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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR FIRST-ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH TWO DEVIATING ARGUMENTS

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Abstract. In this paper, we use the coincidence degree theory to establish the existence and uniqueness of T-periodic solutions for the first-order neutral functional differential equation, with two deviating arguments,

$$(x(t)+Bx(t-\delta))' = g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_2(t))) + p(t).$$

1. Introduction

Consider the first-order neutral functional differential equation (NFDE), with two deviating arguments,

$$(x(t)+Bx(t-\delta))' = g_1(t,x(t-\tau_1(t))) + g_2(t,x(t-\tau_2(t))) + p(t), \tag{1.1}$$

where τ_1, τ_2, $p:\mathbb{R}\to\mathbb{R}$ and $g_1, g_2 : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions, B and δ are constants, τ_1, τ_2 and p are T-periodic, g_1 and g_2 are T-periodic in the first argument, $|B| \neq 1$ and $T > 0$.

The above equation has been used for the study of distributed networks containing lossless transmission lines [6, 7]. Hence, in recent years, the problem of the existence of periodic solutions for (1.1) has been extensively studied. For more details, we refer the reader to [1, 2, 4, 5, 6, 7, 9, 12] and the references cited therein. However, to the best of our knowledge, there exist no results for the existence and uniqueness of periodic solutions of (1.1).

The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of T-periodic solutions of (1.1). The results of this paper are new and they complement previously known results. An illustrative example is given in Section 4.

For ease of exposition, throughout this paper we will adopt the following notation:

$$|x|_k = \Big(\int_0^T |x(t)|^k dt\Big)^{1/k}, \quad |x|_\infty = \max_{t\in[0,T]} |x(t)|.$$

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Let $X = \{x | x \in C(\mathbb{R}, \mathbb{R}), x(t+T) = x(t), \text{ for all } t \in \mathbb{R}\}$ be a Banach space with the norm $\|x\|_X = |x|_\infty$. Define the two linear operators

$$\begin{gathered} A : X \to X, \quad (Ax)(t) = x(t) + Bx(t-\delta); \\ L : D(L) \subset X \to X, \quad Lx = (Ax)', \end{gathered} \tag{1.2}$$

where $D(L) = \{x | x \in X, x' \in C(\mathbb{R}, \mathbb{R})\}$.

We also define the nonlinear operator $N : X \to X$ by

$$Nx = g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) + p(t).$$

By Hale's terminology [4], a solution $u(t)$ of (1.1) is that $u \in C(\mathbb{R}, \mathbb{R})$ such that $Au \in C^1(\mathbb{R}, \mathbb{R})$ and (1.1) is satisfied on $\mathbb{R}$. In general, $u \notin C^1(\mathbb{R}, \mathbb{R})$. But from [9, Lemma 1], in view of $|B| \neq 1$, it is easy to see that $(Ax)' = Ax'$. So a T-periodic solution $u(t)$ of (1.1) must be such that $u \in C^1(\mathbb{R}, \mathbb{R})$. Meanwhile, according to [9, Lemma 1], we can easily get that $\ker L = \mathbb{R}$, and $\operatorname{Im} L = \{x \in X : \int_0^T x(s)ds = 0\}$. Therefore, the operator L is a Fredholm operator with index zero. Define the continuous projectors $P : X \to \ker L$ and $Q : X \to X/ImL$ by setting

$$\begin{gathered} Px(t) = \frac{1}{T}\int_0^T x(s)ds, \\ Qx(t) = \frac{1}{T}\int_0^T x(s)ds. \end{gathered}$$

Hence, $\operatorname{Im} P = \ker L$ and $\ker Q = \operatorname{Im} L$. Set $L_P = L|_{D(L)\cap \mathrm{KerP}}$, then L_P has continuous inverse L_P^{-1} defined by

$$L_P^{-1}y(t) = A^{-1}\Big(\frac{1}{T}\int_0^T sy(s)ds + \int_0^t y(s)ds\Big). \tag{1.3}$$

Therefore, it is easy to see from (1) and (1.3) that N is L-compact on $\overline{\Omega}$, where Ω is an open bounded set in X.

2. Preliminary Results

In view of (1.2) and (1), the operator equation

$$Lx = \lambda Nx$$

is equivalent to the equation

$$x'(t) + Bx'(t-\delta) = \lambda[g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) + p(t)],$$

where $\lambda \in (0,1)$.

For convenience of use, we introduce the Continuation Theorem [2] as follows.

Lemma 2.1. *Let X be a Banach space. Suppose that $L : D(L) \subset X \to X$ is a Fredholm operator with index zero and $N : \overline{\Omega} \to X$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

1. *$Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;*
2. *$Nx \notin ImL$, for all $x \in \partial\Omega \cap \ker L$;*
3. *For the Brower degree, $\deg\{QN, \Omega \cap \ker L, 0\} \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \cap D(L)$.

By using a similar argument of the proof of [8, Lemma 2.5], from [3, Theorem 225], we can obtain the following Lemma.

Lemma 2.2. *Let* $x(t) \in X \cap C^1(\mathbb{R},\mathbb{R})$ *. Suppose that there exists a constant* $D \geq 0$ *such that*

$$|x(\tau_0)| \leq D, \tau_0 \in [0,T].$$

Then

$$|x|_2 \leq \frac{T}{\pi}|x'|_2 + \sqrt{T}D. \tag{2.1}$$

Lemma 2.3 ([10])**.** *Let* $\mu \in [0,T]$ *be a constant,* $\overline{\delta} \in C(\mathbb{R},\mathbb{R})$ *be periodic with period* T*, and* $\sup_{t\in[0,T]} |\overline{\delta}(t)| \leq \mu$*. Then for any* $h \in C^1(\mathbb{R},\mathbb{R})$ *which is periodic with period* T*, we have*

$$\int_0^T |h(s) - h(s - \overline{\delta}(s))|^2 ds \leq 2\mu^2 \int_0^T |h'(s)|^2 ds. \tag{2.2}$$

For the next lemma we need the following conditions

(H) *For* $i = 1,2$*, there exist a constants* μ_i *and an integers* K_i *such that*

$$\mu_i = \sup_{t\in[0,T]} |\tau_i(t) - K_iT| \leq T.$$

(A0) *One of the following conditions holds:*
(1) $(g_i(t,u_1) - g_i(t,u_2))(u_1 - u_2) > 0$*, for* $i = 1,2, u_i \in \mathbb{R}$*, for all* $t \in \mathbb{R}$ *and* $u_1 \neq u_2$*;*
(2) $(g_i(t,u_1) - g_i(t,u_2))(u_1 - u_2) < 0$*, for* $i = 1,2, u_i \in \mathbb{R}$*, for all* $t \in \mathbb{R}$ *and* $u_1 \neq u_2$*;*

(A0') *One of the following conditions holds:*
(1) there exists constants b_1 *and* b_2 *such that* $b_1(\sqrt{2}\mu_1 + \frac{T}{\pi}) + b_2(\sqrt{2}\mu_2 + \frac{T}{\pi}) < 1 - |B|$*, and*

$$|g_i(t,u_1) - g_i(t,u_2)| \leq b_i|u_1 - u_2|, \text{ for } i = 1,2, u_i \in \mathbb{R}, \forall t \in \mathbb{R},$$

(2) There exists constants b_1 *and* b_2 *such that* $b_1(\sqrt{2}\mu_1 + \frac{T}{\pi}) + b_2(\sqrt{2}\mu_2 + \frac{T}{\pi}) < |B| - 1$*, and*

$$|g_i(t,u_1) - g_i(t,u_2)| \leq b_i|u_1 - u_2|, \quad \text{for } i = 1,2, u_i \in \mathbb{R}, \forall t \in \mathbb{R}.$$

Lemma 2.4. *Under assumptions (A0) and (A0'), Equation* (1.1) *has at most one* T*-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1). Then

$$(x_1(t) + Bx_1(t-\delta))' - g_1(t, x_1(t - \tau_1(t))) - g_2(t, x_1(t - \tau_2(t))) = p(t)$$

and

$$(x_2(t) + Bx_2(t-\delta))' - g_1(t, x_2(t - \tau_1(t))) - g_2(t, x_2(t - \tau_2(t))) = p(t).$$

This implies

$$\begin{aligned}&[(x_1(t) - x_2(t)) + B(x_1(t-\delta) - x_2(t-\delta))]' - (g_1(t, x_1(t - \tau_1(t))) \\ &- g_1(t, x_2(t - \tau_1(t)))) - (g_2(t, x_1(t - \tau_2(t))) - g_2(t, x_2(t - \tau_2(t)))) = 0.\end{aligned} \tag{2.3}$$

Set $Z(t) = x_1(t) - x_2(t)$. Then, from (2.3), we obtain

$$\begin{aligned}&Z'(t) + BZ'(t-\delta) - (g_1(t, x_1(t - \tau_1(t))) \\ &- g_1(t, x_2(t - \tau_1(t)))) - (g_2(t, x_1(t - \tau_2(t))) - g_2(t, x_2(t - \tau_2(t)))) = 0.\end{aligned} \tag{2.4}$$

Thus, integrating (2.4) from 0 to T, we have

$$\begin{aligned}&\int_0^T [(g_1(t,x_1(t-\tau_1(t)))-g_1(t,x_2(t-\tau_1(t))))\\&+(g_2(t,x_1(t-\tau_2(t)))-g_2(t,x_2(t-\tau_2(t))))]dt=0.\end{aligned}$$

Therefore, in view of integral mean value theorem, it follows that there exists a constant $\gamma\in[0,T]$ such that

$$\begin{aligned}&(g_1(\gamma,x_1(\gamma-\tau_1(\gamma)))-g_1(\gamma,x_2(\gamma-\tau_1(\gamma))))\\&+(g_2(\gamma,x_1(\gamma-\tau_2(\gamma)))-g_2(\gamma,x_2(\gamma-\tau_2(\gamma))))=0.\end{aligned}\tag{2.5}$$

From (A0), (2.5) implies

$$(x_1(\gamma-\tau_1(\gamma))-x_2(\gamma-\tau_1(\gamma)))(x_1(\gamma-\tau_2(\gamma))-x_2(\gamma-\tau_2(\gamma)))\le 0.$$

Since $Z(t)=x_1(t)-x_2(t)$ is a continuous function on $\mathbb{R}$, it follows that there exists a constant $\xi\in\mathbb{R}$ such that

$$Z(\xi)=0.\tag{2.6}$$

Let $\xi=nT+\widetilde{\gamma}$, where $\widetilde{\gamma}\in[0,T]$ and n is an integer. Then, (2.6) implies that there exists a constant $\widetilde{\gamma}\in[0,T]$ such that

$$Z(\widetilde{\gamma})=Z(\xi)=0.\tag{2.7}$$

Then, from Lemma 2.2, using Schwarz inequality and the inequality

$$|Z(t)|=|Z(\widetilde{\gamma})+\int_{\widetilde{\gamma}}^t Z'(s)ds|\le\int_0^T|Z'(s)|ds,\ \text{for all } t\in[0,T],$$

we obtain

$$|Z|_\infty\le\sqrt{T}|Z'|_2,\quad\text{and}\quad |Z|_2\le\frac{T}{\pi}|Z'|_2.\tag{2.8}$$

Now, we consider two cases.

Case (i). If (A0')(1) holds, multiplying both sides of (2.4) by $Z'(t)$ and then integrating them from 0 to T, using (H), (2.2), (2.8) and Schwarz inequality, we have

$$\begin{aligned}&|Z'|_2^2\\&=\int_0^T|Z'(t)|^2dt\\&=-B\int_0^T Z'(t)Z'(t-\delta)dt+\int_0^T(g_1(t,x_1(t-\tau_1(t)))-g_1(t,x_2(t-\tau_1(t))))Z'(t)dt\\&\quad+\int_0^T(g_2(t,x_1(t-\tau_2(t)))-g_2(t,x_2(t-\tau_2(t))))Z'(t)dt\\&\le|B||Z'|_2^2+b_1\int_0^T|x_1(t-\tau_1(t))-x_2(t-\tau_1(t))||Z'(t)|dt\\&\quad+b_2\int_0^T|x_1(t-\tau_2(t))-x_2(t-\tau_2(t))||Z'(t)|dt\\&\le|B||Z'|_2^2+b_1\int_0^T|Z(t-\tau_1(t))-Z(t)||Z'(t)|dt+b_1\int_0^T|Z(t)||Z'(t)|dt\\&\quad+b_2\int_0^T|Z(t-\tau_2(t))-Z(t)||Z'(t)|dt+b_2\int_0^T|Z(t)||Z'(t)|dt\end{aligned}$$

$$
\begin{aligned}
&\le |B||Z'|_2^2 + b_1(\int_0^T |Z(t-\tau_1(t)) - Z(t)|^2 dt)^{\frac{1}{2}}|Z'|_2 + b_1|Z|_2|Z'|_2\\
&\quad + b_2(\int_0^T |Z(t-\tau_2(t)) - Z(t)|^2 dt)^{\frac{1}{2}}|Z'|_2 + b_2|Z|_2|Z'|_2\\
&= |B||Z'|_2^2 + b_1(\int_0^T |Z(t-(\tau_1(t)-K_1T)) - Z(t)|^2 dt)^{\frac{1}{2}}|Z'|_2 + b_1|Z|_2|Z'|_2\\
&\quad + b_2(\int_0^T |Z(t-(\tau_2(t)-K_2T)) - Z(t)|^2 dt)^{\frac{1}{2}}|Z'|_2 + b_2|Z|_2|Z'|_2\\
&\le [|B| + b_1(\sqrt{2}\mu_1 + \frac{T}{\pi}) + b_2(\sqrt{2}\mu_2 + \frac{T}{\pi})]|Z'|_2^2.
\end{aligned}
$$

From (2.8) and (A0')(1), the above inequalit implies

$$Z(t) \equiv Z'(t) \equiv 0, \quad \text{for all } t \in \mathbb{R}.$$

Hence, $x_1(t) \equiv x_2(t)$, for all $t \in \mathbb{R}$. Therefore, (1.1) has at most one T-periodic solution.

Case (ii). If (A0')(2) holds, multiplying both sides of (2.4) by $Z'(t-\delta)$ and then integrating them from 0 to T, using (H), (2.2), (2.8) and Schwarz inequality, we have

$$
\begin{aligned}
&|B||Z'|_2^2\\
&= |\int_0^T B|Z'(t-\delta)|^2 dt|\\
&= |-\int_0^T Z'(t)Z'(t-\delta)dt\\
&\quad + \int_0^T (g_1(t, x_1(t-\tau_1(t))) - g_2(t, x_2(t-\tau_1(t))))Z'(t-\delta)dt\\
&\quad + \int_0^T (g_2(t, x_1(t-\tau_2(t))) - g_2(t, x_2(t-\tau_2(t))))Z'(t-\delta)dt|\\
&\le |Z'|_2^2 + b_1\int_0^T |x_1(t-\tau_1(t)) - x_2(t-\tau_1(t))||Z'(t-\delta)|dt\\
&\quad + b_2\int_0^T |x_1(t-\tau_2(t)) - x_2(t-\tau_2(t))||Z'(t-\delta)|dt\\
&\le |Z'|_2^2 + b_1\int_0^T |Z(t-\tau_1(t)) - Z(t)||Z'(t-\delta)|dt + b_1\int_0^T |Z(t)||Z'(t-\delta)|dt\\
&\quad + b_2\int_0^T |Z(t-\tau_2(t)) - Z(t)||Z'(t-\delta)|dt + b_2\int_0^T |Z(t)||Z'(t-\delta)|dt\\
&\le |Z'|_2^2 + b_1(\int_0^T |Z(t-\tau_1(t)) - Z(t)|^2 dt)^{\frac{1}{2}}|Z'|_2 + b_1|Z|_2|Z'|_2\\
&\quad + b_2(\int_0^T |Z(t-\tau_2(t)) - Z(t)|^2 dt)^{\frac{1}{2}}|Z'|_2 + b_2|Z|_2|Z'|_2\\
&= |Z'|_2^2 + b_1(\int_0^T |Z(t-(\tau_1(t)-K_1T)) - Z(t)|^2 dt)^{\frac{1}{2}}|Z'|_2 + b_1|Z|_2|Z'|_2
\end{aligned}
$$

$$
\begin{aligned}
&+ b_2\Big(\int_0^T |Z(t-(\tau_2(t)-K_2T)) - Z(t)|^2 dt\Big)^{\frac{1}{2}} |Z'|_2 + b_2|Z|_2|Z'|_2 \\
&\le [1 + b_1(\sqrt{2}\mu_1 + \frac{T}{\pi}) + b_2(\sqrt{2}\mu_2 + \frac{T}{\pi})]|Z'|_2^2
\end{aligned}
$$

Then using the methods similar to those used in Case (i), from the above inequality, (2.8), and (A0')(2), we can conclude that (1.1) has at most one T-periodic solution. The proof of Lemma 2.4 is now complete. □

For the next lemma we use the following assumptions:

(A1) $x(g_1(t,x) + g_2(t,x) + p(t)) > 0$, for all $t \in \mathbb{R}, |x| \ge d$;
(A2) $x(g_1(t,x) + g_2(t,x) + p(t)) < 0$, for all $t \in \mathbb{R}, |x| \ge d$.

Lemma 2.5. *Assume (A0) and that there exists a positive constant d such that one of the two conditions (A1) or (A2) holds. If $x(t)$ is a T-periodic solution of* (2)*, then*

$$
|x|_\infty \le d + \sqrt{T}|x'|_2. \tag{2.9}
$$

Proof. Let $x(t)$ be a T-periodic solution of (2). Then, integrating (2) from 0 to T, we have

$$
\int_0^T [g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) + p(t)]dt = 0.
$$

This implies that there exists a constant $t_1 \in \mathbb{R}$ such that

$$
g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) + p(t_1) = 0. \tag{2.10}
$$

We show next the following Claim: If $x(t)$ is a T-periodic solution of (2), then there exists a constant $t_2 \in \mathbb{R}$ such that

$$
|x(t_2)| \le d. \tag{2.11}
$$

Assume, by way of contradiction, that (2.11) does not hold. Then

$$
|x(t)| > d, \quad \text{for all } t \in \mathbb{R},
$$

which, together with (A1), (A2) and (2.10), implies that one of the following relations holds:

$$
x(t_1-\tau_1(t_1)) > x(t_1-\tau_2(t_1)) > d; \tag{2.12}
$$

$$
x(t_1-\tau_2(t_1)) > x(t_1-\tau_1(t_1)) > d; \tag{2.13}
$$

$$
x(t_1-\tau_1(t_1)) < x(t_1-\tau_2(t_1)) < -d; \tag{2.14}
$$

$$
x(t_1-\tau_2(t_1)) < x(t_1-\tau_1(t_1)) < -d. \tag{2.15}
$$

If (2.12) holds, in view of (A0)(1), (A0)(2), (A1) and (A2), we shall consider four cases as follows.

Case (i). If (A1) and (A0)(1) hold, according to (2.12), we obtain

$$
\begin{aligned}
0 &< g_1(t_1, x(t_1-\tau_2(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) + p(t_1) \\
&< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) + p(t_1),
\end{aligned}
$$

which contradicts (2.10). This contradiction implies that (2.11) holds.

Case (ii). If (A1) and (A0)(2) hold, according to (2.12), we obtain

$$
\begin{aligned}
0 &< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_1(t_1))) + p(t_1) \\
&< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) + p(t_1),
\end{aligned}
$$

which contradicts (2.10). This contradiction implies that (2.11) holds.
Case (iii). If (A2) and (A0)(1) hold, according to (2.12), we obtain

$$\begin{aligned}
&g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) + p(t_1)\\
&< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_1(t_1))) + p(t_1) < 0,
\end{aligned}$$

which contradicts (2.10). This contradiction implies that (2.11) holds.
Case (iv). If (A2) and (A0)(2) hold, according to (2.12), we obtain

$$\begin{aligned}
&g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) + p(t_1)\\
&< g_1(t_1, x(t_1-\tau_2(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) + p(t_1) < 0,
\end{aligned}$$

which contradicts (2.10). This contradiction implies that (2.11) holds.

If (2.13) (or (2.14), or (2.15)) holds, using the methods similar to those used in Case (i) - Case (iv), we can show that (2.11) holds. This completes the proof of the Claim.

Let $t_2 = mT + t_0$, where $t_0 \in [0, T]$ and m is an integer. Then, using Schwarz inequality and the inequality

$$|x(t)| = |x(t_0) + \int_{t_0}^{t} x'(s)ds| \le d + \int_0^T |x'(s)|ds, \quad \text{for all } t \in [0, T],$$

we obtain

$$|x|_\infty = \max_{t\in[0,T]} |x(t)| \le d + \sqrt{T}|x'|_2.$$

This completes the proof. □

3. Main Results

Theorem 3.1. *Assume that (H), (A0), (A0') and either (A1) or (A2). Then* (1.1) *has a unique T-periodic solution.*

Proof. From Lemma 2.4, together with (H), (A0) and (A0'), it i s easy to see that (1.1) has at most one T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that (1.1) has at least one T-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible T-periodic solutions of (2) is bounded.

Let $x(t)$ be a T-periodic solution of equation (2). In view of (A0')(1) and (A0')(2), we shall consider two cases as follows.
Case (i). If (A0')(1) holds, multiplying both sides of (2) by $x'(t)$ and then integrating them from 0 to T, from (2.1), (2.2), (2.11), (H), (A0')(1) and the Schwarz

inequality, we have

$$
\begin{aligned}
&|x'|_2^2\\
&= \int_0^T |x'(t)|^2 dt\\
&= -\int_0^T Bx'(t-\delta)x'(t)dt + \lambda\int_0^T g_1(t, x(t-\tau_1(t)))x'(t)dt\\
&\quad + \lambda\int_0^T g_2(t, x(t-\tau_2(t)))x'(t)dt + \lambda\int_0^T p(t)x'(t)dt\\
&\le |B||x'|_2^2 + |p|_2|x'|_2 + \lambda\int_0^T (g_1(t, x(t-\tau_1(t))) - g_1(t, x(t)) + g_1(t, x(t))\\
&\quad - g_1(t,0))x'(t)dt + \lambda\int_0^T (g_2(t, x(t-\tau_2(t))) - g_2(t, x(t)) + g_2(t, x(t))\\
&\quad - g_2(t,0))x'(t)dt + \lambda\int_0^T g_1(t,0)x'(t)dt + \lambda\int_0^T g_2(t,0)x'(t)dt\\
&\quad |B||x'|_2^2 + |p|_2|x'|_2 + b_1(\int_0^T |x(t-\tau_1(t)) - x(t)|^2 dt)^{\frac{1}{2}}|x'|_2 + b_1|x|_2|x'|_2\\
&\quad + b_2(\int_0^T |x(t-\tau_2(t)) - x(t)|^2 dt)^{\frac{1}{2}}|x'|_2 + b_2|x|_2|x'|_2\\
&\quad + (\max_{t\in[0,T]}|g_1(t,0)| + \max_{t\in[0,T]}|g_2(t,0)|)\sqrt{T}|x'|_2\\
&= |B||x'|_2^2 + |p|_2|x'|_2 + b_1(\int_0^T |x(t-(\tau_1(t)-K_1T)) - x(t)|^2 dt)^{\frac{1}{2}}|x'|_2 + b_1|x|_2|x'|_2\\
&\quad + b_2(\int_0^T |x(t-(\tau_2(t)-K_2T)) - x(t)|^2 dt)^{\frac{1}{2}}|x'|_2 + b_2|x|_2|x'|_2\\
&\quad + (\max_{t\in[0,T]}|g_1(t,0)| + \max_{t\in[0,T]}|g_2(t,0)|)\sqrt{T}|x'|_2\\
&\le [|B| + b_1(\sqrt{2}\mu_1 + \frac{T}{\pi}) + b_2(\sqrt{2}\mu_2 + \frac{T}{\pi})]|x'|_2^2 + |p|_2|x'|_2\\
&\quad + (b_1d + b_2d + \max_{t\in[0,T]}|g_1(t,0)| + \max_{t\in[0,T]}|g_2(t,0)|)\sqrt{T}|x'|_2.
\end{aligned}
\tag{3.1}
$$

Now, let

$$
D_1 = \frac{|p|_2 + (b_1d + b_2d + \max_{t\in[0,T]}|g_1(t,0)| + \max_{t\in[0,T]}|g_2(t,0)|)\sqrt{T}}{1 - |B| - (b_1(\sqrt{2}\mu_1 + \frac{T}{\pi}) + b_2(\sqrt{2}\mu_2 + \frac{T}{\pi}))}.
$$

In view of (2.9) and (3.1), we obtain

$$
|x'|_2 \le D_1, |x|_\infty \le d + \sqrt{T}D_1. \tag{3.2}
$$

Case (ii). If (A0')(2) holds, multiplying both sides of (2) by $x'(t-\delta)$ and then integrating them from 0 to T, from (2.1), (2.2), (2.9), (2.11), (A0')(2) and the inequality of Schwarz, we have

$$
\begin{aligned}
|B||x'|_2^2 &= |\int_0^T B|x'(t-\delta)|^2 dt| \\
&= |-\int_0^T x'(t-\delta)x'(t)dt + \lambda\int_0^T g_1(t,x(t-\tau_1(t)))x'(t-\delta)dt \\
&\quad + \lambda\int_0^T g_2(t,x(t-\tau_2(t)))x'(t-\delta)dt + \lambda\int_0^T p(t)x'(t-\delta)dt| \\
&\le |x'|_2^2 + |p|_2|x'|_2 + |\lambda\int_0^T (g_1(t,x(t-\tau_1(t))) - g_1(t,x(t)) + g_1(t,x(t)) \\
&\quad - g_1(t,0))x'(t-\delta)dt + \lambda\int_0^T (g_2(t,x(t-\tau_2(t))) - g_2(t,x(t)) + g_2(t,x(t)) \\
&\quad - g_2(t,0))x'(t-\delta)dt + \lambda\int_0^T g_1(t,0)x'(t)dt + \lambda\int_0^T g_2(t,0)x'(t-\delta)dt| \\
&\le |x'|_2^2 + |p|_2|x'|_2 + b_1(\int_0^T |x(t-\tau_1(t)) - x(t)|^2 dt)^{\frac{1}{2}}|x'|_2 + b_1|x|_2|x'|_2 \\
&\quad + b_2(\int_0^T |x(t-\tau_2(t)) - x(t)|^2 dt)^{\frac{1}{2}}|x'|_2 + b_2|x|_2|x'|_2 \\
&\quad + (\max_{t\in[0,T]}|g_1(t,0)| + \max_{t\in[0,T]}|g_2(t,0)|)\sqrt{T}|x'|_2 \\
&\le [1 + b_1(\sqrt{2}\mu_1 + \frac{T}{\pi}) + b_2(\sqrt{2}\mu_2 + \frac{T}{\pi})]|x'|_2^2 + |p|_2|x'|_2 \\
&\quad + (b_1 d + b_2 d + \max_{t\in[0,T]}|g_1(t,0)| + \max_{t\in[0,T]}|g_2(t,0)|)\sqrt{T}|x'|_2.
\end{aligned}
\tag{3.3}
$$

Now, let

$$
\overline{D}_1 = \frac{|p|_2 + (b_1 d + b_2 d + \max_{t\in[0,T]}|g_1(t,0)| + \max_{t\in[0,T]}|g_2(t,0)|)\sqrt{T}}{|B| - 1 - (b_1(\sqrt{2}\mu_1 + \frac{T}{\pi}) + b_2(\sqrt{2}\mu_2 + \frac{T}{\pi}))}.
$$

In view of (2.9) and (3.3), we obtain

$$
|x'|_2 \le \overline{D}_1, |x|_\infty \le d + \sqrt{T}\overline{D}_1. \tag{3.4}
$$

If $x \in \Omega_1 = \{x \in \ker L \cap X : Nx \in ImL\}$, then there exists a constant M_1 such that

$$
x(t) \equiv M_1 \quad \text{and} \quad \int_0^T [g_1(t,M_1) + g_2(t,M_1) + p(t)]dt = 0. \tag{3.5}
$$

Thus,

$$
|x(t)| \equiv |M_1| < d, \quad \text{for all } x(t) \in \Omega_1. \tag{3.6}
$$

Let $M = (D_1 + \overline{D}_1)\sqrt{T} + d + 1$. Set

$$
\Omega = \{x | x \in X, |x|_\infty < M\}.
$$

It is easy to see from (1) and (1.3) that N is L-compact on $\overline{\Omega}$. We have from (3.5), (3.6) and the fact $M > \max\{D_1\sqrt{T} + d, \overline{D}_1\sqrt{T} + d, d\}$ that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, define the continuous functions

$$H_1(x,\mu) = (1-\mu)x + \mu \cdot \frac{1}{T}\int_0^T [g_1(t,x) + g_2(t,x) + p(t)]dt; \mu \in [0\,1],$$

$$H_2(x,\mu) = -(1-\mu)x + \mu \cdot \frac{1}{T}\int_0^T [g_1(t,x) + g_2(t,x) + p(t)]dt; \mu \in [0\,1].$$

If (A1) holds, then

$$xH_1(x,\mu) \neq 0 \quad \text{for all} x \in \partial\Omega \cap \ker L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned}\deg\{QN, \Omega\cap\ker L, 0\} &= \deg\{\frac{1}{T}\int_0^T [g_1(t,x) + g_2(t,x) + p(t)]dt, \Omega\cap\ker L, 0\} \\ &= deg\{x, \Omega\cap\ker L, 0\} \neq 0.\end{aligned}$$

If (A2) holds, then $xH_2(x,\mu) \neq 0$ for all $x \in \partial\Omega\cap\ker L$. Hence, using the homotopy invariance theorem, we obtain

$$\begin{aligned}\deg\{QN, \Omega\cap\ker L, 0\} &= \deg\{\frac{1}{T}\int_0^T [g_1(t,x) + g_2(t,x) + p(t)]dt, \Omega\cap\ker L, 0\} \\ &= \deg\{-x, \Omega\cap\ker L, 0\} \neq 0.\end{aligned}$$

In view of all the discussions above and Lemma 2.1, Theorem 3.1 is proved. □

4. Concluding remarks

Example 4.1. The first-order neutral functional differential

$$(x(t) + \frac{1}{8}x(t-\delta))' = -\frac{3}{8}x(t - \frac{\sqrt{2}}{64}\sin^2 t) + \frac{1}{32}[1 - x(t - \frac{\sqrt{2}}{64}\cos^2 t)] + e^{\cos t} \tag{4.1}$$

has a unique 2π-periodic solution.

From (4.1), we have $B = \frac{1}{8}$, $g_1(x) = -\frac{3}{8}x$, $g_2(x) = \frac{1}{32}[1-x]$ and $p(t) = e^{\cos t}$. Then, $\mu_1 = \sup_{t\in[0,T]}|\frac{\sqrt{2}}{64}\sin^2 t| = \frac{\sqrt{2}}{64} < 2\pi$, $\mu_2 = \sup_{t\in[0,T]}|\frac{\sqrt{2}}{64}\cos^2 t| = \frac{\sqrt{2}}{64} < 2\pi$, $b_1 = \frac{3}{8}$, $b_2 = \frac{1}{32}$. It is straight forward to check that all the conditions needed in Theorem 3.1 are satisfied. Therefore, (4.1) has a unique 2π-periodic solution.

Remark 4.2. Equation (4.1) is a very simple version of first order NFDE. Since $B \neq 0$, all the results in the references and their references can not be applicable to (4.1) to obtain the existence and uniqueness of 2π-periodic solutions. This implies that the results of this paper are essentially new.

Remark 4.3. By using the methods similarly to those used for (1.1), we can deal with the NFDE with multiple deviating arguments, for example

$$(x(t) + Bx(t-\delta))' = \sum_{i=1}^{n} g_i(t, x(t-\tau_i(t))) + p(t), \tag{4.2}$$

where $\tau_i (i = 1,2,\dots,n)$, $p : \mathbb{R} \to \mathbb{R}$ and $g_i (i = 1,2,\dots,n) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions, $\tau_i (i = 1,2,\dots,n)$ and p are T-periodic, $g_i, i = 1,2,\dots,n$, are T-periodic in the first argument, and $T > 0$. One may also establish the results similarly to those in Theorem 3.1 under some minor additional assumptions on $g_i(t,x) (i = 1,2,\dots,n)$.

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