

## CONVEXITY OF LEVEL SETS FOR SOLUTIONS TO NONLINEAR ELLIPTIC PROBLEMS IN CONVEX RINGS

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ABSTRACT. We find suitable assumptions for the quasi-concave envelope  $u^*$  of a solution (or a subsolution)  $u$  of an elliptic equation  $F(x, u, \nabla u, D^2u) = 0$  (possibly fully nonlinear) to be a viscosity subsolution of the same equation. We apply this result to study the convexity of level sets of solutions to elliptic Dirichlet problems in a convex ring  $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ .

### 1. INTRODUCTION

The main purpose of this paper is to investigate on conditions which guarantee that, in a Dirichlet problem of elliptic type, relevant geometric properties of the domain are inherited by the level sets of its solutions.

In particular, let  $\Omega = \Omega_0 \setminus \overline{\Omega}_1$  be a convex ring, i.e.  $\Omega_0$  and  $\Omega_1$  are convex, bounded and open subsets of  $\mathbb{R}^n$  such that  $\overline{\Omega}_1 \subset \Omega_0$ ; we consider the Dirichlet problem

$$\begin{aligned} F(x, u, \nabla u, D^2u) &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega_0 \\ u &= 1 && \text{on } \partial\Omega_1, \end{aligned} \tag{1.1}$$

where  $F(x, t, p, A)$  is a real operator acting on  $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n$ , of elliptic type. Here  $\nabla u$  and  $D^2u$  are the gradient and the Hessian matrix of the function  $u$ , respectively, and  $\mathcal{S}_n$  is the set of real symmetric  $n \times n$  matrices.

We prove that, under suitable assumptions on  $F$ , every classical solution of problem (1.1) has convex level sets. This problem has been studied in many papers; we recall, for instance, [1, 4, 5, 7, 8, 12, 15, 16] and the monograph [11] by Kawohl.

The method adopted here is a generalization of the one introduced in [5] and it follows an idea suggested by Kawohl in [11]. It makes use of the *quasi-concave envelope*  $u^*$  of a function  $u$ : roughly speaking,  $u^*$  is the function whose superlevel sets are the convex hulls of the corresponding superlevel sets of  $u$  (we systematically extend  $u \equiv 1$  in  $\Omega_1$ ). We look for conditions that imply  $u = u^*$ . Notice that  $u^* \geq u$  holds by definition (to obtain  $u^*$  we enlarge the superlevel sets of  $u$ ), then it suffices to prove the reverse inequality; the latter can be obtained by a suitable comparison principle, if we prove that  $u^*$  is a viscosity subsolution of problem (1.1). In this way we reduce ourselves to the following question, which has its own interest:

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Can we find suitable assumptions on  $F$  that force  $u^*$  to be a viscosity subsolution of (1.1)?

A positive answer is contained in Theorem 3.1, which is the main result of the present paper. An immediate consequence is Proposition 3.3, which directly applies to operators of the form

$$F(x, u(x), \nabla u(x), D^2u(x)) = L(\nabla u(x), D^2u(x)) - f(x, u(x), \nabla u(x)). \quad (1.2)$$

This paper supplements the results of [5], in which the authors considered only operators whose principal part can be decomposed in a tangential and a normal part (with respect to the level sets of the solution), like the Laplacian, the  $p$ -Laplacian and the mean curvature operator. Here we treat more general operators, including, for instance, Pucci's extremal operators. Moreover, let us mention that the method presented here could be suitable to prove more than the mere convexity of level sets of a solution  $u$ ; indeed, under appropriate boundary behaviour of  $u$  (which we do not determine explicitly in this paper), the same proof of Theorem 3.1 may be used to obtain the  $p$ -concavity of  $u$  for some  $p < 0$  (i.e. the convexity of  $u^p$ ); see Remark 5.1.

Notice that we assume  $|\nabla u| > 0$  in  $\Omega$ , which is a typical assumption for this kind of investigations. Finding geometric properties of level sets of  $u$  without this assumption is partly an open problem; contributions to this question can be found in [11] and [12].

Finally, let us remind that an analogous technique was developed by one of the author in [18] to investigate the starshapedness of level sets of solutions to problem (1.1) when  $\Omega$  is a starshaped ring.

The paper is organized as follows: in §2 we introduce notation and we briefly recall some notions from viscosity theory; in §3 we state the principal result of the paper, Theorem 3.1, and we provide some examples and applications; in §4 we collect some tools which will be used in the proof of Theorem 3.1, which is developed in §5.

## 2. PRELIMINARIES

Let  $n \geq 2$ , for  $x, y \in \mathbb{R}^n$  ( $n$ -dimensional euclidean space) and  $r > 0$ ,  $B(x, r)$  is the euclidean ball of radius  $r$  centered at  $x$ , i.e.

$$B(x, r) = \{z \in \mathbb{R}^n : |z - x| < r\}.$$

With the symbol  $\otimes$  we denote the direct product between vectors in  $\mathbb{R}^n$ , that is, for  $x, y \in \mathbb{R}^n$ ,  $x \otimes y$  is the  $n \times n$  matrix with entries  $(x_i y_j)$  for  $i, j = 1, \dots, n$ .

For a natural number  $m$  and  $a \in \mathbb{R}^m$ , by  $a \geq 0$  ( $> 0$ ) we mean  $a_i \geq 0$  ( $> 0$ ) for  $i = 1, \dots, m$ ; moreover we set

$$\Lambda_m = \{(\lambda_1, \dots, \lambda_m) \in [0, 1]^m : \sum_{i=1}^m \lambda_i = 1\}.$$

For  $A \subset \mathbb{R}^n$ , we denote by  $\bar{A}$  its closure and by  $\partial A$  its boundary.

Throughout the paper  $\Omega_0$  and  $\Omega_1$  will be non-empty, open, convex, bounded subsets of  $\mathbb{R}^n$ , such that  $\bar{\Omega}_1 \subset \Omega_0$ ,  $\Omega$  will denote the convex ring  $\Omega_0 \setminus \bar{\Omega}_1$  and  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  will be a function such that  $u = 0$  on  $\partial\Omega_0$  and  $u = 1$  on  $\partial\Omega_1$ ; we systematically extend  $u \equiv 1$  in  $\Omega_1$ . The gradient and the Hessian matrix of  $u$  are written as  $\nabla u$  and  $D^2u$ , respectively.

Finally,  $\mathcal{S}_n$  is the set of real symmetric  $n \times n$  matrices,  $\mathcal{S}_n^+$  ( $\mathcal{S}_n^{++}$ ) is the subset of  $\mathcal{S}_n$  of positive semidefinite (definite) matrices.

Next we recall few notions from viscosity theory and we refer the reader to [6] for more details.

An operator  $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n \rightarrow \mathbb{R}$  is said *proper* if

$$F(x, s, p, A) \leq F(x, t, p, A) \quad \text{whenever } s \geq t, \quad (2.1)$$

and it is said *strictly proper* if the inequality sign in (2.1) is strict whenever  $s > t$ .

Let  $\Gamma$  be a convex cone in  $\mathcal{S}_n$  with vertex at the origin and containing  $\mathcal{S}_n^{++}$ , then  $F$  is said *degenerate elliptic* in  $\Gamma$  if

$$F(x, t, p, A) \leq F(x, t, p, B), \quad \text{for every } A, B \in \Gamma \text{ such that } A \leq B, \quad (2.2)$$

where  $A \leq B$  means that  $B - A \in \mathcal{S}_n^+$ .

We put  $\Gamma_F = \cup \Gamma$ , where the union is extended to every cone  $\Gamma$  such that  $F$  is degenerate elliptic in  $\Gamma$ ; when we say that  $F$  is degenerate elliptic, we mean that  $F$  is degenerate elliptic in  $\Gamma_F \neq \emptyset$ .

If  $F$  is a degenerate elliptic operator, we say that a function  $u \in C^2(\Omega)$  is *admissible* for  $F$  if  $D^2u(x) \in \Gamma_F$  for every  $x \in \Omega$ . For instance, if  $F$  is the Laplacian, then every  $C^2$  function is admissible for  $F$ ; if  $F$  is the Monge-Ampère operator  $\det(D^2u)$ , then convex functions only are admissible for  $F$ .

Let  $u$  be an upper semicontinuous function and  $\phi$  a continuous function in  $\Omega$  and consider  $x_0 \in \Omega$ : we say that  $\phi$  *touches  $u$  from above at  $x_0$*  if

$$\phi(x_0) = u(x_0) \quad \text{and} \quad \phi(x) \geq u(x) \quad \text{in a neighbourhood of } x_0.$$

Analogously, if  $u$  is lower semicontinuous, we say that  $\phi$  *touches  $u$  from below at  $x_0$*  if

$$\phi(x_0) = u(x_0) \quad \text{and} \quad \phi(x) \leq u(x) \quad \text{in a neighbourhood of } x_0.$$

An upper semicontinuous function  $u$  is a *viscosity subsolution* of the equation  $F = 0$  if, for every  $C^2$  function  $\phi$  touching  $u$  from above at any point  $x \in \Omega$ , it holds

$$F(x, u(x), \nabla\phi(x), D^2\phi(x)) \geq 0.$$

A lower semicontinuous function  $u$  is a *viscosity supersolution* of  $F = 0$  if, for every admissible  $C^2$  function  $\phi$  touching  $u$  from below at any point  $x \in \Omega$ , it holds

$$F(x, u(x), \nabla\phi(x), D^2\phi(x)) \leq 0.$$

A *viscosity solution* is a continuous function which is, at the same time, subsolution and supersolution of  $F = 0$ . In our hypotheses, a *classical solution* is always a viscosity solution and a viscosity solution is a classical solution if it is regular enough.

The technique we use to prove our main result requires the use of the *comparison principle* for viscosity solutions. We say that an operator  $F$  satisfies the comparison principle if the following statement holds:

$$\begin{aligned} \text{Let } u \in C(\bar{\Omega}) \text{ and } v \in C(\bar{\Omega}) \text{ be, respectively, a viscosity supersolution and a} \\ \text{viscosity subsolution of } F = 0 \text{ such that } u \geq v \text{ on } \partial\Omega; \text{ then } u \geq v \text{ in } \bar{\Omega}. \end{aligned} \quad (2.3)$$

The research of conditions which force  $F$  to satisfy a comparison principle is a difficult and current field of investigation (see, for instance, [10, 13, 14]); we consider only operators that satisfy the comparison principle.

## 3. THE MAIN RESULT AND SOME APPLICATIONS

To state our main result, we recall the notion of quasi-concave envelope of a function  $u$  (refer to [5]). Given a convex ring  $\Omega$  and a function  $u \in C(\overline{\Omega})$ , the *quasi-concave envelope* of  $u$  is defined by

$$u^*(x) = \max \left\{ \min\{u(x_1), \dots, u(x_{n+1})\} : x_1, \dots, x_{n+1} \in \overline{\Omega}, \right. \\ \left. x = \sum_{i=1}^{n+1} \lambda_i x_i, \text{ for some } \lambda \in \Lambda_{n+1} \right\}. \quad (3.1)$$

It is almost straightforward that the superlevel sets of  $u^*$  are the convex hulls of the corresponding superlevel sets of  $u$ ; hence  $u^*$  is the smallest quasi-concave function greater or equal than  $u$  (we recall that the convex hull of a set  $A \subseteq \mathbb{R}^n$  is the intersection of all convex subsets of  $\mathbb{R}^n$  containing  $A$  and that a real function  $u$  is said *quasi-concave* if its superlevel sets are all convex).

**Theorem 3.1.** *Let  $\Omega = \Omega_0 \setminus \overline{\Omega}_1$  be a convex ring and let  $F(x, u, \theta, A)$  be a proper, continuous and degenerate elliptic operator in  $\Omega \times (0, 1) \times \mathbb{R}^n \times \Gamma_F$ . Assume that there exists  $\tilde{p} < 0$  such that, for every  $p \leq \tilde{p}$  and for every  $\theta \in \mathbb{R}^n$ , the application*

$$(x, t, A) \rightarrow F(x, t^{\frac{1}{p}}, t^{\frac{1}{p}-1}\theta, t^{\frac{1}{p}-3}A) \text{ is concave in } \Omega \times (1, +\infty) \times \Gamma_F. \quad (3.2)$$

*If  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  is an admissible classical solution of (1.1) such that  $|\nabla u| > 0$  in  $\Omega$ , then  $u^*$  is a viscosity subsolution of (1.1).*

The proof of the above theorem is contained in §5.

A direct consequence of Theorem 3.1 is the following criterion which immediately applies to problem (1.1).

**Proposition 3.2.** *Under the hypothesis of Theorem 3.1, if a viscosity comparison principle holds for  $F$ , then all the superlevel sets of  $u$  are convex (once we extend  $u \equiv 1$  in  $\Omega_1$ ).*

*Proof.* Indeed, Theorem 3.1 and the comparison principle ensure that

$$u^* \leq u \quad \text{in } \Omega.$$

The reverse inequality follows from the definition of  $u^*$ , hence  $u = u^*$ . □

In the following proposition we rewrite explicitly a particular case of Theorem 3.1, which directly applies to some interesting problems.

**Proposition 3.3.** *Assume that  $f(x, u, \theta)$  is a continuous function in  $\Omega \times (0, 1) \times \mathbb{R}^n$ , non-decreasing in  $u$ , and that  $L(\theta, A)$  is a continuous elliptic operator, concave with respect to  $A$ . Moreover, assume that there exist  $\alpha, \beta \in \mathbb{R}$  such that*

$$L(r\theta, A) \geq r^\alpha L(\theta, A), \quad (3.3)$$

$$L(\theta, sA) \geq s^\beta L(\theta, A), \quad (3.4)$$

for every  $r, s > 0$  and  $(\theta, A) \in \mathbb{R}^n \times \Gamma_L$ .

Let  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  be an admissible classical solution of

$$L(\nabla u(x), D^2 u(x)) = f(x, u(x), \nabla u(x)) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega_0 \\ u = 1 \quad \text{on } \partial\Omega_1, \quad (3.5)$$

such that  $|\nabla u| > 0$  in  $\Omega$ .

If there exists  $\tilde{p} < 0$  such that, for every  $p \leq \tilde{p}$  and for every fixed  $\theta \in \mathbb{R}^n$ , the application

$$t^{(1-\frac{1}{p})\alpha+(3-\frac{1}{p})\beta} f(x, t^{\frac{1}{p}}, t^{\frac{1}{p}-1}\theta) \tag{3.6}$$

is convex with respect to  $(x, t) \in \Omega \times (1, +\infty)$ , then  $u^*$  is a viscosity subsolution of (3.5).

The above proposition is only a particular case of Theorem 3.1.

Examples of this kind are the Laplace operator ( $\alpha = 0, \beta = 1$ ), the  $q$ -Laplace operator ( $\alpha = q - 2, \beta = 1$ ) and the mean curvature operator ( $\alpha = 0, \beta = 1$ ). These operators, whose principal part can be naturally decomposed in a tangential and normal part with respect to the level sets of the solution, have been already treated in [5]. There, convexity for superlevel sets of solutions of (3.5), in the just mentioned cases, is proved under the assumption  $t^{\alpha+3\beta} f(x, u, \frac{\theta}{t})$  is convex with respect to  $(x, t)$  for every  $(u, \theta) \in (0, 1) \times \mathbb{R}^n$ .

Notice that letting  $p \rightarrow -\infty$ , (3.6) yields  $t^{\alpha+3\beta} f(x, 1, \frac{\theta}{t})$  being convex with respect to  $(x, t)$ .

Other examples of operators, which our results apply to, are for instance Pucci's extremal operators. For sake of completeness, we briefly recall the definitions and main properties of these operators.

Pucci's extremal operators were introduced by Pucci in [17] and they are perturbations of the usual Laplacian. Given two numbers  $0 < \lambda \leq \Lambda$  and a real symmetric  $n \times n$  matrix  $M$ , whose eigenvalues are  $e_i = e_i(M)$ , for  $i = 1, \dots, n$ , the Pucci's extremal operators are

$$\mathcal{M}_{\lambda, \Lambda}^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i \tag{3.7}$$

and

$$\mathcal{M}_{\lambda, \Lambda}^-(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i. \tag{3.8}$$

We observe that  $\mathcal{M}^+$  and  $\mathcal{M}^-$  are uniformly elliptic, with ellipticity constant  $\lambda$  and  $n\Lambda$  and they are positively homogeneous of degree 1; moreover  $\mathcal{M}^-$  is concave and  $\mathcal{M}^+$  is convex with respect to  $M$  (see [3], for instance).

#### 4. THE $(p, \lambda)$ -ENVELOPE OF A FUNCTION

Before proving Theorem 3.1, we need some preliminary definitions and results. First of all we recall the notion of  $p$ -means; for more details we refer to [9].

Given  $a = (a_1, \dots, a_m) > 0$ ,  $\lambda \in \Lambda_m$  and  $p \in [-\infty, +\infty]$ , the quantity

$$M_p(a, \lambda) = \begin{cases} [\lambda_1 a_1^p + \lambda_2 a_2^p + \dots + \lambda_m a_m^p]^{1/p} & \text{for } p \neq -\infty, 0, +\infty \\ \max\{a_1, \dots, a_m\} & p = +\infty \\ a_1^{\lambda_1} \dots a_m^{\lambda_m} & p = 0 \\ \min\{a_1, a_2, \dots, a_m\} & p = -\infty \end{cases} \tag{4.1}$$

is the  $p$ -(weighted) mean of  $a$ .

For  $a \geq 0$ , we define  $M_p(a, \lambda)$  as above if  $p \geq 0$  and we set  $M_p(a, \lambda) = 0$  if  $p < 0$  and  $a_i = 0$ , for some  $i \in \{1, \dots, m\}$ .

A simple consequence of Jensen's inequality is that, for a fixed  $0 \leq a \in \mathbb{R}^m$  and  $\lambda \in \Lambda_m$ ,

$$M_p(a, \lambda) \leq M_q(a, \lambda) \quad \text{if } p \leq q. \quad (4.2)$$

Moreover, it is easily seen that

$$\lim_{p \rightarrow +\infty} M_p(a, \lambda) = \max\{a_1, \dots, a_m\} \quad (4.3)$$

and

$$\lim_{p \rightarrow -\infty} M_p(a, \lambda) = \min\{a_1, \dots, a_m\}. \quad (4.4)$$

Let us fix  $\lambda \in \Lambda_{n+1}$  and consider  $p \in [-\infty, +\infty]$ .

**Definition 4.1.** Given a convex ring  $\Omega = \Omega_0 \setminus \overline{\Omega}_1$  and  $u \in C(\overline{\Omega})$ , the  $(p, \lambda)$ -envelope of  $u$  is the function  $u_{p,\lambda} : \overline{\Omega} \rightarrow \mathbb{R}_+$  defined as follows

$$\begin{aligned} &u_{p,\lambda}(x) \\ &= \sup\{M_p(u(x_1), \dots, u(x_{n+1}), \lambda) : x_i \in \overline{\Omega}, i = 1, \dots, n+1, x = \sum_{i=1}^{n+1} \lambda_i x_i\}. \end{aligned} \quad (4.5)$$

For convenience, we will refer to  $u_{-\infty,\lambda}$  as  $u_\lambda^*$ .

Notice that, since  $\overline{\Omega}$  is compact and  $M_p$  is continuous, the supremum of the definition is in fact a maximum. Hence, for every  $x \in \overline{\Omega}$ , there exist  $(x_{1,p}, \dots, x_{n+1,p}) \in \overline{\Omega}^{n+1}$  such that

$$x = \sum_{i=1}^{n+1} \lambda_i x_{i,p}, \quad u_{p,\lambda}(x) = \left( \sum_{i=1}^{n+1} \lambda_i u(x_{i,p})^p \right)^{1/p}. \quad (4.6)$$

An immediate consequence of the definition is that

$$u_{p,\lambda}(x) \geq u(x), \quad \forall x \in \overline{\Omega}, \quad p \in [-\infty, +\infty]; \quad (4.7)$$

moreover, from (4.2), we have

$$u_{p,\lambda}(x) \leq u_{q,\lambda}(x), \quad \text{for } p \leq q, \quad x \in \Omega. \quad (4.8)$$

For the rest of this article, we restrict ourselves to the case  $p \in [-\infty, 0)$  and we collect in the following lemmas some helpful properties of  $u_{p,\lambda}$  and  $u_\lambda^*$ .

**Lemma 4.1.** Let  $p \in (-\infty, 0)$  and  $\lambda \in \Lambda_{n+1}$ ; given a convex ring  $\Omega = \Omega_0 \setminus \overline{\Omega}_1$  and a function  $u \in C(\overline{\Omega})$  such that  $u = 0$  on  $\partial\Omega_0$ ,  $u = 1$  on  $\partial\Omega_1$  and  $u \in (0, 1)$  in  $\Omega$ , then  $u_{p,\lambda} \in C(\overline{\Omega})$  and

$$u_{p,\lambda} \in (0, 1) \quad \text{in } \Omega, \quad u_{p,\lambda} = 0 \quad \text{on } \partial\Omega_0, \quad u_{p,\lambda} = 1 \quad \text{on } \partial\Omega_1. \quad (4.9)$$

*Proof.* The proof of (4.9) is almost straightforward. For the continuity of  $u_{p,\lambda}$  in  $\Omega$ ,

$$\begin{aligned} &u_{p,\lambda}^p(x) \\ &= \min \left\{ \lambda_1 u(x_1)^p + \dots + \lambda_{n+1} u(x_{n+1})^p : x_i \in \overline{\Omega}, i = 1, \dots, n+1, x = \sum_{i=1}^{n+1} \lambda_i x_i \right\} \end{aligned}$$

is the infimal convolution of  $u^p$  with itself for  $(n+1)$  times; then refer to [19, Corollary 2.1] to conclude that  $u_{p,\lambda}^p \in C(\Omega)$ . Hence  $u_{p,\lambda} \in C(\Omega)$ , since  $u_{p,\lambda} > 0$  in  $\Omega$ ; then (4.1) and (4.9) easily yield continuity up to the boundary of  $\Omega$ .  $\square$

**Remark 4.1.** If  $u$  is a function satisfying the hypotheses of the previous lemma and if we consider  $x \in \Omega$ , by (4.6) and (4.7), we get

$$x_{i,p} \notin \partial\Omega_0, \quad \text{for } i = 1, \dots, n+1,$$

otherwise it should be  $u_{p,\lambda}(x) = 0$  by definition of  $p$ -means.

**Lemma 4.2.** Let  $\lambda \in \lambda_{n+1}$ ; given a convex ring  $\Omega = \Omega_0 \setminus \overline{\Omega_1}$  and a function  $u \in C^1(\Omega) \cap C(\overline{\Omega})$  such that  $u = 0$  on  $\partial\Omega_0$ ,  $u = 1$  on  $\partial\Omega_1$  and  $|\nabla u| > 0$  in  $\Omega$ , then  $u_\lambda^* \in C(\overline{\Omega})$ ,

$$u_\lambda^* = 0 \quad \text{on } \partial\Omega_0, \quad u_\lambda^* = 1 \quad \text{on } \partial\Omega_1, \quad u_\lambda^* \in (0, 1) \quad \text{in } \Omega.$$

Moreover, for every  $x \in \Omega$ , there exist  $x_1, \dots, x_{n+1} \in \Omega$  such that

$$x = \sum_{i=1}^{n+1} \lambda_i x_i, \quad u_\lambda^*(x) = u(x_1) = \dots = u(x_{n+1}). \quad (4.10)$$

*Proof.* The hypothesis  $|\nabla u| > 0$  in  $\Omega$  guarantees that  $u \in (0, 1)$  in  $\Omega$ ; then we notice that the superlevel sets  $\Omega_{t,\lambda}^* = \{x \in \Omega : u_\lambda^*(x) \geq t\}$  of  $u_\lambda^*$  are characterized by

$$\Omega_{t,\lambda}^* = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i : x_i \in \Omega_t, i = 1, \dots, n+1 \right\},$$

where  $\Omega_t = \{u \geq t\}$ . Then, we can argue exactly as in [5, Section 2 and 3] where the same is proved for the quasi-concave envelope  $u^*$  of  $u$  (see also [2] and [16]).  $\square$

**Remark 4.2.** It is not hard to see that (4.10) holds for every  $(x_1, \dots, x_{n+1})$  realizing the maximum in (4.5), for  $p = -\infty$ .

**Remark 4.3.** It holds

$$u^*(x) = \sup \{u_\lambda^*(x) : \lambda \in \Lambda_{n+1}\}. \quad (4.11)$$

and the sup above is in fact a maximum as  $\Lambda_{n+1}$  is compact.

For further convenience, we also set

$$u_p(x) = \sup \{u_{p,\lambda}(x) : \lambda \in \Lambda_{n+1}\}$$

and we notice that the above supremum is in fact a maximum and that  $u_p$  is the smallest  $p$ -concave function greater or equal to  $u$ . We recall that, for  $p \neq 0$ , a non-negative function  $u$  is said  $p$ -concave if  $\frac{p}{|p|}u^p$  is concave ( $u$  is called *log-concave* if  $\log u$  is concave, which corresponds to the case  $p = 0$ ).

**Theorem 4.3.** Under the assumptions of Lemma 4.1, we have

$$u_{p,\lambda} \rightarrow u_\lambda^* \quad \text{uniformly in } \overline{\Omega}. \quad (4.12)$$

*Proof.* The function  $u_{p,\lambda} - u_\lambda^* \geq 0$  in  $\overline{\Omega}$ , since it is continuous in  $\overline{\Omega}$ , then it admits maximum in  $\overline{\Omega}$ . Let  $\bar{x}_p \in \overline{\Omega}$  such that

$$u_{p,\lambda}(\bar{x}_p) - u_\lambda^*(\bar{x}_p) = \max_{x \in \overline{\Omega}} |u_{p,\lambda}(x) - u_\lambda^*(x)|.$$

To get (4.12) it suffices to prove that

$$u_{p,\lambda}(\bar{x}_p) - u_\lambda^*(\bar{x}_p) \rightarrow 0, \quad \text{for } p \rightarrow -\infty.$$

For every  $p < 0$ , let us consider the points  $x_{1,p}, \dots, x_{n+1,p} \in \overline{\Omega}$ , given by (4.6), such that

$$\bar{x}_p = \sum_{i=1}^{n+1} \lambda_i x_{i,p}, \quad u_{p,\lambda}(\bar{x}_p) = \left[ \sum_{i=1}^{n+1} \lambda_i u(x_{i,p})^p \right]^{1/p}. \quad (4.13)$$

For every negative number  $q > p$ , by (4.2) and the definition of  $u_\lambda^*$ , we have

$$\begin{aligned} u_{p,\lambda}(\bar{x}_p) - u_\lambda^*(\bar{x}_p) &= \left[ \sum_{i=1}^{n+1} \lambda_i u(x_{i,p})^p \right]^{1/p} - u_\lambda^*(\bar{x}_p) \\ &\leq \left[ \sum_{i=1}^{n+1} \lambda_i u(x_{i,p})^q \right]^{1/q} - \min \{u(x_{1,p}), \dots, u(x_{n+1,p})\}. \end{aligned}$$

Since  $\overline{\Omega}$  is closed, it follows that  $x_{i,p} \rightarrow \bar{x}_i \in \overline{\Omega}$  (up to subsequences), for  $i = 1, \dots, n+1$ . Then, letting  $p \rightarrow -\infty$  we get

$$\lim_{p \rightarrow -\infty} (u_{p,\lambda}(\bar{x}_p) - u_\lambda^*(\bar{x}_p)) \leq \left[ \sum_{i=1}^{n+1} \lambda_i u(\bar{x}_i)^q \right]^{1/q} - \min \{u(\bar{x}_1), \dots, u(\bar{x}_{n+1})\}. \quad (4.14)$$

The thesis follows passing to the limit for  $q \rightarrow -\infty$  and by (4.4).  $\square$

## 5. PROOF OF MAIN THEOREM

Let  $u$  and  $\Omega$  be as in the statement of Theorem 3.1. First, we fix  $\lambda \in \Lambda_{n+1}$  and  $p < 0$  and we prove that, for every  $\bar{x} \in \Omega$ , there exists a  $C^2$  function  $\varphi_{p,\lambda}$  which touches the  $(p, \lambda)$ -envelope  $u_{p,\lambda}$  of  $u$  from below at  $\bar{x}$  and such that

$$F(\bar{x}, u_{p,\lambda}(\bar{x}), \nabla \varphi_{p,\lambda}(\bar{x}), D^2 \varphi_{p,\lambda}(\bar{x})) \geq 0. \quad (5.1)$$

Clearly this implies that  $u_{p,\lambda}$  is a viscosity subsolution of (1.1); then, by Theorem 4.3 and the fact that viscosity subsolutions pass to the limit under uniform convergence on compact sets, it follows that  $u_\lambda^*$  is a viscosity subsolution of (1.1) too.

Then, as  $u^*(x)$  is the supremum (with respect to  $\lambda \in \Lambda_{n+1}$ ) of  $u_\lambda^*(x)$ , by [6, Lemma 4.2] we conclude that also  $u^*$  is a viscosity subsolution of (1.1).

Let us consider  $\bar{x} \in \Omega$ . By (4.6) and Remark 4.1, there exist  $x_{1,p}, \dots, x_{n+1,p} \in \overline{\Omega} \setminus \partial\Omega_0$  such that

$$\bar{x} = \lambda_1 x_{1,p} + \dots + \lambda_{n+1} x_{n+1,p}, \quad u_{p,\lambda}(\bar{x})^p = \lambda_1 u(x_{1,p})^p + \dots + \lambda_{n+1} u(x_{n+1,p})^p. \quad (5.2)$$

We suppose, for the moment, that  $x_{i,p} \in \Omega$ , for  $i = 1, \dots, n+1$ . In this case, by the Lagrange Multipliers Theorem, we have

$$\nabla[u(x_{1,p})^p] = \dots = \nabla[u(x_{n+1,p})^p]. \quad (5.3)$$

We introduce a new function  $\varphi_{p,\lambda} : B(\bar{x}, r) \rightarrow \mathbb{R}$ , for a small enough  $r > 0$ , defined as follows:

$$\varphi_{p,\lambda}(x) = [\lambda_1 u(x_{1,p} + a_{1,p}(x - \bar{x}))^p + \dots + \lambda_{n+1} u(x_{n+1,p} + a_{n+1,p}(x - \bar{x}))^p]^{1/p}, \quad (5.4)$$

where

$$a_{i,p} = \frac{u(x_{i,p})^p}{u_{p,\lambda}(\bar{x})^p}, \quad \text{for } i = 1, \dots, n+1. \quad (5.5)$$

The following facts trivially hold:

$$(1) \sum_{i=1}^{n+1} \lambda_i a_i = 1;$$



- (2)  $x = \sum_{i=1}^{n+1} \lambda_i (x_{i,p} + a_{i,p}(x - \bar{x}))$ , for every  $x \in B(\bar{x}, r)$ ;  
 (3)  $\varphi_{p,\lambda}(\bar{x}) = u_{p,\lambda}(\bar{x})$ ;  
 (4)  $\varphi_{p,\lambda}(x) \leq u_{p,\lambda}(x)$  in  $B(\bar{x}, r)$  (this follows from 2 and from the definition of  $u_{p,\lambda}$ ).

In particular, 3 and 4 mean that  $\varphi_{p,\lambda}$  touches from below  $u_{p,\lambda}$  at  $\bar{x}$ . A straightforward calculation yields

$$\begin{aligned} \nabla \varphi_{p,\lambda}(\bar{x}) &= \varphi_{p,\lambda}(\bar{x})^{1-p} [\lambda_1 u(x_{1,p})^{p-1} a_{1,p} \nabla u(x_{1,p}) + \dots \\ &\quad + \lambda_{n+1} u(x_{n+1,p})^{p-1} a_{n+1,p} \nabla u(x_{n+1,p})] \\ &= \varphi_{p,\lambda}(\bar{x})^{1-p} \sum_{i=1}^{n+1} \lambda_i u(x_{i,p})^{p-1} \frac{u(x_{i,p})^p}{\varphi_{p,\lambda}(\bar{x})^p} \nabla u(x_{i,p}). \end{aligned}$$

Then, by (5.3) and the definition of  $\varphi_{p,\lambda}$ , we have

$$\begin{aligned} \nabla \varphi_{p,\lambda}(\bar{x}) &= \varphi_{p,\lambda}(\bar{x})^{1-p} u(x_{i,p})^{p-1} \nabla u(x_{i,p}) \sum_{i=1}^{n+1} \lambda_i \frac{u(x_{i,p})^p}{\varphi_{p,\lambda}(\bar{x})^p} \\ &= \varphi_{p,\lambda}(\bar{x})^{1-p} u(x_{i,p})^{p-1} \nabla u(x_{i,p}) \quad i = 1, \dots, n+1. \end{aligned} \quad (5.6)$$

Moreover,

$$\begin{aligned} D^2 \varphi_{p,\lambda}(\bar{x}) &= (1-p) \varphi_{p,\lambda}(\bar{x})^{-1} \nabla \varphi_{p,\lambda}(\bar{x}) \otimes \nabla \varphi_{p,\lambda}(\bar{x}) \\ &\quad - (1-p) \varphi_{p,\lambda}(\bar{x})^{1-p} \sum_{i=1}^{n+1} \lambda_i u(x_{i,p})^{p-2} a_{i,p}^2 \nabla u(x_{i,p}) \otimes \nabla u(x_{i,p}) \\ &\quad + \varphi_{p,\lambda}(\bar{x})^{1-p} \sum_{i=1}^{n+1} \lambda_i u(x_{i,p})^{p-1} a_{i,p}^2 D^2 u(x_{i,p}). \end{aligned} \quad (5.7)$$

Taking in to account (5.6) and (5.5), we obtain

$$\begin{aligned} D^2 \varphi_{p,\lambda}(\bar{x}) &= \sum_{i=1}^{n+1} \lambda_i \frac{u(x_{i,p})^{3p-1}}{\varphi_{p,\lambda}(\bar{x})^{3p-1}} D^2 u(x_{i,p}) + (1-p) \varphi_{p,\lambda}(\bar{x})^{-1} \nabla \varphi_{p,\lambda}(\bar{x}) \\ &\quad \otimes \nabla \varphi_{p,\lambda}(\bar{x}) \left[ 1 - \varphi_{p,\lambda}(\bar{x})^{-p} \sum_{i=1}^{n+1} \lambda_i u(x_{i,p})^p \right]. \end{aligned}$$

The quantity in square brackets is equal to 0 by the definition of  $\varphi_{p,\lambda}$ . Then

$$D^2 \varphi_{p,\lambda}(\bar{x}) = \sum_{i=1}^{n+1} \lambda_i \frac{u(x_{i,p})^{3p-1}}{\varphi_{p,\lambda}(\bar{x})^{3p-1}} D^2 u(x_{i,p}). \quad (5.8)$$

Thanks to (5.6) and (5.8), for  $p \leq \tilde{p}$ , applying assumption (3.2), we get

$$\begin{aligned} & F\left(\bar{x}, u_{p,\lambda}(\bar{x}), \nabla\varphi_{p,\lambda}(\bar{x}), D^2(\varphi_{p,\lambda}(\bar{x}))\right) \\ &= F\left(\bar{x}, [u_{p,\lambda}(\bar{x})]^{\frac{1}{p}}, [\varphi_{p,\lambda}(\bar{x})]^{\frac{1}{p}-1} \varphi_{p,\lambda}(\bar{x})^{p-1} \nabla\varphi_{p,\lambda}(\bar{x}), \right. \\ &\quad \left. [\varphi_{p,\lambda}(\bar{x})]^{\frac{1}{p}-3} \varphi_{p,\lambda}(\bar{x})^{3p-1} D^2(\varphi_{p,\lambda}(\bar{x}))\right) \\ &\geq \sum_{i=1}^n \lambda_i F\left(x_{i,p}, [u(x_{i,p})]^{\frac{1}{p}}, [u(x_{i,p})]^{\frac{1}{p}-1} \varphi_{p,\lambda}(\bar{x})^{p-1} \nabla\varphi_{p,\lambda}(\bar{x}), D^2u(x_{i,p})\right) \\ &= \sum_{i=1}^n \lambda_i F(x_{i,p}, u(x_{i,p}), \nabla u(x_{i,p}), D^2u(x_{i,p})) = 0 \end{aligned}$$

since  $u$  is a classical solution of  $F = 0$ . Then (5.1) is proved for every  $\bar{x} \in \Omega$  such that the points  $x_{1,p}, x_{2,p}, \dots, x_{n+1,p}$  determined by (5.2) are contained in  $\Omega$ .

In order to conclude our proof, we prove the following lemma.

**Lemma 5.1.** *Under the assumptions of Theorem 3.1, for every compact  $K \subset \Omega$ , there exists  $\bar{p} = \bar{p}(K) < 0$  such that, if  $p \leq \bar{p}$  and  $x \in K$ , the points  $x_{i,p}$ ,  $i = 1, \dots, n+1$ , given by (5.2) are all contained in  $\Omega$ .*

*Proof.* Let  $x \in K$ : the points  $x_{i,p}$ ,  $i = 1, \dots, n+1$ , determined by (5.2) are in  $\bar{\Omega} \setminus \partial\Omega_0$ , by Remark 4.1. Hence we have only to prove that no one of them belongs to  $\partial\Omega_1$ .

We argue by contradiction. We suppose that there exist two sequences  $\{p_m\} \subseteq (-\infty, 0)$  and  $\{\xi_m\} \subseteq K$  such that  $p_m \rightarrow -\infty$  and

$$u_{p_m,\lambda}(\xi_m) > M_{p_m}(u(y_1), \dots, u(y_{n+1}), \lambda)$$

for every  $(y_1, \dots, y_{n+1}) \in \Omega^{n+1}$  such that  $\xi_m = \sum_{i=1}^{n+1} \lambda_i y_i$ . Then

$$u_{p_m,\lambda}(\xi_m) = M_{p_m}(u(\bar{x}_{1,p_m}), \dots, u(\bar{x}_{n+1,p_m}), \lambda),$$

with  $\bar{x}_{i,p_m} \in \partial\Omega_1$ , for some  $i = 1, \dots, n+1$  and  $\xi_m = \sum_{i=1}^{n+1} \lambda_i \bar{x}_{i,p_m}$ . Without leading the generality of the proof, we may suppose that

$$\bar{x}_{1,p_m} \in \partial\Omega_1, \quad \text{for every } m \in \mathbb{N}.$$

The following facts hold for  $m \rightarrow +\infty$ , up to subsequences:

- (1)  $\xi_m \rightarrow x \in K$ ,
- (2)  $\bar{x}_{1,p_m} \rightarrow \bar{x}_1 \in \partial\Omega_1$ ,
- (3)  $\bar{x}_{2,p_m} \rightarrow \bar{x}_2 \in \bar{\Omega}, \dots, \bar{x}_{n+1,p_m} \rightarrow \bar{x}_{n+1} \in \bar{\Omega}$ ,
- (4)  $x = \sum_{i=1}^{n+1} \lambda_i \bar{x}_i$ .

Collecting all these information, by (4.2), for  $p_m < q < 0$  we get

$$\begin{aligned} u_{p_m,\lambda}(\xi_m) &= M_{p_m}(u(\bar{x}_{1,p_m}), \dots, u(\bar{x}_{n+1,p_m}), \lambda) \\ &\leq M_q(u(\bar{x}_{1,p_m}), \dots, u(\bar{x}_{n+1,p_m}), \lambda). \end{aligned}$$

If we let  $m \rightarrow +\infty$ , by Theorem 4.3 and the continuity of  $u$ ,  $u_{p,\lambda}$  and  $M_p$ , we obtain

$$u_\lambda^*(x) \leq M_q(u(\bar{x}_1), \dots, u(\bar{x}_{n+1}), \lambda).$$

Now let  $q \rightarrow -\infty$ , then

$$u_\lambda^*(x) \leq \min\{u(\bar{x}_1), \dots, u(\bar{x}_{n+1})\}.$$

In particular, by definition of  $u_\lambda^*$ , it has to be

$$u_\lambda^*(x) = \min\{u(\bar{x}_1), \dots, u(\bar{x}_{n+1})\}, \quad \text{with } \bar{x}_1 \in \partial\Omega_1.$$

This contradicts Lemma 4.2 and Remark 4.2.  $\square$

Finally, we obtained that  $u_{p,\lambda}$  is a viscosity subsolution of (1.1) for every compact subset  $K$  of  $\Omega$ , for  $p \leq \min\{\tilde{p}, \bar{p}\}$ . The arbitrariness of  $K$  ensures that  $u_{p,\lambda}$  is a viscosity subsolution of (1.1) in the whole  $\Omega$ . Proof is now complete.

**Remark 5.1.** If, for some  $p \in \mathbb{R}$ , we had that, for every  $\lambda \in \Lambda_{n+1}$  and for every  $x \in \Omega$ , the points  $x_{i,p}$ ,  $i = 1, \dots, n+1$ , given by (5.2), are all inside  $\Omega$ , then we would obtain that  $u_{p,\lambda}$  is a subsolution of (1.1). Hence  $u_p(x)$  is a subsolution and finally, by the comparison principle, it holds  $u \equiv u_p$ , which means that  $u$  is  $p$ -concave (that is more than saying that it is quasi-concave).

Notice that we already know that  $x_{i,p} \notin \partial\Omega_0$  for  $i = 1, \dots, n+1$  (see Remark 4.1); hence, to prove  $p$ -concavity of  $u$ , one has only to find conditions which rule out the chance that  $x_{i,p} \in \partial\Omega_1$  for some  $i \in \{1, \dots, n+1\}$ .

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