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ANALYSIS OF A SINGLE SPECIES WITH DIFFUSION IN A POLLUTED ENVIRONMENT

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Abstract. In this paper, the effect of diffusion on the permanence and extinction in the polluted environment is studied by a single population diffusive system in two patches. Assume that the two patches are a protective patch and a non-protective patch. We examine the effects of protective patch and conclude that it is effective for the conservation of a population facing polluted environment. The conditions for the permanence and extinction of the population are obtained.

1. Introduction

Biological resources are renewable resources. In recent years, many countries have already realized that the pollution of the environment is a very urgent problem. Specialists coming from all kinds of fields have studied and solved it. One of the most meaningful question in mathematical biology is the permanence and extinction of a population in a polluted environment. Organisms are often exposed to a polluted environment and take up toxicant. Therefore, it is important to study the effects of a toxicant and diffusion on populations and to find a theoretical threshold value, which determines permanence or extinction of a population or community.

In order to prevent the biological resources from destruction and protect the environment, all kinds of measures have been proposed. Establishing protective patch as for a resource population is applied widely. The practical effects of the protective patch on the polluted population is worth examination.

Since Hallam and his colleagues proposed a toxicant-population model in the early 1980s [1]-[3], many authors have studied the mathematical models with toxicant effect [4, 5]. In this paper, pollution together with diffusional migration is taken into account comprehensively. It is particularly interested in the managers who need to deal with the size and control of barriers in protective patch [6, 7]. The organization of this paper is as follows. In the next section, we formulate our model as a system of non-autonomous ordinary differential equations, and describe

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our hypotheses. In section 3, we determine the equilibria of two autonomous systems. In section 4, sufficient conditions are obtained for permanence and extinction of population.

2. The Model

Let $N(t)$ be the density of population in region Ω at time t; $C_0(t)$ be the toxicant density in a body; $C_e(t)$ be the toxicant density of environment; $u(t)$ be the exogenous toxicant input rate, which is nonnegative, continuous and bounded function in the internal $[0, \infty)$.

The basic assumption is that compared with the number of the individuals, the content of the environment is large enough, the uptake and egestion by the organisms can be neglected. Equation of the polluted population reads

$$\begin{gathered}\dot{N}(t) = N(t)[r(t) - d(t)C_0(t) - a(t)N(t)],\\ \dot{C}_0(t) = k(t)C_e(t) - g(t)C_0(t) - m(t)C_0(t),\\ \dot{C}_e(t) = -h(t)C_e(t) + u(t),\end{gathered} \tag{2.1}$$

where $r(t), d(t)C_0(t), a(t)$ are the intrinsic growth rate, death rate, density restriction respectively, $k(t)C_e(t)$ represents the uptake of the toxicant from the environment by the population, $g(t)C_0(t)$ represents the toxicant quantity input to the environment from the population due to egestion, $m(t)C_0(t)$ represents the metabolic processes and other losses, $h(t)C_e(t)$ represents the losses of the toxicant from the environment due to egestion.

To protect the population resources, Ω is divided into two patches Ω_1 and Ω_2. Pollution is permitted in Ω_1 and is inhibited in Ω_2. We call Ω_2 the protective patch. The densities of population in Ω_1 and Ω_2 are denoted by $x(t), y(t)$ respectively, $b(t)$ is the density restriction of the population. The mathematical model of the polluted population with protective patch can be described as

$$\begin{gathered}\dot{x}(t) = x(t)[r(t) - d(t)C_0(t) - a(t)x(t)] + D(t)(y(t) - x(t)),\\ \dot{y}(t) = y(t)[r(t) - b(t)y(t)] + D(t)(x(t) - y(t)),\\ \dot{C}_0(t) = k(t)C_e(t) - g(t)C_0(t) - m(t)C_0(t),\\ \dot{C}_e(t) = -h(t)C_e(t) + u(t).\end{gathered} \tag{2.2}$$

The initial conditions are $x_0 = x(0) > 0$, $y_0 = y(0) > 0$, $0 \le C_0(0) \le 1$, $0 \le C_e(0) \le 1$. Since the difference of densities between patch Ω_1 and Ω_2 exists, the diffusive migration can occur between the two patches, which is assumed to be $D(t)$. The coefficients in the models are all nonnegative, continuous and bounded functions in the internal $[0, \infty)$.

To simplify our representations, we introduce the following notations in this paper: if $f(t)$ is a nonnegative, continuous and bounded functions in the internal $(-\infty, \infty)$,

$$f^u = \max_{t\in R} f(t), \quad f^l = \min_{t\in R} f(t).$$

Considering the realistic situation, the toxicant density of single body or the environment can't be greater than 1, or any population wiil not survive. So we should give some conditions, such that

$$0 \le C_0(t) \le 1, \quad 0 \le C_e(t) \le 1, \quad \text{for all } t \ge 0.$$

Lemma 2.1. *The set*

$$\{(x(t), y(t), C_0(t), C_e(t)) : x(t) > 0, y(t) > 0, C_0(t) > 0, C_e(t) > 0\}$$

is an invariant region of system (2.2)

Lemma 2.2. *For* (2.2)*, if* $k^u \leq g^l + m^l$*,* $u^u \leq h^l$*, then* $0 \leq C_0(t) \leq 1$*,* $0 \leq C_e(t) \leq 1$*, for all* $t \geq 0$*.*

Proof. According to Lemma 2.1, we have $0 \leq C_0(t)$, $0 \leq C_e(t)$, for all $t \geq 0$. Now we are going to prove that $C_0(t) \leq 1$, $C_e(t) \leq 1$, for all $t \geq 0$.

If the conclusion is false, then the maximum interval is $[0, T]$. At least one of the following cases will happen:

(1) $C_0(t) = 1$, $C_e(t) < 1$;
(2) $C_0(t) < 1$, $C_e(t) = 1$;
(3) $C_0(t) = 1$, $C_e(t) = 1$.

We will prove that none of this cases will happen. (1) $C_0(t) = 1$, $C_e(t) < 1$: Using the condition $k^u \leq g^l + m^l$, we get

$$\frac{dC_0(t)}{dt}|_{t=T} = k(t)C_e(t) - g(t)C_0(t) - m(t)C_0(t) \leq 0,$$

thus $\exists t_1 > 0$, such that $C_0(t) \leq 1$, $C_e(t) < 1$, for all $t \in [T, T + t_1]$. This is the contradiction with the definition of the interval $[0, T]$. So there is no T such that $C_e(t) < 1$, $t \in [0, T]$; $C_0(t) < 1$, $t \in [0, T)$ and $C_0(T) = 1$.

With the same reasoning as in case (1), for cases (2) and (3), as far as t which keeps $C_0(t) \leq 1$ and $C_e(t) \leq 1$ is concerned, the interval $[0, T]$ can be extended rightwards. This contradicts the property of T. Therefore, there is no such T, furthermore $0 \leq C_0(t) \leq 1$, $0 \leq C_e(t) \leq 1$, for all $t \geq 0$. □

It is clear $C_0(t)$ and $C_e(t)$ can be easily solved formally from the last two equations of the system (2.2),

$$C_0(t) = e^{-\int(m(s)+g(s))ds}[\int k(s)e^{\int(m(s)+g(s))ds}C_e(s)ds + C_0(0)],$$
$$C_e(t) = e^{-\int(h(s))ds}[\int u(s)e^{\int(h(s))ds}ds + C_e(0)],$$

Substituting $C_e(t)$ in $C_0(t)$, we can express $C_0(t)$ in term of some bounded continuous functions; therefore, the system (2.1) may be simplified as follows:

$$\begin{aligned} \dot{x}(t) &= x(t)[r(t) - d(t)C_0(t) - a(t)x(t)] + D(t)(y(t) - x(t)), \\ \dot{y}(t) &= y(t)[r(t) - b(t)y(t)] + D(t)(x(t) - y(t)). \end{aligned} \tag{2.3}$$

The initial conditions are $x_0 = x(0) > 0$, $y_0 = y(0) > 0$. For the simplified model (2.3), because the $C_0(t)$ may be regarded as a known function of t, we need only to impose the conditions of the diffusive coefficient $D(t)$ and the toxicant density in a body $C_0(t)$ in order to investigate the threshold between permanence and extinction of the populations. There is toxicant in patch Ω_1, but not in patch Ω_2 of systems (2.2), (2.3). Assume patch Ω_2 is the protective patch in order to the conservation of population resources in the polluted environment, though in some case the extinction can not be eliminated.

Considering the biological significance, we study system (2.3) in the region

$$R_+^2 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}.$$

Defintion 2.3. [6]/ A solution $x(t)$ of the system (2.3) is said to be permanent if for any $x(0) > 0$, there exist positive constants $0 < \delta < \varepsilon$ (independent of $x(0)$) such that $\delta < x(t) < \varepsilon$, then $x(t)$ is said to be uniformly permanent for large enough t. $x(t)$ is said to go to extinction if $\lim_{t\to+\infty} x(t) = 0$.

Defintion 2.4 ([8])**.** The differential equation

$$\dot{x}(t) = F(t,x), \quad x \in \mathbb{R}^n,$$

is said to be cooperative if the off-diagonal elements of $D_xF(t,x)$ are nonnegative, where $D_xF(t,x)$ is the $n \times n$ matrix derivative of F with respect to x.

Theorem 2.5 (Kamke)**.** *Let $x(t)$ and $y(t)$be the solutions of*

$$\dot{x}(t) = F(t,x)$$
$$\dot{y}(t) = G(t,x)$$

respectively, where both systems are assumed to have the uniqueness property for initial value problems. Assume both $x(t)$ and $y(t)$ belong to a domain $D \subseteq \mathbb{R}^n$ for $[t_0,t_1]$ in which one of the two systems is cooperative and

$$F(t,z) \le G(t,z) \quad (t,z) \in [t_0,t_1] \times D.$$

if $x(t_0) \le y(t_0)$ then $x(t_1) \le y(t_1)$. If $F = G$ and $x(t_0) < y(t_0)$ then $x(t_1) < y(t_1)$.

Lemma 2.6. *The set R_+^2 is an invariant region of system* (2.3)*.*

Lemma 2.7. *Solutions of system* (2.3) *with the positive initial conditions are uniformly bounded and ultimately uniformly bounded.*

Proof. Let

$$\Delta > \max\{\frac{r^u - d^lC_0^l}{a^l}, \frac{r^u}{b^l}\},$$
$$\dot{x}(t)|_{x=\Delta, y<\Delta} \le x(r^l - d^uC_0^u - a^ux) + D(t)(y-x) < 0,$$
$$\dot{y}(t)|_{y=\Delta, x<\Delta} \le y(r^l - b^uy) + D(t)(x-y) < 0,$$

(i) If $\max\{x(0),y(0)\} \le \Delta$, then $\max\{x(t),y(t)\} \le \Delta$ for $t \ge 0$.
(ii) If $\max\{x(0),y(0)\} > \Delta$, then there exists $\mu > 0$, $\max\{x(t),y(t)\} > \Delta$, for $t \in [0,\mu)$.

When $\max\{x(t),y(t)\} = x(t)$, letting $\alpha = a^l(\frac{r^u - d^lC_0^l}{a^l} - \Delta) < 0$, we have

$$\begin{aligned}\dot{x}(t) &= x(t)[r(t) - d(t)C_0(t) - a(t)x(t)] + D(t)(y(t) - x(t))\\ &\le a^lx(\frac{r^u - d^lC_0^l}{a^l} - x)\\ &< \alpha x\,.\end{aligned}$$

Then $x(t)$ is monotone decreasing with speed α, so there exists $T_1 = \frac{-1}{\alpha}\ln\frac{\Delta}{x(0)}$, such that $x(t) < \Delta$ for $t \ge T_1$.

When $\max\{x(t),y(t)\} = y(t)$, letting $\alpha = b^l(\frac{r^u}{b^l} - \Delta) < 0$, we have

$$\begin{aligned}\dot{y}(t) &= y(t)[r(t) - b(t)y(t)] + D(t)(x(t) - y(t))\\ &\le b^lx(\frac{r^u}{b^l} - \Delta)\\ &< \alpha y\,.\end{aligned}$$

Then $y(t)$ is monotone decreasing with speed α, so there exists $T_2 = \frac{-1}{\alpha}\ln\frac{\Delta}{y(0)}$, such that $y(t) < \delta$ for $t \geq T_2$.

So that $\max\{x(t), y(t)\}$ is monotonically decreasing with speed α in the interval $[0, \mu)$. For all $t^* \in [0, +\infty)$, if $\max\{x(t^*), y(t^*)\} > \Delta$, there exists μ, such that $\max\{x(t), y(t)\}$ is monotonically decreasing with speed α in the interval $[t^*, t^* + \mu)$. Then there is $T^* > t^*$, with $\max\{x(t), y(t)\} < \Delta$, for $t > T^*$.

Solutions of system (2.3) with positive initial conditions are uniformly bounded and ultimately uniformly bounded. □

3. Two Cooperative Systems

In this section we consider two autonomous systems generated by the system (2.3):

$$\begin{aligned} \dot{x} &= x[r^u - D^l - d^l C_0^l - a^l x] + D^u y := P_1(x, y), \\ \dot{y} &= y[r^u - D^l - b^l y] + D^u x := Q_1(x, y), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \dot{x} &= x[r^l - D^u - d^u C_0^u - a^u x] + D^l y := P_2(x, y), \\ \dot{y} &= y[r^l - D^u - b^u y] + D^l x := Q_2(x, y). \end{aligned} \tag{3.2}$$

Obviously, systems (3.1) and (3.2) are cooperative. Now we study the existence and the stability of equilibria of (3.1), which are solutions of

$$\begin{aligned} l_1 :\quad & x[r^u - D^l - d^l C_0^l - a^l x] + D^u y = 0, \\ l_2 :\quad & y[r^u - D^l - b^l y] + D^u x = 0. \end{aligned} \tag{3.3}$$

We are only interested in the non-negative equilibria, they are the intersection of the isoclines l_1, l_2. The graph of l_1 and l_2 are parabolas. l_1 is symmetric to line $x = -\frac{r^u - D^l - d^l C_0^l}{2a^l}$ and l_2 is symmetric to line $y = -\frac{r^u - D^l}{2b^l}$. We denote the intersection in the first quadrant by (x^*, y^*).

Let k_i $(i = 1, 2)$ denote the slope of the tangent line of l_i at $(0, 0)$. Clearly $k_1 = \frac{d^l C_0^l + D^l - r^u}{D^u}$, $k_2 = \frac{D^u}{D^l - r^u}$.

Condition 1: If $r^u < D^l$ and $(D^u)^2 < (D^l - r^u)(d^l C_0^l + D^l - r^u)$, then $0 < k_1, 0 < k_2$ and $k_1 > k_2$, the curves l_1, l_2 do not intersect in the positive quadrant. That is to say, the unique non-negative equilibrium is $(0, 0)$ (see Fig. 1(a)).

Condition 2: If $r^u < D^l$ and $(D^u)^2 > (D^l - r^u)(d^l C_0^l + D^l - r^u)$, then $0 < k_1, 0 < k_2$ and $k_1 < k_2$, the curves l_1, l_2 intersect in the positive quadrant. That is to say, the unique positive equilibrium is (x^*, y^*) in the positive quadrant. At the same time, the unique nonnegative equilibrium $(0, 0)$ exists(see Fig. 1(b)).

Condition 3: If $r^u \geq D^l$, the existence of the unique positive equilibrium (x^*, y^*) and the nonnegative equilibrium $(0, 0)$ can be proved (see Fig.1(c)(d)(e)).

Theorem 3.1. *The point* $(0, 0)$ *is always an equilibrium of system* (3.1). *If* $r^u < D^l$ *and* $(D^u)^2 < (D^l - r^u)(d^l C_0^l + D^l - r^u)$, *then* $(0, 0)$ *is the unique nonnegative equilibrium, it is a stable node.*

Proof. Obviously, $(0, 0)$ is an equilibrium system of (3.1). The Jacobian matrix corresponding to the linearized system of (3.1), at $(0, 0)$, is

$$J(0, 0) = \begin{pmatrix} r^u - D^l - d^l C_0^l & D^u \\ D^u & r^u - D^l \end{pmatrix}.$$

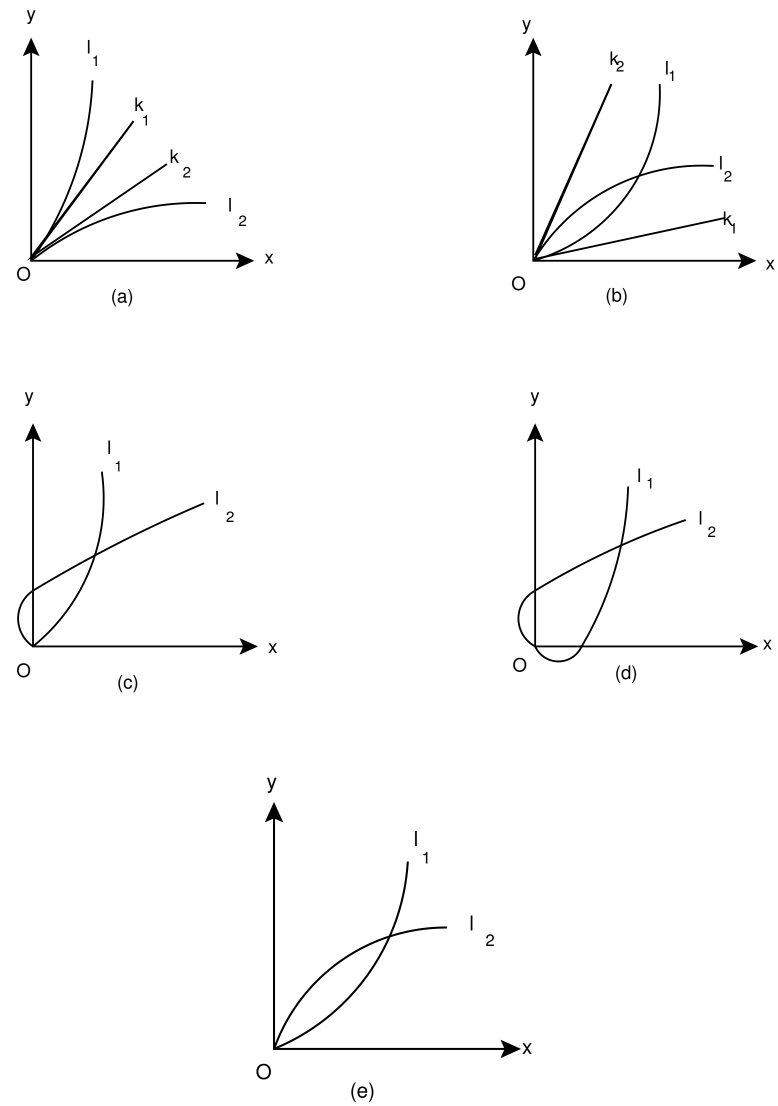

FIGURE 1.

Hence, the stability of $(0,0)$ is determined by the characteristic equation's eigenvalues λ_1, λ_2 that satisfy

$$\lambda^2 - [2r^u - 2D^l - d^l C_0^l]\lambda + (r^u - D^l)(r^u - D^l - d^l C_0^l) - (D^u)^2 = 0.$$

Solving it produces

$$\begin{aligned}\Delta &= [2r^u - 2D^l - d^l C_0^l]^2 - 4(r^u - D^l)(r^u - D^l - d^l C_0^l) + 4(D^u)^2 \\ &= (d^l C_0^l)^2 + 4(D^u)^2 > 0.\end{aligned}$$

Assuming without loss of generality that $\lambda_1 < \lambda_2$, we have

$$\lambda_1 + \lambda_2 = 2r^u - 2D^l - d^l C_0^l < 0,$$
$$\lambda_1 \lambda_2 = (r^u - D^l)(r^u - D^l - d^l C_0^l) - (D^u)^2 > 0.$$

Hence, when the condition of the theorem is satisfied, which implies $\lambda_1 < \lambda_2 < 0$, therefore $(0, 0)$ is a stable node. This completes the proof. □

Theorem 3.2. *If $r^u < D^l$, and $(D^u)^2 > (D^l - r^u)(d^l C_0^l + D^l - r^u)$, or $r^u \geq D^l$, then there exists a unique positive equilibrium (x^*, y^*) of (3.1) which is a stable node.*

Proof. our previous discussion establishes the existence of a positive equilibrium. Now, we analyze the local geometric properties of (x^*, y^*). The Jacobian matrix of (x^*, y^*) is

$$\begin{aligned} J(x^*, y^*) &= \begin{pmatrix} r^u - D^l - d^l C_0^l - 2a^l x^* & D^u \\ D^u & r^u - D^l - 2b^l y^* \end{pmatrix} \\ &= \begin{pmatrix} -\frac{D^u y^*}{x^*} - a^l x^* & D^u \\ D^u & -\frac{D^u x^*}{y^*} - b^l y^* \end{pmatrix}. \end{aligned}$$

Hence, the stability of (x^*, y^*) determined by the characteristic equation's eigenvalues

$$\lambda^2 + [\frac{D^u y^*}{x^*} + a^l x^* + \frac{D^u x^*}{y^*} + b^l y^*]\lambda + (\frac{D^u y^*}{x^*} + a^l x^*)(\frac{D^u x^*}{y^*} + b^l y^*) - (D^u)^2 = 0.$$

$$\begin{aligned} \Delta &= [\frac{D^u y^*}{x^*} + a^l x^* + \frac{D^u x^*}{y^*} + b^l y^*]^2 - 4(\frac{D^u y^*}{x^*} + a^l x^*)(\frac{D^u x^*}{y^*} + b^l y^*) + 4(D^u)^2 \\ &= [(\frac{D^u y^*}{x^*} + a^l x^*) - (\frac{D^u x^*}{y^*} + b^l y^*)]^2 + 4(D^u)^2 > 0. \end{aligned}$$

Under the condition of the theorem, it produces $\lambda_1 + \lambda_2 < 0$, $\lambda_1 \lambda_2 > 0$. We have $\lambda_1 < \lambda_2 < 0$. (x^*, y^*) is a stable node. The proof is completed. □

Theorem 3.3. *Each trajectory of (3.1) starting in R_+^2 is positive-going bounded.*

Proof. We want to construct an outer boundary of a positive invariant region which contains (x^*, y^*). Let AB and BC be the line segments of $L_1 : x = p$, $L_2 : y = q$, and (p, q) is an arbitrary fixed point in R_+^2 satisfying $p > x^*$ and

$$\frac{r^u - D^l + \sqrt{(r^u - D^l)^2 + 4b^l D^u p}}{2b^l} < q < \frac{(a^l p + d^l C_0^l + D^l - r^u)p}{D^u}$$

where the intersections of the straight line L_1 and l_1, l_2 are $F(p, y_2)$ and $E(p, y_1)$ respectively.

$$y_1 = \frac{r^u - D^l + \sqrt{(r^u - D^l)^2 + 4b^l D^u p}}{2b^l},$$
$$y_2 = \frac{(a^l p + d^l C_0^l + D^l - r^u)p}{D^u}$$

The domain is enclosed by $OABCO$, (see Fig. 2). By the sign of $\dot{x}, \dot{y}$, we can say

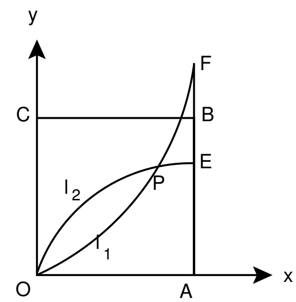

FIGURE 2. with $A(p,0), B(p,q), C(0,q), E(p,y_1), F(p,y_2), P(x^*,y^*)$

that the trajectory starting from (p,q) of (3.1) can not leave the confined set.

$$\begin{aligned}
\dot{x}\mid_{x=0} &= D^u y > 0,\\
\dot{y}\big|_{y=0} &= D^u x > 0,\\
\dot{x}\big|_{x=p} &= p(r^u - Dl - d^l C_0^l - a^l p) + D^u y < 0,\\
\dot{y}\big|_{y=q} &= q(r^u - D^l - b^l q) + D^u x < 0.
\end{aligned}$$

This completes the proof. □

Theorem 3.4. *For system* (3.1)*, if* $r^u < D^l$ *and* $(D^u)^2 < (D^l - r^u)(d^l C_0^l + D^l - r^u)$*, then* $(0,0)$ *is the unique nonnegative equilibrium, it is globally asymptotically stable.*

Proof. By theorem 3.3, we can easily prove that in $OABCO$,

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = -2r^u - 2D^l - d^l C_0^l - 2a^l x - 2b^l y < 0$$

then by Poincare-Bendixon theorem there are no limit cycles in $OABCO$, and $(0,0)$ is the unique positive equilibrium which is stable node in $OABCO$, so it is globally asymptotically stable. This completes the proof. □

Theorem 3.5. *For system* (3.1)*, if* $r^u < D^l$ *and* $(D^u)^2 > (D^l - r^u)(d^l C_0^l + D^l - r^u)$*, or* $r^u \ge D^l$ *then* (x^*, y^*) *is the unique positive equilibrium, it is globally asymptotically stable.*

Proof. We construct the Liapunov function

$$V(x,y) = \alpha(x - x^* - x^* \ln \frac{x}{x^*}) + \beta(y - y^* - y^* \ln \frac{y}{y^*}),$$

where α, β are positive constants. Calculating the derivative of $V(x,y)$ along (3.1), we have

$$\begin{aligned} V'_{(3.1)}(x,y) &= \alpha(x-x^*)\frac{\dot{x}}{x} + \beta(y-y^*)\frac{\dot{y}}{y} \\ &= -\alpha a^l(x-x^*)^2 - \beta b^l(y-y^*)^2 \\ &\quad + \alpha D^u(x-x^*)(\frac{y}{x} - \frac{y^*}{x^*}) + \beta D^u(y-y^*)(\frac{x}{y} - \frac{x^*}{y^*}) \\ &= -x^* a^l(x-x^*)^2 - y^* b^l(y-y^*)^2 \\ &\quad - D^u[\sqrt{\frac{y}{x}}(x-x^*) - \sqrt{\frac{x}{y}}(y-y^*)]^2 \le 0, \end{aligned}$$

In fact, we choose that $\alpha = x^*$, $\beta = y^*$. We can see that in the domain $OABCO$, $V'_{(3.1)} = 0$ if and only if $x = x^*$, $y = y^*$. Hence (x^*, y^*) is globally asymptotically stable. This completes the proof. □

The specific computation is similar to above-proved theorems, for the system (3.2) has two equilibria $O(0,0)$ and (x^{**}, y^{**}).

Theorem 3.6. *The point $(0,0)$ is always an equilibrium of system (3.2). If $r^l < D^u$, $(D^l)^2 < (D^u - r^l)(d^u C_0^u + D^u - r^l)$, then $(0,0)$ is the unique nonnegative equilibrium, which is a stable node and globally asymptotically stable.*

Theorem 3.7. *If $r^l < D^u$ and $(D^l)^2 > (D^u - r^l)(d^u C_0^u + D^u - r^l)$, or $r^l \ge D^u$, then there exists a unique positive equilibrium (x^*, y^*) of system (3.2) which is a stable node and globally asymptotically stable.*

In other words, for the systems (3.1), (3.2), when the only nonnegative equilibrium $(0,0)$ exists, it is stable node and is globally asymptotically stable. If the $(0,0)$ is unstable, then there exists a unique positive equilibrium which is globally asymptotically stable.

4. Permanence and Extinction

In this section, we study the permanence and extinction of population of system (2.3).

Theorem 4.1. *(1) If $r^u < D^l$ and $(D^l)^2 > (D^u - r^l)(d^u C_0^u + D^u - r^l)$, or $r^l \ge D^u$, then the system (2.3) is permanent; (2) If $r^u < D^l$ and $(D^u)^2 < (D^l - r^u)(d^l C_0^l + D^l - r^u)$, system (2.3) goes to extinction.*

Proof. From the conditions (1) of the theorem, we know that $r^l < D^u$ and $(D^u)^2 > (D^u l - r^u)(d^l C_0^l + D^l - r^u)$, or $r^u \ge D^l$ holds. By the above discussion of the theorem 3.5 and 3.7, we know that the system (3.1) and (3.2) have the globally asymptotically stable positive equilibria (x^*, y^*) and (x^{**}, y^{**}), the trivial equilibrium $(0,0)$ is unstable. We construct a positively invariant region for system (2.3).

where p_1, p_2, q_1, q_2 are positive constants satisfying

$$p_1 < \min\{x^*, x^{**}\}, \quad p_2 > \max\{x^*, x^{**}\},$$

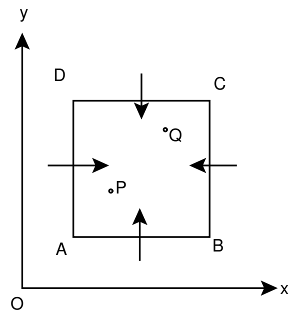

FIGURE 3. The rectangle $ABCD$ with $A(p_1, q_1)$, $B(p_2, q_1)$, $C(p_2, q_2)$, $D(p_1, q_2)$ and $P(x^{**}, y^{**})$, $Q(x^*, y^*)$

$$\begin{aligned}
&\frac{p_1}{D^l}(a^u p_1 + d^u C_0^u + D^u - r^l) \\
&< q_1 \\
&< \min\{y^*, y^{**}, \frac{-(r^l - D^u) + \sqrt{(r^l - D^u)^2 + 4b^u D^l p_1}}{2b^u}\},
\end{aligned}$$

and

$$\begin{aligned}
&\max\{y^*, y^{**}, \frac{-(r^u - D^l) + \sqrt{(r^u - D^l)^2 + 4b^l D^u p_2}}{2b^l}\} \\
&< q_2 \\
&< \frac{p_2}{D^u}(a^l p_2 + d^l C_0^l + D^l - r^u).
\end{aligned}$$

Since

$$\begin{aligned}
\dot{x}(t)\big|_{x=p_1} &\ge p_1(r^l - D^u - d^u C_0^u + a^u p_1) + D^l y\big|_{q_1 \le y \le q_2} > 0, \\
\dot{x}(t)\big|_{x=p_2} &\le p_2(r^u - D^l - d^l C_0^l + a^l p_2) + D^u y\big|_{q_1 \le y \le q_2} < 0, \\
\dot{y}(t)\big|_{y=q_1} &\ge q_1(r^l - D^u - b^u q_1) + D^l x\big|_{p_1 \le x \le p_2} > 0, \\
\dot{y}(t)\big|_{y=q_2} &\le q_2(r^u - D^l - b^l q_2) + D^u x\big|_{p_1 \le x \le p_2} < 0,
\end{aligned}$$

So the compact confined set $ABCD$ in R_+^2, the phase trajectories of the system (2.3) starting from the boundary always point into the enclosed domain. According to the Kamke theorem and definition 2.3, for any positive solution $(x(t), y(t))$ of (2.3) with positive initial value, there exists a time T, when $(x(t), y(t))$ goes in the $ABCD$ and never leaves for all $t > T$. Hence the system (2.3) is permanent.

Let $(x(t), y(t))$ be an arbitrary positive solution of system (2.3) with the positive initial value; $(x^*(t), y^*(t))$ and $(x^{**}(t), y^{**}(t))$ are the same of systems (3.1) and (3.2) respectively. Choose initial value $x^{**}(0) = x(0) = x^*(0)$, $y^{**}(0) = y(0) = y^*(0)$, If the condition (2) of the theorem exists, then the conditions $r^l < D^u$ and $(D^l)^2 < (D^u - r^l)(d^u C_0^u + D^u - r^l)$ hold, according to the Theorem 3.4 and 3.6, we know that the systems (3.1) and (3.2) have unique trivial equilibrium $(0, 0)$ which is globally asymptotically stable. By the Kamke theorem, $x^{**}(t) \le x(t) \le x^*(t)$,

$y^{**}(t) \le y(t) \le y^*(t)$, for $t \ge 0$. Furthermore, we have

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} y(t) = 0,$$

then system (2.3) is extinctive. □

Discussion. The toxic of the polluted population comes from the environment. Suppose environment of patch 1 is polluted–non-protective patch, and patch 2 is the ecological protective patch. In order to protect the existence of the polluted population, we can use the artificial method, the one's own purification function of the population, toxin in the body of the individuals in patch 1 can be removed, then we will put them into the protective patch. Set up ecological protective patch need a large number of financial resources, material resources and manpower, so when the scale of the protective patch is too large, individuals of some populations are put back non-protective patch. So, they will be polluted by the toxic in the patch1. The scale of the protection zone can be regulated through diffusive coefficient $D(t)$.

These conditions can simplify the mathematical model, I supposed that no toxic effects in the protective patch. The nest step in these investigations would be to consider systems in which there are toxic in the protective patch.

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