,Z) \\ &= \alpha\langle Z, Z\rangle - \alpha\langle Y, G(Y)\rangle + \langle X, H(X)\rangle + \langle X, F(X,Y)Z\rangle \\ &\quad - \alpha\langle F(X,Y)Z, Y\rangle + \langle X, G(Y)\rangle - \alpha\langle H(X), Y\rangle \\ &\quad + \alpha\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma + \alpha\frac{d}{dt}\int_0^1 \sigma\langle F(X,\sigma Y)Y, Y\rangle d\sigma. \end{aligned} \tag{2.4}$$

Check that

$$\begin{aligned}\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma &= \int_0^1 \sigma\langle J_H(\sigma X)Y, X\rangle d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle d\sigma \\ &= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle H(\sigma X), Y\rangle d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle d\sigma \\ &= \sigma\langle H(\sigma X), Y\rangle\big|_0^1 = \langle H(X), Y\rangle\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \sigma\langle F(X,\sigma Y)Y, Y\rangle d\sigma \\ &= \int_0^1 \langle \sigma F(X,\sigma Y)Z, Y\rangle d\sigma + \int_0^1 \langle \sigma F(X,\sigma Y)Y, Z\rangle d\sigma \\ &\quad + \int_0^1 \langle \sigma J(F(X,\sigma Y)Y \mid X)Y, Y\rangle d\sigma + \int_0^1 \langle \sigma^2 J(F(X,\sigma Y)Z \mid Y)Y, Y\rangle d\sigma \\ &= \int_0^1 \langle \sigma F(X,\sigma Y)Z, Y\rangle d\sigma + \int_0^1 \langle \sigma J(F(X,\sigma Y)Y \mid X)Y, Y\rangle d\sigma \\ &\quad + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma F(X,\sigma Y)Z, Y\rangle d\sigma \\ &= \sigma^2\langle F(X,Y)Z, Y\rangle\big|_0^1 + \int_0^1 \langle \sigma J(F(X,\sigma Y)Y \mid X)Y, Y\rangle d\sigma \\ &= \langle F(X,Y)Z, Y\rangle + \int_0^1 \langle \sigma J(F(X,\sigma Y)Y \mid X)Y, Y\rangle d\sigma.\end{aligned} \tag{2.6}$$

Combining the estimates (2.5) and (2.6) with (2.4), we obtain

$$\begin{aligned}\dot V_0 &= \alpha\langle Z, Z\rangle - \alpha\langle Y, G(Y)\rangle + \langle X, H(X)\rangle \\ &\quad + \langle X, F(X,Y)Z\rangle + \langle X, G(Y)\rangle + \alpha\int_0^1 \langle \sigma J(F(X,\sigma Y)Y \mid X)Y, Y\rangle d\sigma.\end{aligned} \tag{2.7}$$

Since

$$G(0) = 0, \quad \frac{\partial}{\partial\sigma}G(\sigma Y) = J_G(\sigma Y)Y,$$

it follows that

$$G(Y) = \int_0^1 J_G(\sigma Y)Y\, d\sigma.$$

Thus, assumption (i) of Theorem 2.3 shows that

$$\begin{aligned}\alpha\langle Y, G(Y)\rangle &= \alpha\int_0^1 \langle Y, J_G(\sigma Y)Y\rangle d\sigma \\ &\le -\alpha a_2\int_0^1 \langle Y, Y\rangle d\sigma \\ &= -\alpha a_2\langle Y, Y\rangle = -\alpha a_2\|Y\|^2.\end{aligned} \tag{2.8}$$

By noting assumption (i) of Theorem 2.3 and then combining the estimate (2.8) with (2.7) we can easily find that

$$\dot V_0 \ge \alpha\|Z\|^2 + \alpha a_2\|Y\|^2 + a_3\|X\|^2 + \langle X, F(X,Y)Z\rangle + \langle X, G(Y)\rangle. \tag{2.9}$$

Now, for some constants k_1 and k_2 conveniently chosen later, we have

$$\begin{aligned}\langle X, G(Y)\rangle &= \frac{1}{2}\left\|k_1X + k_1^{-1}G(Y)\right\|^2 - \frac{1}{2}k_1^2\langle X, X\rangle - \frac{1}{2}k_1^{-2}\langle G(Y), G(Y)\rangle\\ &\geq -\frac{1}{2}k_1^2\langle X, X\rangle - \frac{1}{2k_1^2}\overline{a}_2^2\langle Y, Y\rangle\\ &= -\frac{1}{2}k_1^2\|X\|^2 - \frac{1}{2k_1^2}\overline{a}_2^2\|Y\|^2\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}&\langle X, F(X,Y)Z\rangle\\ &= \frac{1}{2}\left\|k_2X + k_2^{-1}F(X,Y)Z\right\|^2 - \frac{1}{2}k_2^2\langle X, X\rangle - \frac{1}{2k_2^2}\langle F(X,Y)Z, F(X,Y)Z\rangle\\ &\geq -\frac{1}{2}k_2^2\langle X, X\rangle - \frac{1}{2k_2^2}\langle F(X,Y)Z, F(X,Y)Z\rangle\\ &\geq -\frac{1}{2}k_2^2\|X\|^2 - \frac{1}{2k_2^2}a_1^2\|Z\|^2.\end{aligned} \tag{2.11}$$

From the estimates (2.9)-(2.11), we deduce that

$$\dot{V}_0 \geq [a_3 - \frac{1}{2}k_1^2 - \frac{1}{2}k_2^2]\|X\|^2 + [\alpha a_2 - \frac{1}{2k_1^2}\overline{a}_2^2]\|Y\|^2 + [\alpha - \frac{1}{2k_2^2}a_1^2]\|Z\|^2.$$

Let

$$k_1^2 = \min\{\frac{a_3}{2}, \frac{\overline{a}_2^2}{a_2\alpha}\}, k_2^2 = \min\{\frac{a_3}{2}, a_1^2\alpha^{-1}\}.$$

Then

$$\begin{aligned}\dot{V}_0 &\geq \left(\frac{a_3}{2}\right)\|X\|^2 + \left(\frac{\alpha a_2}{2}\right)\|Y\|^2 + \left(\frac{3\alpha}{4}\right)\|Z\|^2\\ &\geq k\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right) > 0,\end{aligned}$$

where

$$k = \min\{\frac{a_3}{2}, \frac{\alpha a_2}{2}, \frac{\alpha}{2}\}.$$

Thus, assumption (i) of Theorem 2.3 shows that $\dot{V}_0(t) \geq 0$ for all $t \geq 0$, that is, $\dot{V}_0$ is positive semi-definite. Furthermore, the equality $\dot{V}_0 = 0$ ($t \geq 0$) necessarily implies that $Y = 0$ for all $t \geq 0$. Hence, we obtain that $X = \xi$ (a constant vector), $Z = \dot{Y} = 0$ for all $t \geq 0$. Substituting the estimates

$$X = \xi, \quad Y = Z = 0$$

in the system (1.2) it follows that $H(\xi) = 0$ which necessarily implies that $\xi = 0$ because of $H(0) = 0$. So

$$X = Y = Z = 0 \quad \text{for all } t \geq 0.$$

Therefore, the function V_0 has the entire requisite Krasovskii's criteria [10] if assumption (i) in Theorem 2.3 holds. This proves part (i) of Theorem 2.3.

Similarly, for the proof of part (i') of Theorem 2.3, we consider the Lyapunov function $V_1 = V_1(X, Y, Z)$ defined by:

$$\begin{aligned}2V_1 = &-2\overline{\alpha}\int_0^1\langle H(\sigma X), X\rangle d\sigma + 2\overline{\alpha}\langle Y, Z\rangle + \overline{\alpha}\int_0^1\sigma\langle F(X,\sigma Y)Y, Y\rangle d\sigma\\ &- \langle Y, Y\rangle + 2\langle X, Z\rangle,\end{aligned} \tag{2.12}$$

where $\overline{\alpha}$ is a positive constant.

When we follow the lines indicated in the proof of part (i) of Theorem 2.3, we can easily obtain

$$V_1(0,0,0)=0, V_1(\overline{\varepsilon},0,0) \geq \frac{a_3}{2}\left\|\overline{\varepsilon}\right\|^2 > 0$$

for all arbitrary $\overline{\varepsilon} \neq 0, \overline{\varepsilon} \in \mathbb{R}^n$ and

$$\dot{V}_1 \geq \overline{k}\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right) > 0,$$

where $\overline{k}$ is a certain positive constant. This proves the proof of the part (i') of Theorem 2.3. The basic properties of $V_0(X,Y,Z)$ and $V_1(X,Y,Z)$, which we have proved just above, justify that the zero solution of the system (1.2) is unstable. See [18, theorem 1.15], see also [10]. The system (1.2) is equivalent to the differential equation (1.1). It follows thus the original statement of the theorem. □

Example: As a special case of the system (1.2), let us choose, for the case $n = 3, F, G$ and H that appeared in (1.2) as follows:

$$F(X,Y) = \begin{bmatrix} 1 & -5x_1^2 + \frac{1}{1+y_1^2} & x_3 + 2y_3 \\ 0 & \frac{1}{1+x_2^2+y_2^2} & 0 \\ 0 & 0 & \frac{1}{2+x_3^4+y_3^4} \end{bmatrix},$$

$$G(Y) = \begin{bmatrix} -y_1 - y_1^3 \\ -y_2 - y_2^3 \\ -y_3 - y_3^3 \end{bmatrix}, \quad H(X) = \begin{bmatrix} x_1 + x_1^3 \\ x_2 + x_2^3 \\ x_3 + x_3^3 \end{bmatrix}$$

Then, clearly, the eigenvalues of the matrix $F(X,Y)$ are

$$\lambda_1(F(X,Y)) = 1, \quad \lambda_2(F(X,Y)) = \frac{1}{1+x_2^2+y_2^2} \leq 1,$$
$$\lambda_3(F(X,Y)) = \frac{1}{2+x_3^4+y_3^4} \leq 1.$$

Next, observe that

$$J_G(Y) = \begin{bmatrix} -1-3y_1^2 & 0 & 0 \\ 0 & -1-3y_2^2 & 0 \\ 0 & 0 & -1-3y_3^2 \end{bmatrix},$$

and hence $\lambda_1(J_G(Y)) = -1-3y_1^2$, $\lambda_2(J_G(Y)) = -1-3y_2^2$, $\lambda_3(J_G(Y)) = -1-3y_3^2$. Clearly, $-1 \leq \lambda_1(J_G(Y)) \leq -\frac{1}{2}$, $-1 \leq \lambda_2(J_G(Y)) \leq -\frac{1}{2}$ and $-1 \leq \lambda_3(J_G(Y)) \leq -\frac{1}{2}$. Finally, we have that

$$J_H(X) = \begin{bmatrix} 1+3x_1^2 & 0 & 0 \\ 0 & 1+3x_2^2 & 0 \\ 0 & 0 & 1+3x_3^2 \end{bmatrix},$$

and $\lambda_1(J_H(X)) = 1+3x_1^2 \geq 1 > 0$, $\lambda_2(J_H(X)) = 1+3x_2^2 \geq 1 > 0$, $\lambda_3(J_H(X)) = 1+3x_3^2 \geq 1 > 0$. Thus all the conditions of part (i) of Theorem 2.3 are satisfied.

The next theorem is our second main result.

Theorem 2.4. *Further to the basic assumptions imposed on F, G and H appeared in (1.2), suppose that there are constants a_1, a_2 and a_3 such that one of the following conditions is satisfied:*

(i) $\lambda_i(F(X,Y)) \leq -a_1 < 0$, $\lambda_i(J_G(Y)) \leq a_2$ *and* $\lambda_i(J_H(X)) \geq a_3 > 0$, $(i = 1,2,\dots,n)$, *for all* $X, Y \in \mathbb{R}^n$.

(i') $\lambda_i(F(X,Y)) \geq a_1 > 0$, $\lambda_i(J_G(Y)) \leq a_2$ *and* $\lambda_i(J_H(X)) \leq -a_3 < 0$, $(i = 1, 2, \dots, n)$, *for all* $X, Y \in \mathbb{R}^n$.

Then the zero solution $X = 0$ *of the system* (1.2) *is unstable.*

Proof. Consider the function $V_2 = V_2(X,Y,Z)$ defined by

$$2V_2 = \beta\langle Z, Z\rangle + 2\beta\langle Y, H(X)\rangle + 2\beta\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma + \langle Y, Y\rangle - 2\langle X, Z\rangle, \quad (2.13)$$

where β is a positive constant. Observe that $V_2(0,0,0) = 0$. It is also clear from assumption (i) of Theorem 2.4 that

$$V_2(0,0,\varepsilon) \geq \beta\langle\varepsilon,\varepsilon\rangle = \beta\|\varepsilon\|^2 > 0$$

for all arbitrary $\varepsilon \in \mathbb{R}^n$, $\varepsilon \neq 0$. So that in every neighborhood of $(0,0,0)$ there exists a point (ξ,η,ζ) such that $V_2(\xi,\eta,\zeta) > 0$ for all ξ,η and ζ in $\mathbb{R}^n$. Next, let $(X,Y,Z) = (X(t),Y(t),Z(t))$ be an arbitrary solution of the system (1.2). An easy calculation from (2.13) and (1.2) yields that

$$\begin{aligned}
\dot{V}_2 &= \frac{d}{dt}V_2(X,Y,Z) \\
&= -\beta\langle Z, F(X,Y)Z\rangle + \beta\langle Y, J_H(X)Y\rangle + \langle X, H(X)\rangle \\
&\quad + \langle X, F(X,Y)Z\rangle + \langle X, G(Y)\rangle - \beta\langle G(Y), Z\rangle + \beta\frac{d}{dt}\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma.
\end{aligned}$$

But

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma &= \int_0^1 \sigma\langle J_G(\sigma Y)Z, Y\rangle d\sigma + \int_0^1 \langle G(\sigma Y), Z\rangle d\sigma \\
&= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle G(\sigma Y), Z\rangle d\sigma + \int_0^1 \langle G(\sigma Y), Z\rangle d\sigma \qquad (2.14)\\
&= \sigma\langle G(\sigma Y), Z\rangle\big|_0^1 = \langle G(Y), Z\rangle.
\end{aligned}$$

Therefore, by using (2.14) and assumption (i) of Theorem 2.4, we get

$$\begin{aligned}
\dot{V}_2 &= -\beta\langle Z, F(X,Y)Z\rangle + \beta\langle Y, J_H(X)Y\rangle + \langle X, H(X)\rangle \\
&\quad + \langle X, F(X,Y)Z\rangle + \langle X, G(Y)\rangle \qquad (2.15)\\
&\geq \beta a_1\|Z\|^2 + \beta a_3\|Y\|^2 + a_3\|X\|^2 + \langle X, F(X,Y)Z\rangle + \langle X, G(Y)\rangle.
\end{aligned}$$

Similarly, as shown just above for some constants $\overline{k}_1$ and $\overline{k}_2$ conveniently chosen later, we can easily obtain from (2.15) that

$$\dot{V}_2 \geq \big(a_3 - \frac{1}{2}\overline{k}_1^2 - \frac{1}{2}\overline{k}_2^2\big)\|X\|^2 + \big(\beta a_3 - \frac{1}{2}\overline{k}_1^{-2}a_2^2\big)\|Y\|^2 + \big(\beta a_1 - \frac{1}{2}\overline{k}_2^{-2}a_1^2\big)\|Z\|^2.$$

Let

$$\overline{k}_1^2 = \min\big\{\frac{a_3}{2}, \frac{a_2^2}{\beta a_3}\big\}, \overline{k}_2^2 = \min\big\{\frac{a_3}{2}, \frac{a_1}{\beta}\big\}.$$

Hence

$$\begin{aligned}
\dot{V}_2 &\geq \left(\frac{a_3}{2}\right)\|X\|^2 + \left(\frac{\beta a_3}{2}\right)\|Y\|^2 + \left(\frac{\beta a_1}{2}\right)\|Z\|^2 \\
&\geq \overline{k}\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right) > 0,
\end{aligned}$$

where

$$\overline{k} = \min\big\{\frac{a_3}{2}, \frac{\beta a_3}{2}, \frac{\beta a_1}{2}\big\}.$$

The rest of the proof of part (i) of Theorem 2.4 is the same as the proof of part (i) of Theorem 2.3 just proved above and hence it is omitted the details. □

Finally, for the proof of part (i') of Theorem 2.4, we consider the Lyapunov function

$$V_3(X,Y,Z) = V_2(X,Y,Z) - 2\beta \int_0^1 \langle H(\sigma X), X\rangle d\sigma,$$

where $V_2(X,Y,Z)$ is defined as the same the function in (2.13). The remaining of the proof can be verified proceeding exactly along the lines indicated just in the proof of Theorem 2.3. Hence we omit the detailed proof.

References

[1] R. Bellman; *Introduction to matrix analysis*, reprint of the second edition (1970), With a foreword by Gene Golub. Classics in Applied mathematics, 19. Society for Industrial and Applied mathematics (SIAM), Philadelphia, PA, 1997.

[2] H. Bereketoğlu; *On the instability of trivial solutions of a class of eighth-order differential equations*, Indian J. Pure. Appl. Math., 22 (1991), no.3, 199-202.

[3] H. Bereketoğlu, C. Kart; *Some Results for a Certain Third-Order Nonlinear Ordinary Differential equation*, Bull. Math. Soc. Sc. Math. Roumanie, Tome 39 (87), No.1-4, 77-83, 1996.

[4] J.O.C. Ezeilo; *An instability theorem for a certain fourth order differential equation*, Bull. London Math.Soc., 10 (1978), no.2, 184-185.

[5] J.O.C. Ezeilo; *Instability theorems for certain fifth-order differential equations*, Math. Proc.Cambridge. Philos. Soc., 84 (1978), no.2, 343-350.

[6] J.O.C. Ezeilo; *A further instability theorem for a certain fifth-order differential equation*, Math. Proc.Cambridge Philos. Soc., 86 (1979), no.3, 491-493.

[7] J.O.C. Ezeilo; *Extension of certain instability theorems for some fourth and fifth order differential equations*, Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis., Mat. Natur., (8) 66 (1979), no.4, 239-242.

[8] J.O.C. Ezeilo; *An instability theorem for a certain sixth order differential equation*, J. Austral. Math. Soc. Ser. A, 32 (1982), no.1, 129-133.

[9] L. A. Kipnis, *On the instability of a particular third-order linear system*, J. Appl. Math. Mech., 38, 868-869 ; translated from Prikl. Mat. Meh. ,38 (1974, 1975), 921-922 (Russian).

[10] N. N. Krasovskii, *On conditions of inversion of A. M. Lyapunov's theorems on instability for stationary systems of differential equations*, (Russian) Dokl. Akad. Nauk. SSSR (N.S.), 101, (1955), 17-20.

[11] Z.H. Liao, D.Lu; *Instability of solution for the third order linear differential equation with varied coefficient*, Appl. Math. Mech. (English Ed.), no. 10, 9 (1988), 969-984; translated from Appl. Math. Mech., no. 10, 9 (1988), 909-923 (Chinese).

[12] W. J. Li, Y.H. Yu; *Instability theorems for some fourth-order and fifth-order differential equations* (Chinese), J. Xinjiang Univ. Natur. Sci., 7 (1990), no.2, 7-10.

[13] W. J. Li, K. C. Duan; *Instability theorems for some nonlinear differential systems of fifth order*, J. Xinjiang Univ. Natur. Sci., 17 (2000), no.3, 1-5.

[14] G. A. Losprime; *Finding the regions of stability and instability of a differential equation of the third order with periodic coefficients*, (Russian) Ukrain. Mat. Z., 18(1966), no.4, 110-116.

[15] D.Lu, Z. H.Liao; *Instability of solution for the fourth order linear differential equation with varied coefficient*, Appl. Math. Mech. (English Ed.), no. 5, 14 (1993), 481-497; translated from Appl. Math. Mech., no. 5, 14(1993), 455-469 (Chinese).

[16] D. Lu; *Instability of solution for a class of the third order nonlinear differential equation*, Appl. Math. Mech. (English Ed.), no. 12, 16 (1995), 1185-1200; translated from Appl. Math. Mech ., no. 12, 16(1995), 1101-1114 (Chinese).

[17] A. M. Lyapunov; *Stability of Motion*, Academic Press, 1966.

[18] R. Reissig, G. Sansone, R. Conti; *Non-linear Differential Equations of Higher Order*, Translated from the German. Noordhoff International Publishing, Leyden, (1974).

[19] A. I. Sadek; *An instability theorem for a certain seventh-order differential equation*, Ann. Differential Equations, 19 (2003), no. 1, 1-5.

[20] A.I. Sadek; *Instability results for certain systems of fourth and fifth order differential equations*, Appl. Math.Comput., 145 (2003), no.2-3, 541-549.
[21] W.A. Skrapek; *Instability results for fourth-order differential equations*, Proc. Roy. Soc. Edinburgh Sect. A ,85 (1980), no. 3-4, 247-250.
[22] W.A. Skrapek; *Some instability theorems for third order ordinary differential equations*, Math. Nachr.,96 (1980), 113-117.
[23] H. O. Tejumola; *Instability and periodic solutions of certain nonlinear differential equations of orders six and seven*, Ordinary differential equations (Abuja, 2000), 56-65, Proc. Natl. Math. Cent. Abuja. Niger., 1.1 , Natl. Math.Cent., Abuja, 2000.
[24] A. Tiryaki; *Extension of an instability theorem for a certain fourth order differential equation*, Bull. Inst. Math. Acad. Sinica., 16 (1988), no.2, 163-165.
[25] A. Tiryaki; *An instability theorem for a certain sixth order differential equation*, Indian J. Pure. Appl. Math., 21 (1990), no.4, 330-333.
[26] A. Tiryaki; *Extension of an instability theorem for a certain fifth order differential equation*, National Mathematics Symposium (Trabzon, 1987), J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math. Phys., 11, (1988, 1989), 225-227.
[27] C. Tunç; *An instability theorem for a certain vector differential equation of the fourth order*, Journal of Inequalities in Pure and Applied Mathematics, 5(2004), no.1, 1-5.
[28] C. Tunç; *An instability result for certain system of sixth order differential equations*, Applied Mathematics and Computation, 157(2004), no.2, 477-481.
[29] C. Tunç; *On the instability of solutions of certain nonlinear vector differential equations of the fifth order*, Panamerican Mathematical Journal, 14(2004), no.4, 25-30.
[30] C. Tunç; *On the instability of certain sixth-order nonlinear differential equations*, Electron. J. Diff. Eqns., Vol.2004, (2004), no. 117, 1-6.
[31] C. Tunç; *An instability result for a certain non-autonomous vector differential equation of fifth order*, Panamerican Mathematical Journal, 15 (3), (2005), 51-58.
[32] C. Tunç; *Instability of solutions of a certain non-autonomous vector differential equation of eighth order*, Analls of Differential Equations, Vol. 22, no. 1, (2006), 7-12
[33] C. Tunç; *New results about instability of nonlinear ordinary vector differential equations of sixth and seventh orders*, Dynamics of Continuous, Discrete and Impulsive Systems; DCDIS Series A: Mathematical Analysis, (2006), (in press).
[34] C. Tunç; *A further instability result for a certain vector differential equation of fourth order*, International Journal of Mathematics, Game Theory, and Algebra, (2006), (in press).
[35] C. Tunç; *Some instability results on certain third order nonlinear vector differential equations*, Bulletin of the Institute of Mathematics, Academia Sinica, (2006), (in press).
[36] C. Tunç, E. Tunç; *A result on the instability of solutions of certain non-autonomous vector differential equations of fourth order*, East-West Journal of Mathematics, 6(2), (2004), 127-134.
[37] C. Tunç, E. Tunç; *Instability of solutions of certain nonlinear vector differential equations of order seven*, Iranian Journal of Science and Technology, Transaction A, 29(A3), 2005, 515-521.
[38] C. Tunç, E. Tunç; *An instability theorem for a class of eighth order differential equations*, Differential Equations(Differ. Uravn.), Tom. 42, no.1, (2006), 150-154.
[39] C. Tunç, H. Şevli; *On the istability of solutions of certain fifth order nonlinear differential equations*, Memoirs on Differential Equations and Mathematical Physics, 35 (2005), 147-156.
[40] E. Tunç; *Instability of solutions of certain nonlinear vector differential equations of third order*, Electronic J. Differential Equations, Vol. 2004, no.51, (2004), 1-6. (electronic).

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