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A SEMILINEAR ELLIPTIC PROBLEM INVOLVING NONLINEAR BOUNDARY CONDITION AND SIGN-CHANGING POTENTIAL

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Abstract. In this paper, we study the multiplicity of nontrivial nonnegative solutions for a semilinear elliptic equation involving nonlinear boundary condition and sign-changing potential. With the help of the Nehari manifold, we prove that the semilinear elliptic equation:

$$-\Delta u + u = \lambda f(x)|u|^{q-2}u \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = g(x)|u|^{p-2}u \quad \text{on } \partial\Omega,$$

has at least two nontrivial nonnegative solutions for λ is sufficiently small.

1. Introduction

In this paper, we consider the multiplicity of nontrivial nonnegative solutions for the following semilinear elliptic equation

$$\begin{aligned} -\Delta u + u &= \lambda f(x)|u|^{q-2}u \quad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} &= g(x)|u|^{p-2}u \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $1 < q < 2 < p < \frac{2(N-1)}{N-2}$, $\lambda > 0$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\frac{\partial}{\partial\nu}$ is the outer normal derivative and $f, g : \overline{\Omega} \to \mathbb{R}$ are continuous functions which change sign in $\overline{\Omega}$. Associated with (1.1), we consider the energy functional J_λ in $H^1(\Omega)$,

$$J_\lambda(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{\lambda}{q}\int_\Omega f|u|^q dx - \frac{1}{p}\int_{\partial\Omega} g|u|^p ds.$$

where ds is the measure on the boundary and $\|u\|_{H^1}^2 = \int_\Omega |\nabla u|^2 + u^2 dx$. It is well known that J_λ is of C^1 in $H^1(\Omega)$ and the solutions of equation (1.1) are the critical points of the energy functional J_λ.

The fact that the number of solutions of equation (1.1) is affected by the nonlinear boundary conditions has been the focus of a great deal of research in recent years. Garcia-Azorero, Peral and Rossi [10] have investigated (1.1) when $f \equiv g \equiv 1$.

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They found that there exist positive numbers Λ_1, Λ_2 with $\Lambda_1 \leq \Lambda_2$ such that equation (1.1) admits at least two positive solutions for $\lambda \in (0, \Lambda_1)$ and no positive solution exists for $\lambda > \Lambda_2$. Also see Chipot-Chlebik-Fila-Shafrir [4], Chipot-Shafrir-Fila [5], Flores-del Pino [8], Hu [11], Pierrotti-Terracini [14] and Terraccini [16] where problems similar to equation (1.1) have been studied.

The purpose of this paper is to consider the multiplicity of nontrivial nonnegative solutions of equation (1.1) with sign-changing potential. We prove that equation (1.1) has at least two nontrivial nonnegative solutions for λ is sufficiently small.

Theorem 1.1. *There exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, equation* (1.1) *has at least two nontrivial nonnegative solutions.*

Among the other interesting problems which are similar of equation (1.1), Ambrosetti-Brezis-Cerami [3] have investigated the equation

$$\begin{gathered} -\Delta u = \lambda |u|^{q-2}u + |u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $1 < q < 2 < p \leq \frac{2N}{N-2}$. They proved that there exists $\lambda_0 > 0$ such that (1.2) admits at least two positive solutions for $\lambda \in (0, \lambda_0)$, has a positive solution for $\lambda = \lambda_0$, and no positive solution for $\lambda > \lambda_0$. Actually, Adimurthi-Pacella-Yadava [1], Damascelli-Grossi-Pacella [6], Ouyang-Shi [13] and Tang [17] proved that there exists $\lambda_0 > 0$ such that equation (1.2) in the unit ball $B^N(0;1)$ has exactly two positive solutions for $\lambda \in (0, \lambda_0)$, has exactly one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Generalizations of the result of equation (1.2) were done by Ambrosetti-Azorero-Peral [2], de Figueiredo-Gossez-Ubilla [9] and Wu [18].

This paper is organized as follows. In section 2, we give some notation and preliminaries. In section 3, we prove that (1.1) has at least two nontrivial nonnegative solutions for λ is sufficiently small.

2. Notation and Preliminaries

Throughout this section, we denote by S_p, C_p the best Sobolev embedding and trace constant for the operators $H^1(\Omega) \hookrightarrow L^p(\Omega)$, $H^1(\Omega) \hookrightarrow L^p(\partial\Omega)$, respectively. Now, we consider the Nehari minimization problem: For $\lambda > 0$,

$$\alpha_\lambda = \inf\{J_\lambda(u) : u \in \mathbf{M}_\lambda\},$$

where $\mathbf{M}_\lambda = \{u \in H^1(\Omega)\backslash\{0\} : \langle J'_\lambda(u), u\rangle = 0\}$. Define

$$\psi_\lambda(u) = \langle J'_\lambda(u), u\rangle = \|u\|^2_{H^1} - \lambda \int_\Omega f|u|^q dx - \int_{\partial\Omega} g|u|^p ds.$$

Then for $u \in \mathbf{M}_\lambda$,

$$\langle \psi'_\lambda(u), u\rangle = 2\|u\|^2_{H^1} - \lambda q \int_\Omega f|u|^q dx - p\int_{\partial\Omega} g|u|^p ds.$$

Similarly to the method used in Tarantello [15], we split $\mathbf{M}_\lambda$ into three parts:

$$\begin{aligned} \mathbf{M}^+_\lambda &= \{u \in \mathbf{M}_\lambda : \langle \psi'_\lambda(u), u\rangle > 0\}, \\ \mathbf{M}^0_\lambda &= \{u \in \mathbf{M}_\lambda : \langle \psi'_\lambda(u), u\rangle = 0\}, \\ \mathbf{M}^-_\lambda &= \{u \in \mathbf{M}_\lambda : \langle \psi'_\lambda(u), u\rangle < 0\}. \end{aligned}$$

Then, we have the following results.

Lemma 2.1. *There exists $\lambda_1 > 0$ such that for each $\lambda \in (0, \lambda_1)$ we have $\mathbf{M}_\lambda^0 = \phi$.*

Proof. We consider the following two cases.
Case (I): $u \in \mathbf{M}_\lambda$ and $\int_{\partial\Omega} g|u|^p ds \le 0$. We have

$$\lambda \int_\Omega f|u|^q dx = \|u\|_{H^1}^2 - \int_{\partial\Omega} g|u|^p ds.$$

Thus,

$$\begin{aligned}\langle \psi_\lambda'(u), u\rangle &= 2\|u\|_{H^1}^2 - \lambda q \int_\Omega f|u|^q dx - p\int_{\partial\Omega} g|u|^p ds \\ &= (2-q)\|u\|_{H^1}^2 + (q-p)\int_{\partial\Omega} g|u|^p ds > 0\end{aligned}$$

and so $u \in \mathbf{M}_\lambda^+$.
Case (II): $u \in \mathbf{M}_\lambda$ and $\int_{\partial\Omega} g|u|^p ds > 0$. Suppose that $\mathbf{M}_\lambda^0 \ne \phi$ for all $\lambda > 0$. If $u \in \mathbf{M}_\lambda^0$, then we have

$$\begin{aligned}0 = \langle \psi_\lambda'(u), u\rangle &= 2\|u\|_{H^1}^2 - \lambda q \int_\Omega f|u|^q dx - p\int_{\partial\Omega} g|u|^p ds \\ &= (2-q)\|u\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u|^p ds.\end{aligned}$$

Thus,

$$\|u\|_{H^1}^2 = \frac{p-q}{2-q}\int_{\partial\Omega} g|u|^p ds \tag{2.1}$$

and

$$\lambda \int_\Omega f|u|^q dx = \|u\|_{H^1}^2 - \int_{\partial\Omega} g|u|^p ds = \frac{p-2}{2-q}\int_{\partial\Omega} g|u|^p ds. \tag{2.2}$$

Moreover,

$$\begin{aligned}(\frac{p-2}{p-q})\|u\|_{H^1}^2 &= \|u\|_{H^1}^2 - \int_{\partial\Omega} g|u|^p ds \\ &= \lambda \int_\Omega f|u|^q dx \\ &\le \lambda \|f\|_{L^{p^*}} \|u\|_{L^p}^q \\ &\le \lambda \|f\|_{L^{p^*}} S_p^q \|u\|_{H^1}^q,\end{aligned}$$

where $p^* = \frac{p}{p-q}$. This implies

$$\|u\|_{H^1} \le \Big[\lambda(\frac{p-q}{p-2})\|f\|_{L^{p^*}} S_p^q\Big]^{1/(2-q)}. \tag{2.3}$$

Let $I_\lambda : \mathbf{M}_\lambda \to \mathbb{R}$ be given by

$$I_\lambda(u) = K(p,q)\Big(\frac{\|u\|_{H^1}^{2(p-1)}}{\int_{\partial\Omega} g|u|^p ds}\Big)^{1/(p-2)} - \lambda \int_\Omega f|u|^q dx,$$

where $K(p,q)=(\frac{2-q}{p-q})^{(p-1)/(p-2)}(\frac{p-2}{2-q})$. Then $I_\lambda(u)=0$ for all $u\in\mathbf{M}_\lambda^0$. Indeed, from (2.1) and (2.2) it follows that for $u\in\mathbf{M}_\lambda^0$ we have

$$\begin{aligned} I_\lambda(u) &= K(p,q)\Big(\frac{\|u\|_{H^1}^{2(p-1)}}{\int_{\partial\Omega} g|u|^p ds}\Big)^{1/(p-1)} - \lambda\int_\Omega f|u|^q dx \\ &= (\frac{2-q}{p-q})^{\frac{p}{p-1}}(\frac{p-2}{2-q})\Big(\frac{(\frac{p-q}{2-q})^{p-1}\big(\int_{\partial\Omega} g|u|^p ds\big)^{p-1}}{\int_{\partial\Omega} g|u|^p ds}\Big)^{\frac{1}{p-2}} \\ &\quad - \frac{p-2}{2-q}\int_{\partial\Omega} g|u|^p ds = 0. \end{aligned} \tag{2.4}$$

However, by (2.3), the Hölder and Sobolev trace inequality, for $u\in\mathbf{M}_\lambda^0$

$$\begin{aligned} I_\lambda(u) &\geq K(p,q)\Big(\frac{\|u\|_{H^1}^{2(p-1)}}{\int_{\partial\Omega} g|u|^p ds}\Big)^{1/(p-2)} - \lambda S_p^q\|f\|_{L^{p^*}}\|u\|_{H^1}^q \\ &\geq \|u\|_{H^1}^q\Big(K(p,q)\Big(\frac{\|u\|_{H^1}^{2(p-1)}}{C_p^p\|g\|_\infty\|u\|_{H^1}^{p+q(p-2)}}\Big)^{1/(p-2)} - \lambda S_p^q\|f\|_{L^{p^*}}\Big) \\ &\geq \|u\|_{H^1}^q\big\{K(p,q)C_p^{\frac{p}{2-p}}\lambda^{\frac{1-q}{2-q}}\Big[(\frac{p-q}{p-2})\|f\|_{L^{p^*}}S_p^q\Big]^{\frac{1-q}{2-q}} - \lambda S_p^q\|f\|_{L^{p^*}}\big\}. \end{aligned}$$

This implies that for λ sufficiently small we have $I_\lambda(u)>0$ for all $u\in\mathbf{M}_\lambda^0$, this contradicts (2.4). Thus, we can conclude that there exists $\lambda_1>0$ such that for $\lambda\in(0,\lambda_1)$, we have $\mathbf{M}_\lambda^0=\phi$. ☐

By Lemma 2.1, for $\lambda\in(0,\lambda_1)$ we write $\mathbf{M}_\lambda=\mathbf{M}_\lambda^+\cup\mathbf{M}_\lambda^-$ and define

$$\alpha_\lambda^+ = \inf_{u\in\mathbf{M}_\lambda^+} J_\lambda(u);\quad \alpha_\lambda^-(\Omega) = \inf_{u\in\mathbf{M}_\lambda^-} J_\lambda(u).$$

The following lemma shows that the minimizers on $\mathbf{M}_\lambda$ are "usually" critical points for J_λ.

Lemma 2.2. *For $\lambda\in(0,\lambda_1)$. If u_0 is a local minimizer for J_λ on $\mathbf{M}_\lambda$, then $J'_\lambda(u_0)=0$ in $H^*(\Omega)$.*

Proof. If u_0 is a local minimizer for J_λ on $\mathbf{M}_\lambda$, then u_0 is a solution of the optimization problem

$$\text{minimize } J_\lambda(u)\quad\text{subject to } \psi_\lambda(u)=0.$$

Hence, by the theory of Lagrange multipliers, there exists $\theta\in\mathbb{R}$ such that

$$J'_\lambda(u_0)=\theta\psi'_\lambda(u_0)\quad\text{in } H^*(\Omega).$$

Thus,

$$\langle J'_\lambda(u_0),u_0\rangle_{H^1}=\theta\langle\psi'_\lambda(u_0),u_0\rangle_{H^1}. \tag{2.5}$$

By Lemma 2.1, $u_0\in\mathbf{M}_\lambda^+\cup\mathbf{M}_\lambda^-$, we have $\langle\psi'_\lambda(u_0),u_0\rangle_{H^1}\neq 0$ and so by (2.5) $\theta=0$. This completes the proof. ☐

Lemma 2.3. (i) *If $u\in\mathbf{M}_\lambda^+$, then $\int_\Omega f|u|^q dx>0$;*
(ii) *If $u\in\mathbf{M}_\lambda^-$, then $\int_{\partial\Omega} g|u|^p ds>0$.*

Proof. (i) Case (I): $\int_{\partial\Omega} g|u|^p ds \le 0$. We have

$$\lambda \int_\Omega f|u|^q dx = \|u\|_{H^1}^2 - \int_{\partial\Omega} g|u|^p ds > 0.$$

Case (II): $\int_{\partial\Omega} g|u|^p ds > 0$. We have

$$\|u\|_{H^1}^2 - \lambda \int_\Omega f|u|^q dx - \int_{\partial\Omega} g|u|^p ds = 0$$

and

$$\|u\|_{H^1}^2 > \frac{p-q}{2-q} \int_{\partial\Omega} g|u|^p ds.$$

Thus,

$$\lambda \int_\Omega f|u|^q dx = \|u\|_{H^1}^2 - \int_{\partial\Omega} g|u|^p ds > \frac{p-2}{2-q} \int_{\partial\Omega} g|u|^p ds > 0.$$

(ii) Since

$$(2-q)\|u\|_{H^1}^2 - (p-q) \int_{\partial\Omega} g|u|^p ds = \langle \psi_\lambda'(u), u\rangle < 0.$$

It follows that $\int_{\partial\Omega} g|u|^p ds > 0$. This completes the proof. □

For each $u \in \mathbf{M}_\lambda^-$, we write

$$t_{\max} = \Big(\frac{(2-q)\|u\|_{H^1}^2}{(p-q)\int_{\partial\Omega} g|u|^p ds} \Big)^{1/(p-2)} < 1.$$

Then we have the following lemma.

Lemma 2.4. *Let $p^* = \frac{p}{p-q}$ and $\lambda_2 = (\frac{p-2}{p-q})(\frac{2-q}{p-q})^{\frac{2-q}{p-2}} C_p^{\frac{p(2-q)}{2-p}} S_p^{-q} \|f\|_{L^{p^*}}^{-1}$. Then for each $u \in \mathbf{M}_\lambda^-$ and $\lambda \in (0, \lambda_2)$, we have*

- (i) *if $\int_\Omega f|u|^q dx \le 0$, then $J_\lambda(u) = \sup_{t\ge 0} J_\lambda(tu) > 0$;*
- (ii) *if $\int_\Omega f|u|^q dx > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+u \in \mathbf{M}_\lambda^+$ and*

$$J_\lambda(t^+u) = \inf_{0\le t\le t_{\max}} J_\lambda(tu), J_\lambda(u) = \sup_{t\ge t_{\max}} J_\lambda(tu).$$

Proof. Fix $u \in \mathbf{M}_\lambda^-$. Let

$$h(t) = t^{2-q}\|u\|_{H^1}^2 - t^{p-q} \int_{\partial\Omega} g|u|^p ds \quad \text{for } t \ge 0.$$

We have $h(0) = 0$, $h(t) \to -\infty$ as $t \to \infty$, $h(t)$ achieves its maximum at $t_{\max}$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Moreover,

$$\begin{aligned}
&h(t_{\max})\\
&= \Big(\frac{(2-q)\|u\|_{H^1}^2}{(p-q)\int_{\partial\Omega} g|u|^p ds} \Big)^{\frac{2-q}{p-2}} \|u\|_{H^1}^2 - \Big(\frac{(2-q)\|u\|_{H^1}^2}{(p-q)\int_{\partial\Omega} g|u|^p ds} \Big)^{\frac{p-q}{p-2}} \int_{\partial\Omega} g|u|^p ds\\
&= \|u\|_{H^1}^q \Big[(\frac{2-q}{p-q})^{\frac{2-q}{p-2}} - (\frac{2-q}{p-q})^{\frac{p-q}{p-2}} \Big] \Big(\frac{\|u\|_{H^1}^p}{\int_{\partial\Omega} g|u|^p ds} \Big)^{\frac{2-q}{p-2}}\\
&\ge \|u\|_{H^1}^q (\frac{p-2}{p-q})(\frac{2-q}{p-q})^{\frac{2-q}{p-2}} C_p^{\frac{p(2-q)}{2-p}}
\end{aligned}$$

or

$$h(t_{\max}) \ge \|u\|_{H^1}^q (\frac{p-2}{p-q})(\frac{2-q}{p-q})^{\frac{2-q}{p-2}} C_p^{\frac{p(2-q)}{2-p}}. \tag{2.6}$$

(i): $\int_\Omega f|u|^q dx \le 0$. There is a unique $t^- > t_{\max}$ such that $h(t^-) = \lambda \int_\Omega f|u|^q dx$ and $h'(t^-) < 0$. Now,

$$\begin{aligned}
&(2-q)\|t^- u\|_{H^1}^2 - (p-q)\int_{\partial\Omega} |t^- u|^p ds \\
&= (t^-)^{1+q}\Big[(2-q)(t^-)^{1-q}\|u\|_{H^1}^2 - (p-q)(t^-)^{p-q-1}\int_{\partial\Omega} g|u|^p ds\Big] \\
&= (t^-)^{1+q} h'(t^-) < 0,
\end{aligned}$$

and

$$\begin{aligned}
&\langle J'_\lambda(t^- u), t^- u\rangle \\
&= (t^-)^2\|u\|_{H^1}^2 - (t^-)^q \lambda \int_\Omega f|u|^q dx - (t^-)^p \int_{\partial\Omega} g|u|^p ds \\
&= (t^-)^q\Big[h(t^-) - \lambda\int_\Omega f|u|^q dx\Big] = 0.
\end{aligned}$$

Thus, $t^- u \in \mathbf{M}_\lambda^-$ or $t^- = 1$. Since for $t > t_{\max}$, we have

$$(2-q)\|tu\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|tu|^p ds < 0,$$

$$\frac{d^2}{dt^2} J_\lambda(tu) < 0,$$

$$\frac{d}{dt} J_\lambda(tu) = t\|u\|_{H^1}^2 - \lambda t^{q-1}\int_\Omega f|u|^q dx - t^{p-1}\int_{\partial\Omega} g|u|^p ds = 0 \quad \text{for } t = t^-.$$

Thus, $J_\lambda(u) = \sup_{t\ge 0} J_\lambda(tu)$. Moreover,

$$J_\lambda(u) \ge J_\lambda(tu) \ge \frac{t^2}{2}\|u\|_{H^1}^2 - \frac{t^p}{p}\int_{\partial\Omega} g|u|^p ds \quad \text{for all } t \ge 0.$$

By routine computations, $g(t) = \frac{t^2}{2}\|u\|_{H^1}^2 - \frac{t^p}{p}\int_{\partial\Omega} g|u|^p ds$ achieves its maximum at $t_0 = (\|u\|_{H^1}^2 / \int_{\partial\Omega} g|u|^p ds)^{1/(p-2)}$. Thus,

$$J_\lambda(u) \ge \frac{p-2}{2p}\Big(\frac{\|u\|_{H^1}^p}{\int_{\partial\Omega} g|u|^p ds}\Big)^{\frac{2}{p-2}} > 0.$$

(ii): $\int_\Omega f|u|^q dx > 0$. By (2.6) and

$$\begin{aligned}
h(0) = 0 < \lambda \int_\Omega f|u|^q dx &\le \lambda \|f\|_{L^{p^*}} S_p^q \|u\|_{H^1}^q \\
&< \|u\|_{H^1}^q \big(\frac{p-2}{p-q}\big)\big(\frac{2-q}{p-q}\big)^{\frac{2-q}{p-2}} C_p^{\frac{p(2-q)}{2-p}} \\
&\le h(t_{\max}) \quad \text{for } \lambda \in (0,\lambda_2),
\end{aligned}$$

there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$h(t^+) = \lambda\int_\Omega f|u|^q dx = h(t^-),$$

$$h'(t^+) > 0 > h'(t^-).$$

We have $t^+u \in \mathbf{M}_\lambda^+$, $t^-u \in \mathbf{M}_\lambda^-$, and $J_\lambda(t^-u) \geq J_\lambda(tu) \geq J_\lambda(t^+u)$ for each $t \in [t^+, t^-]$ and $J_\lambda(t^+u) \leq J_\lambda(tu)$ for each $t \in [0, t^+]$. Thus, $t^- = 1$ and

$$J_\lambda(u) = \sup_{t\geq 0} J_\lambda(tu), J_\lambda(t^+u) = \inf_{0\leq t\leq t_{\max}} J_\lambda(tu).$$

This completes the proof. □

Next, we establish the existence of nontrivial nonnegative solutions for the equation

$$\begin{gathered} -\Delta u + u = \lambda f(x)|u|^{q-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.7}$$

Associated with equation (2.7), we consider the energy functional

$$K_\lambda(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{\lambda}{q}\int_\Omega f|u|^q dx$$

and the minimization problem

$$\beta_\lambda = \inf\{K_\lambda(u) : u \in \mathbf{N}_\lambda\},$$

where $\mathbf{N}_\lambda = \{u \in H_0^1(\Omega)\backslash\{0\} : \langle K_\lambda'(u), u\rangle = 0\}$. Then we have the following result.

Theorem 2.5. *Suppose that $\lambda > 0$. Then equation* (2.7) *has a nontrivial nonnegative solution v_λ with $K_\lambda(v_\lambda) = \beta_\lambda < 0$.*

Proof. First, we need to show that K_λ is bounded below on $\mathbf{N}_\lambda$ and $\beta_\lambda < 0$. Then for $u \in \mathbf{N}_\lambda$,

$$\|u\|_{H^1}^2 = \lambda\int_\Omega f|u|^q dx \leq \lambda\|f\|_{L^{q^*}} S_p^{-\frac{q}{2}}\|u\|_{H^1}^q.$$

where $p^* = \frac{p}{p-q}$. This implies

$$\|u\|_{H^1} \leq (\lambda\|f\|_{L^{p^*}} S_p^{-\frac{q}{2}})^{\frac{1}{2-q}}. \tag{2.8}$$

Hence,

$$\begin{aligned} K_\lambda(u) &= \frac{1}{2}\|u\|_{H^1} - \frac{\lambda}{q}\int_\Omega f|u|^q dx \\ &= \big(\frac{1}{2} - \frac{1}{q}\big)\|u\|_{H^1}^2 \\ &\leq \big(\frac{1}{2} - \frac{1}{q}\big)\big(\lambda\|f\|_{L^{p^*}} S_p^{-\frac{q}{2}}\big)^{\frac{1}{2-q}} \end{aligned}$$

for all $u \in \mathbf{N}_\lambda$ and $\beta_\lambda < 0$. Let $\{v_n\}$ be a minimizing sequence for K_λ on $\mathbf{N}_\lambda$. Then by (2.8) and the compact imbedding theorem, there exist a subsequence $\{v_n\}$ and v_λ in $H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v_\lambda \quad \text{weakly in } H_0^1(\Omega)$$

and

$$v_n \to v_\lambda \quad \text{strongly in } L^q(\Omega). \tag{2.9}$$

First, we claim that $\int_\Omega f|v_\lambda|^q dx > 0$. If not,

$$K_\lambda(v_n) \geq \frac{1}{2}\|v_\lambda\|_{H^1}^2 - \frac{\lambda}{q}\int_\Omega f|v_\lambda|^q dx + o(1) \geq \frac{1}{2}\|v_\lambda\|_{H^1}^2 + o(1),$$

this contradicts $K_\lambda(v_n) \to \beta_\lambda(\Omega) < 0$ as $n \to \infty$. Thus, $\int_\Omega f|v_\lambda|^q dx > 0$. In particular, $v_\lambda \not\equiv 0$. Now, we prove that $v_n \to v_\lambda$ strongly in $H_0^1(\Omega)$. Suppose otherwise, then $\|v_\lambda\|_{H^1} < \liminf_{n\to\infty} \|v_n\|_{H^1}$ and so

$$\|v_\lambda\|_{H^1}^2 - \lambda \int_\Omega f|v_\lambda|^q dx < \liminf_{n\to\infty} \Big(\|v_n\|_{H^1}^2 - \lambda \int_\Omega f|v_n|^q dx\Big) = 0.$$

Since $\int_\Omega f|v_\lambda|^q dx > 0$, there is a unique $t_0 \neq 1$ such that $t_0 v_\lambda \in \mathbf{N}_\lambda$. Thus,

$$t_0 v_n \rightharpoonup t_0 v_\lambda \quad \text{weakly in } H_0^1(\Omega).$$

Moreover,

$$K_\lambda(t_0 v_\lambda) < K_\lambda(v_\lambda) < \lim_{n\to\infty} K_\lambda(v_n) = \beta_\lambda,$$

which is a contradiction. Hence $v_n \to v_\lambda$ strongly in $H_0^1(\Omega)$. This implies $v_\lambda \in \mathbf{N}_\lambda$ and

$$K_\lambda(v_n) \to K_\lambda(v_\lambda) = \beta_\lambda \qquad \text{as } n \to \infty.$$

Since $K_\lambda(v_\lambda) = K_\lambda(\|v_\lambda\|)$ and $\|v_\lambda\| \in \mathbf{N}_\lambda$, without loss of generality, we may assume that v_λ is a nontrivial nonnegative solution of equation (2.7). □

Then we have the following results.

Lemma 2.6. (i) *$\alpha_\lambda \le \alpha_\lambda^+ \le \beta_\lambda < 0$;*
(ii) *J_λ is coercive and bounded below on $\mathbf{M}_\lambda$ for all $\lambda \in (0, \frac{p-2}{p-q}]$.*

Proof. (i) Let v_λ be a positive solution of equation (2.7) such that $K(v_\lambda) = \beta_\lambda$. Since $v_\lambda \in C^2(\overline{\Omega})$. Then we have $\int_{\partial\Omega} g|v_\lambda|^p ds = 0$ and $v_\lambda \in \mathbf{M}_\lambda^+$. This implies

$$J_\lambda(v_\lambda) = \frac{1}{2}\|v_\lambda\|_{H^1}^2 - \frac{\lambda}{q}\int_\Omega f|v_\lambda|^q dx = \beta_\lambda < 0$$

and so $\alpha_\lambda \le \alpha_\lambda^+ \le \beta_\lambda < 0$.
(ii) For $u \in \mathbf{M}_\lambda$, we have $\|u\|_{H^1}^2 = \lambda \int_\Omega f|u|^q dx + \int_{\partial\Omega} g|u|^p ds$. Then by the Hölder and Young inequalities,

$$\begin{aligned}
J_\lambda(u) &= \frac{p-2}{2p}\|u\|_{H^1}^2 - \lambda\big(\frac{p-q}{pq}\big)\int_\Omega f|u|^q dx \\
&\ge \frac{p-2}{2p}\|u\|_{H^1}^2 - \lambda\big(\frac{p-q}{pq}\big)\|f\|_{L^{p^*}} S_p^q \|u\|_{H^1}^q \\
&\ge \big[\frac{p-2}{2p} - \lambda\big(\frac{p-q}{2p}\big)\big]\|u\|_{H^1}^2 - \lambda\big(\frac{(p-q)(2-q)}{2pq}\big)\big(\|f\|_{L^{p^*}} S_p^q\big)^{\frac{2}{2-q}} \\
&= \frac{1}{2p}\big[(p-2) - \lambda(p-q)\big]\|u\|_{H^1}^2 - \lambda\big(\frac{(p-q)(2-q)}{2pq}\big)\big(\|f\|_{L^{p^*}} S_p^q\big)^{\frac{2}{2-q}}.
\end{aligned}$$

Thus, J_λ is coercive on $\mathbf{M}_\lambda$ and

$$J_\lambda(u) \ge -\lambda\big(\frac{(p-q)(2-q)}{2pq}\big)\big(\|f\|_{L^{p^*}} S_p^q\big)^{\frac{2}{2-q}}$$

for all $\lambda \in (0, \frac{p-2}{p-q}]$. □

3. Proof of Theorem 1.1

First, we will use the idea of Ni-Takagi [12] to get the following results.

Lemma 3.1. *For each $u \in \mathbf{M}_\lambda$, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset H^1(\Omega) \to \mathbb{R}^+$ such that $\xi(0) = 1$, the function $\xi(v)(u-v) \in \mathbf{M}_\lambda$ and*

$$\langle \xi'(0), v\rangle = \frac{2\int_\Omega \nabla u \nabla v dx - \lambda q \int_\Omega f|u|^{q-2}uv dx - p\int_{\partial\Omega} g|u|^{p-2}uv ds}{(2-q)\|u\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u|^p ds} \tag{3.1}$$

for all $v \in H^1(\Omega)$.

Proof. For $u \in \mathbf{M}_\lambda$, define a function $F : \mathbb{R} \times H^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} F_u(\xi, w) &= \langle J'_\lambda(\xi(u-w)), \xi(u-w)\rangle \\ &= \xi^2 \int_\Omega |\nabla(u-w)|^2 + (u-w)^2 dx - \xi^q \lambda \int_\Omega f|u-w|^q dx \\ &\quad - \xi^p \int_{\partial\Omega} g|u-w|^p ds. \end{aligned}$$

Then $F_u(1,0) = \langle J'_\lambda(u), u\rangle = 0$ and

$$\begin{aligned} \frac{d}{d\xi} F_u(1,0) &= 2\|u\|_{H^1}^2 - \lambda q \int_{\partial\Omega} f|u|^q dx - p\int_{\partial\Omega} g|u|^p ds \\ &= (2-q)\|u\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u|^p ds \neq 0. \end{aligned}$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset H^1(\Omega) \to \mathbb{R}$ such that $\xi(0) = 1$,

$$\langle \xi'(0), v\rangle = \frac{2\int_\Omega \nabla u \nabla v dx - \lambda q \int_\Omega f|u|^{q-2}uv dx - p\int_{\partial\Omega} g|u|^{p-2}uv ds}{(2-q)\|u\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u|^p ds}$$

and

$$F_u(\xi(v), v) = 0 \quad \text{for all } v \in B(0;\epsilon)$$

which is equivalent to

$$\langle J'_\lambda(\xi(v)(u-v)), \xi(v)(u-v)\rangle = 0 \qquad \text{for all } v \in B(0;\epsilon),$$

that is $\xi(v)(u-v) \in \mathbf{M}_\lambda$. ◻

Lemma 3.2. *For each $u \in \mathbf{M}_\lambda^-$, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset H^1(\Omega) \to \mathbb{R}^+$ such that $\xi^-(0) = 1$, the function $\xi^-(v)(u-v) \in \mathbf{M}_\lambda^-$ and*

$$\langle (\xi^-)'(0), v\rangle = \frac{2\int_\Omega \nabla u \nabla v dx - \lambda q \int_\Omega f|u|^{q-2}uv dx - p\int_{\partial\Omega} g|u|^{p-2}uv ds}{(2-q)\|u\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u|^p ds} \tag{3.2}$$

for all $v \in H^1(\Omega)$.

Proof. Similar to the argument in Lemma 3.1, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset H^1(\Omega) \to \mathbb{R}$ such that $\xi^-(0) = 1$ and $\xi^-(v)(u-v) \in \mathbf{M}_\lambda$ for all $v \in B(0;\epsilon)$. Since

$$\langle \psi'_\lambda(u), u\rangle = (2-q)\|u\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u|^p ds < 0.$$

Thus, by the continuity of the function ξ^-, we have

$$\begin{aligned}
&\langle \psi_\lambda'(\xi^-(v)(u-v)), \xi^-(v)(u-v)\rangle \\
&= (2-q)\|\xi^-(v)(u-v)\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|\xi^-(v)(u-v)|^p ds < 0
\end{aligned}$$

if ϵ sufficiently small, this implies that $\xi^-(v)(u-v) \in \mathbf{M}_\lambda^-$. □

Proposition 3.3. *Let* $\lambda_0 = \min\{\lambda_1, \lambda_2, \frac{p-1}{p-q}\}$*, Then for* $\lambda \in (0, \lambda_0)$*:*

(i) *There exists a minimizing sequence* $\{u_n\} \subset \mathbf{M}_\lambda$ *such that*

$$\begin{aligned}
J_\lambda(u_n) &= \alpha_\lambda + o(1), \\
J_\lambda'(u_n) &= o(1) \quad \text{in } H^*(\Omega);
\end{aligned}$$

(ii) *there exists a minimizing sequence* $\{u_n\} \subset \mathbf{M}_\lambda^-$ *such that*

$$\begin{aligned}
J_\lambda(u_n) &= \alpha_\lambda^- + o(1), \\
J_\lambda'(u_n) &= o(1) \quad \text{in } H^*(\Omega).
\end{aligned}$$

Proof. (i) By Lemma 2.6 (ii) and the Ekeland variational principle [7], there exists a minimizing sequence $\{u_n\} \subset \mathbf{M}_\lambda$ such that

$$J_\lambda(u_n) < \alpha_\lambda + \frac{1}{n}, \tag{3.3}$$

$$J_\lambda(u_n) < J_\lambda(w) + \frac{1}{n}\|w - u_n\|_{H^1} \quad \text{for each } w \in \mathbf{M}_\lambda. \tag{3.4}$$

By taking n large, from Lemma 2.6 (i), we have

$$\begin{aligned}
J_\lambda(u_n) &= (\frac{1}{2} - \frac{1}{p})\|u_n\|_{H^1}^2 - (\frac{1}{q} - \frac{1}{p})\lambda\int_\Omega f|u_n|^q dx \\
&< \alpha_\lambda + \frac{1}{n} < \frac{\beta_\lambda}{2}.
\end{aligned} \tag{3.5}$$

This implies

$$\|f\|_{L^{p^*}} S_p^q \|u_n\|_{H^1}^q \ge \int_\Omega f|u_n|^q dx > \frac{-pq}{2\lambda(p-q)}\beta_\lambda > 0. \tag{3.6}$$

Consequently, $u_n \neq 0$ and putting together (3.5), (3.6) and the Hölder inequality, we obtain

$$\|u_n\|_{H^1} > \Big[\frac{-pq}{2\lambda(p-q)}\beta_\lambda S_p^{-q}\|f\|_{L^{p^*}}^{-1}\Big]^{1/q} \tag{3.7}$$

$$\|u_n\|_{H^1} < \Big[\frac{2(p-q)}{(p-2)q}\|f\|_{L^{p^*}} S_p^q\Big]^{1/(2-q)} \tag{3.8}$$

Now, we show that

$$\|J_\lambda'(u_n)\|_{H^{-1}} \to 0 \quad \text{as } n \to \infty.$$

Applying Lemma 3.1 with u_n to obtain the functions $\xi_n : B(0;\epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $\xi_n(w)(u_n - w) \in \mathbf{M}_\lambda$. Choose $0 < \rho < \epsilon_n$. Let $u \in H^1(\Omega)$ with $u \not\equiv 0$ and let $w_\rho = \frac{\rho u}{\|u\|_{H^1}}$. We set $\eta_\rho = \xi_n(w_\rho)(u_n - w_\rho)$. Since $\eta_\rho \in \mathbf{M}_\lambda$, we deduce from (3.4) that

$$J_\lambda(\eta_\rho) - J_\lambda(u_n) \ge -\frac{1}{n}\|\eta_\rho - u_n\|_{H^1}$$

and by the mean value theorem, we have

$$\langle J'_\lambda(u_n), \eta_\rho - u_n\rangle + o(\|\eta_\rho - u_n\|_{H^1}) \geq -\frac{1}{n}\|\eta_\rho - u_n\|_{H^1}.$$

Thus,

$$\begin{aligned}&\langle J'_\lambda(u_n), -w_\rho\rangle + (\xi_n(w_\rho) - 1)\langle J'_\lambda(u_n), (u_n - w_\rho)\rangle \\ &\geq -\frac{1}{n}\|\eta_\rho - u_n\|_{H^1} + o(\|\eta_\rho - u_n\|_{H^1}).\end{aligned} \tag{3.9}$$

Since $\xi_n(w_\rho)(u_n - w_\rho) \in \mathbf{M}_\lambda$ and (3.9) it follows that

$$\begin{aligned}&-\rho\langle J'_\lambda(u_n), \frac{u}{\|u\|_{H^1}}\rangle + (\xi_n(w_\rho) - 1)\langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - w_\rho)\rangle \\ &\geq -\frac{1}{n}\|\eta_\rho - u_n\|_{H^1} + o(\|\eta_\rho - u_n\|_{H^1}).\end{aligned}$$

Thus,

$$\begin{aligned}\langle J'_\lambda(u_n), \frac{u}{\|u\|_{H^1}}\rangle \leq &\frac{\|\eta_\rho - u_n\|_{H^1}}{n\rho} + \frac{o(\|\eta_\rho - u_n\|_{H^1})}{\rho} \\ &+ \frac{(\xi_n(w_\rho) - 1)}{\rho}\langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - w_\rho)\rangle.\end{aligned} \tag{3.10}$$

Since $\|\eta_\rho - u_n\|_{H^1} \leq \rho\|\xi_n(w_\rho)\| + \|\xi_n(w_\rho) - 1\|\|u_n\|_{H^1}$ and

$$\lim_{\rho\to 0}\frac{\|\xi_n(w_\rho) - 1\|}{\rho} \leq \|\xi'_n(0)\|,$$

if we let $\rho \to 0$ in (3.10) for a fixed n, then by (3.8) we can find a constant $C > 0$, independent of ρ, such that

$$\langle J'_\lambda(u_n), \frac{u}{\|u\|_{H^1}}\rangle \leq \frac{C}{n}(1 + \|\xi'_n(0)\|).$$

The proof will be complete once we show that $\|\xi'_n(0)\|$ is uniformly bounded in n. By (3.1), (3.8) and the Hölder inequality, we have

$$\langle \xi'_n(0), v\rangle \leq \frac{b\|v\|_{H^1}}{|(2-q)\|u_n\|_{H^1} - (p-q)\int_{\partial\Omega} g|u_n|^p ds|} \quad \text{for some } b > 0.$$

We only need to show that

$$|(2-q)\|u_n\|_{H^1} - (p-q)\int_{\partial\Omega} g|u_n|^p ds| > c \tag{3.11}$$

for some $c > 0$ and n large enough. We argue by contradiction. Assume that there exists a subsequence $\{u_n\}$, we have

$$(2-q)\|u_n\|_{H^1} - (p-q)\int_{\partial\Omega} g|u_n|^p ds = o(1). \tag{3.12}$$

Combining (3.12) with (3.7), we can find a suitable constant $d > 0$ such that

$$\int_{\partial\Omega} g|u_n|^p ds \geq d \quad \text{for } n \text{ sufficiently large.} \tag{3.13}$$

In addition (3.12), and the fact that $u_n \in \mathbf{M}_\lambda$ also give

$$\lambda\int_\Omega f|u_n|^q dx = \|u_n\|^2_{H^1} - \int_{\partial\Omega} g|u_n|^p ds = \frac{p-2}{2-q}\int_{\partial\Omega} g|u_n|^p ds + o(1)$$

and

$$\|u_n\|_{H^1} \leq \Big[\lambda(\frac{p-q}{p-2})\|f\|_{L^{p^*}} S_p^q\Big]^{\frac{1}{2-q}} + o(1). \tag{3.14}$$

This implies

$$I_\lambda(u_n) = K(p,q)\Big(\frac{\|u_n\|_{H^1}^{2(p-1)}}{\int_{\partial\Omega} g|u_n|^p ds}\Big)^{1/(p-2)} - \lambda\int_\Omega f|u_n|^q dx = o(1). \tag{3.15}$$

However, by (3.13), (3.14) and $\lambda \in (0, \lambda_0)$,

$$\begin{aligned} I_\lambda(u_n) &\geq K(p,q)\Big(\frac{\|u_n\|_{H^1}^{2(p-1)}}{\int_{\partial\Omega} g|u_n|^p ds}\Big)^{1/(p-2)} - \lambda S_p^q \|f\|_{L^{p^*}} \|u_n\|_{H^1}^q \\ &\geq \|u_n\|_{H^1}^q \Big(K(p,q)\Big(\frac{\|u_n\|_{H^1}^{2(p-1)}}{C_p^p \|u_n\|_{H^1}^{p+q(p-2)}}\Big)^{1/(p-2)} - \lambda S_p^q \|f\|_{L^{p^*}}\Big) \\ &\geq \|u_n\|_{H^1}^q \Big\{K(p,q) C_p^{\frac{p}{2-p}} \lambda^{\frac{1-q}{2-q}} \big[(\frac{p-q}{p-2})\|f\|_{L^{p^*}} S_p^q\big]^{\frac{1-q}{2-q}} - \lambda\|f\|_{L^{p^*}}\Big\}. \end{aligned}$$

this contradicts (3.15). We get

$$\langle J_\lambda'(u_n), \frac{u}{\|u\|_{H^1}}\rangle \leq \frac{C}{n}.$$

This completes the proof of (i).
(ii) Similarly, by using Lemma 3.2, we can prove (ii). We will omit detailed proof here. □

Now, we establish the existence of a local minimum for J_λ on $\mathbf{M}_\lambda^+$.

Theorem 3.4. *Let $\lambda_0 > 0$ as in Proposition 3.3, then for $\lambda \in (0, \lambda_0)$ the functional J_λ has a minimizer u_0^+ in $\mathbf{M}_\lambda^+$ and it satisfies*

(i) $J_\lambda(u_0^+) = \alpha_\lambda = \alpha_\lambda^+$;
(ii) u_0^+ *is a nontrivial nonnegative solution of equation* (1.1);
(iii) $J_\lambda(u_0^+) \to 0$ *as* $\lambda \to 0$.

Proof. Let $\{u_n\} \subset \mathbf{M}_\lambda$ be a minimizing sequence for J_λ on $\mathbf{M}_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + o(1) \quad \text{and} \quad J_\lambda'(u_n) = o(1) \quad \text{in } H^*(\Omega).$$

Then by Lemma 2.6 and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^+ \in H^1(\Omega)$ such that

$$u_n \rightharpoonup u_0^+ \quad \text{weakly in } H^1(\Omega), u_n \to u_0^+ \quad \text{strongly in } L^p(\partial\Omega)$$

and

$$u_n \to u_0^+ \quad \text{strongly in } L^q(\Omega). \tag{3.16}$$

First, we claim that $\int_\Omega f(x)\|u_0^+\|^q dx \neq 0$. Suppose otherwise, by (3.16) we can conclude that

$$\int_\Omega f|u_n|^q dx \to \int_\Omega f|u_0^+|^q dx = 0 \quad \text{as } n \to \infty$$

and so

$$\|u_n\|_{H^1}^2 = \int_{\partial\Omega} g|u_n|^p ds + o(1).$$

Thus,

$$\begin{aligned} J_\lambda(u_n) &= \frac{1}{2}\|u_n\|_{H^1}^2 - \frac{\lambda}{q}\int_\Omega f|u_n|^q dx - \frac{1}{p}\int_{\partial\Omega} g|u_n|^p ds \\ &= (\frac{1}{2}-\frac{1}{p})\int_{\partial\Omega} g|u_n|^p ds + o(1) \\ &= (\frac{1}{2}-\frac{1}{p})\int_{\partial\Omega} g|u_0^+|^p ds \quad \text{as } n\to\infty, \end{aligned}$$

this contradicts $J_\lambda(u_n) \to \alpha_\lambda < 0$ as $n\to\infty$. Moreover,

$$o(1) = \langle J_\lambda'(u_n), \phi\rangle = \langle J_\lambda'(u_0), \phi\rangle + o(1) \quad \text{for all } \phi \in H^1(\Omega).$$

Thus, $u_0^+ \in \mathbf{M}_\lambda$ is a nonzero solution of equation (1.1) and $J_\lambda(u_0^+) \ge \alpha_\lambda$. Now we prove that $u_n \to u_0^+$ strongly in $H^1(\Omega)$. Suppose otherwise, then $\|u_0^+\|_{H^1} < \liminf_{n\to\infty} \|u_n\|_{H^1}$ and so

$$\begin{aligned} &\|u_0^+\|_{H^1}^2 - \lambda\int_\Omega f|u_0^+|^q dx - \int_{\partial\Omega} g|u_0^+|^p ds \\ &< \liminf_{n\to\infty}\Big(\|u_n\|_{H^1}^2 - \lambda\int_\Omega f|u_n|^q dx - \int_{\partial\Omega} g|u_n|^p ds\Big) = 0, \end{aligned}$$

this contradicts $u_0^+ \in \mathbf{M}_\lambda$. Hence $u_n \to u_0^+$ strongly in $H^1(\Omega)$ and

$$J_\lambda(u_n) \to J_\lambda(u_0^+) = \alpha_\lambda \quad \text{as } n\to\infty.$$

Moreover, we have $u_0^+ \in \mathbf{M}_\lambda^+$. If not, then $u_0^+ \in \mathbf{M}_\lambda^-$ and by Lemma 2.4, there are unique t_0^+ and t_0^- such that $t_0^+u_0^+ \in \mathbf{M}_\lambda^+$ and $t_0^-u_0^+ \in \mathbf{M}_\lambda^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_\lambda(t_0^+u_0^+) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}J_\lambda(t_0^+u_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that $J_\lambda(t_0^+u_0^+) < J_\lambda(\bar{t}u_0^+)$. By Lemma 2.4,

$$J_\lambda(t_0^+u_0^+) < J_\lambda(\bar{t}u_0^+) \le J_\lambda(t_0^-u_0^+) = J_\lambda(u_0^+),$$

which is a contradiction. Since $J_\lambda(u_0^+) = J_\lambda(|u_0^+|)$ and $|u_0^+| \in \mathbf{M}_\lambda^+$, by Lemma 2.2 we may assume that u_0^+ is a nontrivial nonnegative solution of equation (1.1). From Lemma 2.6 it follows that

$$0 > J_\lambda(u_0^+) \ge -\lambda\Big(\frac{(p-q)(2-q)}{2pq}\Big)(\|f\|_{L^{p^*}}S_p^q)^{\frac{2}{2-q}}$$

and so $J_\lambda(u_0^+) \to 0$ as $\lambda \to 0$. □

Next, we establish the existence of a local minimum for J_λ on $\mathbf{M}_\lambda^-$.

Theorem 3.5. *Let $\lambda_0 > 0$ as in Proposition 3.3. Then for $\lambda \in (0,\lambda_0)$ the functional J_λ has a minimizer u_0^- in $\mathbf{M}_\lambda^-$ and satisfies*

(i) *$J_\lambda(u_0^-) = \alpha_\lambda^-$;*
(ii) *u_0^- is a nontrivial nonnegative solution of equation* (1.1).

Proof. By Proposition 3.3 (ii), there exists a minimizing sequence $\{u_n\}$ for J_λ on $\mathbf{M}_\lambda^-$ such that

$$J_\lambda(u_n) = \alpha_\lambda^- + o(1) \quad \text{and} \quad J_\lambda'(u_n) = o(1) \quad \text{in } H^*(\Omega).$$

By Lemma 2.6 and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^- \in H^1(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u_0^- \quad \text{weakly in } H^1(\Omega), \\ u_n \to u_0^- \quad \text{strongly in } L^p(\partial\Omega), \\ u_n \to u_0^- \quad \text{strongly in } L^q(\Omega). \end{gathered}$$

Since $(2-q)\|u_n\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u_n|^p ds < 0$, by the Sobolev trace inequality there exists $C > 0$ such that $\int_{\partial\Omega} g|u_n|^p ds > C$. Moreover,

$$o(1) = \langle J'_\lambda(u_n), \phi\rangle = \langle J'_\lambda(u_0), \phi\rangle + o(1) \quad \text{for all } \phi \in H^1(\Omega)$$

and

$$\begin{aligned} &(2-q)\|u_0\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u_0|^p ds \\ &\le \liminf_{n\to\infty} \Big((2-q)\|u_n\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u_n|^p ds\Big) \le 0. \end{aligned}$$

Thus, $u_0^- \in \mathbf{M}_\lambda^-$ is a nonzero solution of equation (1.1). Now we prove that $u_n \to u_0^-$ strongly in $H^1(\Omega)$. Suppose otherwise, then $\|u_0^-\|_{H^1} < \liminf_{n\to\infty} \|u_n\|_{H^1}$ and so

$$\begin{aligned} &\|u_0^-\|_{H^1}^2 - \lambda\int_\Omega f|u_0^-|^q dx - \int_{\partial\Omega} g|u_0^-|^p ds \\ &< \liminf_{n\to\infty} \Big(\|u_n\|_{H^1}^2 - \lambda\int_\Omega f|u_n|^q dx - \int_{\partial\Omega} g|u_n|^p ds\Big) = 0, \end{aligned}$$

this contradicts $u_0^- \in \mathbf{M}_\lambda^-$. Hence $u_n \to u_0^-$ strongly in $H^1(\Omega)$. This implies

$$J_\lambda(u_n) \to J_\lambda(u_0^-) = \alpha_\lambda^- \quad \text{as } n \to \infty.$$

Since $J_\lambda(u_0^-) = J_\lambda(|u_0^-|)$ and $|u_0^-| \in \mathbf{M}_\lambda^-$, by Lemma 2.2 we may assume that u_0^- is a nontrivial nonnegative solution of equation (1.1). □

Now, we complete the proof of Theorem 1.1. By Theorems 3.4, 3.5, we obtain equation (1.1) has two nontrivial nonnegative solutions u_0^+ and u_0^- such that $u_0^+ \in \mathbf{M}_\lambda^+$ and $u_0^- \in \mathbf{M}_\lambda^-$. Since $\mathbf{M}_\lambda^+ \cap \mathbf{M}_\lambda^- = \phi$, this implies that u_0^+ and u_0^- are different.

References

[1] Adimurthi, F. Pacella, and L. Yadava; *On the number of positive solutions of some semilinear Dirichlet problems in a ball*, Diff. Int. Equations 10 (6) (1997) 1157–1170.

[2] A. Ambrosetti, J. Garcia-Azorero and I. Peral; *Multiplicity results for some nonlinear elliptic equations*, J. Funct. Anal. 137 (1996) 219–242.

[3] A. Ambrosetti, H. Brezis and G. Cerami; *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. 122 (1994) 519–543.

[4] M. Chipot, M. Chlebik, M. Fila and I. Shafrir; *Existence of positive solutions of a semilinear elliptic equation in $\mathbb{R}_+^N$ with a nonlinear boundary condition*, J. Math. Anal. Appl., 223 (1998) 429–471.

[5] M. Chipot, I. Shafrir and M. Fila; *On the solutions to some elliptic equations with nonlinear boundary conditions*, Adv. Diff. Eqns 1 (1) (1996) 91–100.

[6] L. Damascelli, M. Grossi and F. Pacella; *Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle*, Annls Inst. H. Poincaré Analyse Non linéaire 16 (1999) 631–652.

[7] I. Ekeland; *On the variational principle*, J. Math. Anal. Appl. 17 (1974) 324–353.

[8] C. Flores, M. del Pino; *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains*, Comm. Partial Diff. Eqns. 26 (11–12) (2001) 2189–2210.

[9] D. G. de Figueiredo, J. P. Gossez and P. Ubilla; *Local superlinearity and sublinearity for indefinite semilinear elliptic problems*, J. Funct. Anal. 199 (2003) 452–467.
[10] J. Garcia-Azorero, I. Peral and J. D. Rossi; *A convex-concave problem with a nonlinear boundary condition*, J. Diff. Eqns. 198 (2004) 91–128.
[11] B. Hu; *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, Diff. Integral Eqns. 7 (2) (1994), 301–313.
[12] W. M. Ni and I. Takagi; *On the shape of least energy solution to a Neumann problem*, Comm. Pure Appl. Math. 44 (1991) 819–851.
[13] T. Ouyang and J. Shi; *Exact multiplicity of positive solutions for a class of semilinear problem II*, J. Diff. Eqns. 158 (1999) 94–151.
[14] D. Pierrotti and S. Terracini; *On a Neumann problem with critical exponent and critical nonlinearity on the boundary*, Comm. Partial Diff. Eqns. 20 (7–8) (1995) 1155–1187.
[15] G. Tarantello; *On nonhomogeneous elliptic involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (1992) 281–304.
[16] S. Terraccini; *Symmetry properties of positive solutions to some elliptic equations with nonlinear boundary conditions*, Diff. Integral Eqns. 8 (1995) 1911–1922.
[17] M. Tang; *Exact multiplicity for semilinear elliptic Dirichlet problems involving concave and convex nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 705–717.
[18] T. F. Wu; *On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function*, J. Math. Anal. Appl. 318 (2006) 253–270.

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