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A NOTE ON ALMOST PERIODIC SOLUTIONS OF SEMILINEAR EQUATIONS IN BANACH SPACES

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Abstract. In this article, we generalize the main result obtained by Bahaj [1]. Also our proof is shorter than the original proof.

1. Introduction

This article concerns the semilinear equation

$$u'(t) + Au(t) = f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $-A$ generates a C_0-semigroup on a Banach space E and f is a continuous function from $\mathbb{R} \times E$ to E. In [1, Theorem 3.1], the existence and uniqueness of almost periodic solutions to (1.1) was established under the following conditions:

(i) $-A$ generates an analytic semigroup $(S(t))_{t\geq 0}$ on X satisfying $\|T(t)\| \leq e^{-\beta t}$ for some $\beta > 0$;

(ii) $f(t, x) : \mathbb{R} \times D(A^\alpha) \mapsto E$ satisfying

(A1) f is uniformly almost periodic;

(A2) There are numbers $L > 0$ (sufficiently small) and $0 \leq \theta \leq 1$ such that

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leq L(|t_1 - t_2|^\theta + \|x_1 - x_2\|_\alpha)$$

for t_1, t_2 in $\mathbb{R}$ and x_1, x_2 in $D(A^\alpha)$, where $D(A^\alpha)$ ($\alpha \geq 0$) is the domain of the fractional power A^α with the norm $\|x\|_\alpha = \|A^\alpha x\|$.

In this note we generalize that result to an operator $-A$, which generates a C_0 semigroup admitting an exponential dichotomy and to some subspaces of $BC(\mathbb{R})$, the Banach space of bounded, continuous function from $\mathbb{R}$ to E with the sup-norm. Namely, we consider the following subspaces:

$BUC(\mathbb{R})$, the space of bounded, uniformly continuous functions on $\mathbb{R}$;
$AP(\mathbb{R})$, the space of almost periodic functions on $\mathbb{R}$;
$P(\omega)$, the space of ω-periodic functions;
$C_1 := \{f \in BC(\mathbb{R}) : \lim_{t\to\pm\infty} f(t) \text{ exists }\}$;
$C_0 := \{f \in BC(\mathbb{R}) : \lim_{t\to\pm\infty} f(t) = 0$.

Recall, that a function $f(t) : \mathbb{R} \mapsto E$ is called almost periodic if the set $\{f_s : s \in \mathbb{R}\}$ is relatively compact in $BC(\mathbb{R})$, where $f_s(\cdot) := f(s + \cdot)$ is the s-translation of f.

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Note that all above subspaces are Banach spaces with the sup-norm. In this note we prove the following theorem.

Theorem 1.1. *Let $-A$ generate an analytic C_0-semigroup $T(t)$ satisfying $\{i\lambda : \lambda \in \mathbb{R}\} \subset \varrho(-A)$, let $\mathcal{M}$ be one of the above mentioned subspaces of $BC(\mathbb{R})$, and let $f(t,x) : \mathbb{R} \times D(A^\alpha) \mapsto E$, where $0 \le \alpha < 1$, satisfy the following conditions*

(B1) *For each $u \in \mathcal{M}$, the function $t \mapsto f(t, u(t))$ is in $\mathcal{M}$;*
(B2) *For u and v in $D(A^\alpha)$ we have*

$$\|f(t,u) - f(t,v)\| \le L\|A^\alpha u - A^\alpha v\|. \tag{1.2}$$

Then Equation (1.1) *has a unique mild solution (defined below) in $\mathcal{M}$ for a sufficiently small L. Moreover, if $f(t,x)$ satisfies condition* (B1) *and* (A2)*, then this solution is a classical solution.*

It is easy to see that the main result in [1] is a particular case of Theorem 1.1, when $\mathcal{M} = AP(\mathbb{R})$ and $\sigma(-A) \subset \{\lambda \in \mathbb{C} : Re\lambda < -\beta\}$ for some $\beta > 0$.

2. Preparation and Proof of Theorem 1.1

To prove Theorem 1.1, we first consider the linear equation

$$u'(t) + Au(t) = f(t), \quad t \in \mathbb{R}, \tag{2.1}$$

where $-A$ generates a semigroup $(T(t))_{t\ge 0}$. A continuous function u is called a mild solution to (2.1) if it satisfies

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau)d\tau, \quad t \ge s.$$

Similarly, a mild solution to (1.1) is of the form

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau, u(\tau))d\tau, \quad t \ge s.$$

Suppose $\mathcal{M}$ is a closed subspace of $BC(\mathbb{R})$. We say that $\mathcal{M}$ is admissible with respect to (w.r.t. for short) Equation (2.1) if for each function $f \in \mathcal{M}$, Equation (2.1) has a unique mild solution $u \in \mathcal{M}$. Over the last two decades, the study of the admissibility of $BC(\mathbb{R})$ and the above mentioned subspaces w.r.t. Equation (2.1) has been of increasing interest (see e.g. [7] and [8]). Recently, to the nonautonomous equation

$$u'(t) + A(t)u(t) = f(t), \quad t \in \mathbb{R}, \tag{2.2}$$

the admissibility of several spaces, such as $BUC(\mathbb{R})$, $L_p(\mathbb{R})$ and $AP(\mathbb{R})$ has also been intensively investigated (see e.g. [2, 3, 5] and references therein). In both cases, it is involved with the concept so-called *exponential dichotomy* of a C_0-semigroup (or of an evolution family, in nonautonomous case). Recall, a C_0-semigroup $T(t)$ has an exponential dichotomy if there exist a projection operator $P \in B(E)$ and two numbers $M > 0$, $\delta > 0$ such that

(i) $PT(t) = T(t)P$ for all $t \ge 0$;
(ii) $\|T(t)Px\| \le Me^{-\delta t}\|Px\|$ for all $x \in E$ and $t \ge 0$;
(iii) $T(t)(I-P)$ extends to a C_0-group on $N(P)$, the nullspace of P, and $\|T(t)(I-P)x\| \le Me^{\delta t}\|(I-P)x\|$ for all $x \in E$ and $t \le 0$.

We have the following result ([7, Theorem 4]).

Theorem 2.1. *The following three statements are equivalent.*

(i) *Operator $-A$ generates a C_0-semigroup, which admits an exponential dichotomy.*
(ii) *For each function $f \in BC(\mathbb{R})$, Equation (2.1) has a unique mild solution in $BC(\mathbb{R})$.*
(iii) $S = \{\mu \in \mathbb{C} : |\mu| = 1\} \subset \varrho(T(t))$ *for one (all) $t > 0$.*

In this case, the mild solution of Equation (2.1) has the form

$$u(t) := \int_{-\infty}^{\infty} G(t-s)f(s)ds, \quad \text{where } G(t) := \begin{cases} T(t)P & \text{for } t > 0, \\ -T(t)(I-P) & \text{for } t < 0 \end{cases}$$

which is the Green's kernel. Moreover, $u \in \mathcal{M}$ whenever $f \in \mathcal{M}$, where $\mathcal{M}$ is one of the above mentioned subspaces of $BC(\mathbb{R})$ ([7, Theorem 5]). If A now generates an analytic semigroup, then A^α is given by

$$A^\alpha x = A^\alpha P x + e^{\alpha\pi i}(-A)^\alpha (I-P)x.$$

We have the following lemma.

Lemma 2.2. *If $-A$ generates an analytic semigroup, then $u(t) \in D(A^\alpha)$ for $0 < \alpha < 1$, and $\|\tilde{u}\| \le C\|f\|$, where $\tilde{u}(t) := A^\alpha u(t)$, for some $C > 0$.*

Proof. First note that for each $t > 0$, $A^\alpha T(t)$ is a bounded operator and $\|A^\alpha T(t)\| \le M t^{-\alpha} e^{-\beta t}$ for some positive M and β ([6, Theorem 2.6.13]). Hence, $\int_0^\infty \|A^\alpha T(t)\| dt \le M_1 < \infty$. Using this fact we have

$$\begin{aligned} \|A^\alpha u(t)\| &= \| \int_{-\infty}^{\infty} A^\alpha G(t-s) f(s) ds \| \\ &\le \| \int_{-\infty}^{t} A^\alpha T(t-s) P f(s) ds \| + \| \int_{t}^{\infty} (-A)^\alpha T(t-s)(I-P) f(s) ds \| \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 \le \int_{-\infty}^{t} \|A^\alpha T(t-s)\| \cdot \|f\| ds = \int_0^\infty \|A^\alpha T(s)\| ds \cdot \|f\| \le M_1 \|f\|$$

and

$$\begin{aligned} I_2 &\le \int_t^\infty \|(-A)^\alpha T(t-s)(I-P) f(s)\| ds \\ &= \int_0^\infty \|(-A)^\alpha T(-s')(I-P) f(s'+t)\| ds' \\ &\le \int_0^\infty \|(-A)^\alpha T(-s')\| \cdot \|(I-P) f(s'+t)\| ds' \\ &\le M_1 \|f\|. \end{aligned}$$

Hence, $\|A^\alpha u(t)\| \le 2M_1 \|f\|$ for each $t \in \mathbb{R}$. □

We now turn to (1.1). First, we state a preliminary result.

Lemma 2.3. *Let $-A$ generate a C_0-semigroup and B be an invertible operator on E, and $\mathcal{M}$ be a closed subspace of $BC(\mathbb{R})$ with the property: $\mathcal{M}$ is admissible w.r.t. (2.1) and $\tilde{u}(\cdot) := Bu(\cdot) \in \mathcal{M}$ and $\|\tilde{u}\| \le C\|f\|$ for each $f \in \mathcal{M}$. Moreover, suppose $f(t,x) : \mathbb{R} \times D(B) \mapsto E$ satisfying*

(B1) *For every $u \in \mathcal{M}$, the function $t \mapsto f(t, u(t))$ is in $\mathcal{M}$;*

(B2) *For u and v in $D(B)$ we have*

$$\|f(t,u) - f(t,v)\| \le L\|Bu - Bv\|. \tag{2.3}$$

Then Equation (1.1) *has a unique mild solution in $\mathcal{M}$ for L small enough.*

Proof. Let $K : \mathcal{M} \mapsto \mathcal{M}$ be the operator defined as follows: For each $f \in \mathcal{M}$, Kf is the unique mild solution to (2.1). Then K is a linear and bounded operator on $\mathcal{M}$. For each $u \in \mathcal{M}$ put $\tilde{u}(t) := f(t, B^{-1}u(t))$. Define the map $\tilde{K} : \mathcal{M} \mapsto \mathcal{M}$ by

$$(\tilde{K}u)(t) := B(K\tilde{u})(t).$$

By the assumption, $B(K\tilde{u})(\cdot)$ also belongs to $\mathcal{M}$. Hence $\tilde{K}$ is well defined. If u and v are in $\mathcal{M}$, we have

$$\begin{aligned}
\|(\tilde{K}u)(t) - (\tilde{K}v)(t)\| &= \|B(K\tilde{u})(t) - B(K\tilde{v})(t)\| \\
&= \|B[(K\tilde{u})(t) - (K\tilde{v})(t)]\| \\
&\le C \cdot \sup_{t\in\mathbb{R}} \|f(t, B^{-1}u(t)) - f(t, B^{-1}v(t)\| \\
&\le C \cdot L\|u - v\|.
\end{aligned}$$

Hence $\|\tilde{K}u - \tilde{K}v\| \le C \cdot L\|u - v\|$. So $\tilde{K}$ is a contraction map for sufficiently small L. Let $\phi(t)$ be the unique fixed point of $\tilde{K}$, then it is easy to see that $u(t) = B^{-1}\phi(t)$ is the unique mild solution of (1.1) in $\mathcal{M}$. □

Proof of Theorem 1.1. Since $-A$ generates an analytic semigroup, the spectral mapping theorem holds, i.e., $\sigma(T(t)) = e^{t\sigma(-A)}$ ([4, Corollary III.3.12]). Hence, condition $\{i\lambda : \lambda \in \mathbb{R}\} \subset \varrho(-A)$ implies that $(T(t))$ admits an exponential dichotomy, and hence, space $BC(\mathbb{R})$ is admissible w.r.t. Equation (2.1).

Define the operator $\bar{K} : BC(\mathbb{R}) \mapsto BC(\mathbb{R})$ by follows: for each $f \in BC(\mathbb{R})$, $(\bar{K}f)(t) := A^\alpha u(t)$, where $u(t)$ is the unique solution to (2.1).Then $\bar{K}$ is a linear and, by Lemma 2.2, bounded operator. We now apply Lemma 2.3 with $B = A^\alpha$, and it suffices us to complete the proof by showing that $\bar{K}$ leaves all above mentioned subspaces of $BC(\mathbb{R})$ invariant.

Let $f_t(\cdot) := f(\cdot + t)$ be the left translation of a function f. It is easy to see that $K(f_t) = (Kf)_t$, and this yields $\bar{K}(f_t) = (\bar{K}f)_t$. Hence, $P(\omega)$ and $AP(\mathbb{R})$ are invariant w.r.t. $\bar{K}$. Moreover, $\|(\bar{K}f)_t - \bar{K}f\| = \|\bar{K}f_t - \bar{K}f\| \le \|\bar{K}\| \cdot \|f_t - f\|$, which shows that $BUC(\mathbb{R})$ is also invariant w.r.t. $\bar{K}$. Finally, since $Kf(\pm\infty) = A^{-1}f(\pm\infty)$, we have $\bar{K}f(\pm\infty) = A^{\alpha-1}f(\pm\infty)$, and this proves that $\bar{K}$ leaves C_1 and C_0 invariant. □

References

[1] M. Bahaj, O. Sidki: *Almost periodic solutions of semilinear equations with analytic semigroups in Banach spaces.* Electronic Journal of Differential Equations Electronic Vol. **2002** (2002), No. 98, 1–11.

[2] W. Hutter: *Spectral theory and almost periodicity of mild solutions of nonautonomous Cauchy problems.* Ph. D. Thesis, University of Tübingen, 1998.

[3] Y. Latushkin, T. Randolph, R. Schnaubelt: *Exponential dichotomy and mild solutions of nonautonomous equations in Banach spaces.* J. Dynam. Differential Equations 10, No. 3 (1998), 489–510.

[4] K. Engel, R. Nagel: *One-parameter semigroups for linear evolution equations.* Graduate Texts in Mathematics, Springer-Verlag 2000.

[5] L. Maniar, R. Schnaubelt: *Almost periodicity of inhomogeneous parabolic evolution equations.* in G. Ruiz Goldstein, R. Nagel, S. Romanelli (Eds): “Recent Contributions to Evolution Equations”, Marcel Decker, 2003, 299–318.

[6] Pazy A.: *Semigroups of linear operators and applications to partial differential equations*. Springer–Verlag, Berlin, 1983.
[7] J. Pruss: *On the spectrum of C_0-semigroup*. Trans. Amer. Math. Soc. **284**, 1984, 847–857.
[8] Vu Quoc Phong, E. Schuler: *The operator equation $AX - XB = C$, admissibility and asymptotic behavior of differential equations*. J. Differential Equations **145** (1998), 394–419.

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