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ON EXACT SOLUTIONS OF A DEGENERATE QUASILINEAR WAVE EQUATION WITH SOURCE TERM

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Abstract. In this paper, we construct exact solutions of the Cauchy problem of degenerate quasilinear wave equation of Kirchhoff type, and study the effect of the nonlinear terms on the existence of solutions. We construct solutions that exist globally for some initial data, and that blow up in a finite time for some other initial data.

1. Introduction

In this paper, we construct exact solutions of the following initial-value problem of a quasilinear wave equation with nonlinear source term, and study their asymptotic behaviour concerning the time variable,

$$\begin{gathered} (|u'|^{l-2}u')' - M(\|u_x\|_2^2)u_{xx} = \gamma |u|^{p-1}u, \quad \text{in } \mathbb{R}\times[0,+\infty[,\\ u(x,0)=u_0(x), \quad u'(x,0)=u_1(x)\text{in } \mathbb{R}. \end{gathered} \tag{1.1}$$

where $M(r) = r^{\frac{p-1}{2}}$, $\|u_x\|_2^2 = \int_{-\infty}^{\infty} |u_x(x,t)|^2\,dx$, $l > 2$, $p > l-1$ and $\gamma > 0$ are constants.

For problem (1.1), when $l > 2$ and $M \equiv 1$ without source term and with nonlinear dissipation, Benaissa and Mimouni [3] determined suitable relations between l and p, so that the energy decays exponentially or alternatively polynomially. More precisely, they showed that the energy of the solutions decays with exponentially if $l+1 \geq p$ and decays polynomially if $l+1 < p$.

For problem (1.1) (in the case of bounded domain), when $l > 2$ and M is not a constant function, with nonlinear dissipation, Benaissa and Messaoudi [2] have investigated the blowup of solutions. They show that, for suitably chosen initial data and a relation between l and p, any classical solution blows up in finite time.

When $l = 2$, the equation without a source term is often called the wave equation of Kirchhoff type which has been introduced for studying nonlinear vibrations of an elastic string by Kirchhoff [9]. The existence of local and global solutions in Sobolev and Gevrey classes was investigated by many authors; see for example [7, 6, 8].

When $l = 2$, Ebihara, Hoshino and Kurokiba [4] constructed one of the solutions for (1.1) and studied their behavior in t. They have shown that there exists a solution which blows up at a finite time under some initial condition and in the other

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case there exists a global solution which decays with the order $O(t^{-\frac{2}{p-1}})(t \to \infty)$. We think that the interaction of the term $(|u'|^{l-2}u')'$ $(l > 2)$ and the source term $|u|^{p-1}u$ and the velocity $M(r)$ have an effect on the result of [4].

We construct the solution of the form $u(x,t) = v(x)\varphi(t)$ for (1.1) when $M(r) = r^{\frac{p-1}{2}}, p > 1, l > 2$ and $p + 3 = 2l$ and we study the behavior of the solutions as the time t increases.

2. Separation of variables

Let $u(x,t) = v(x)\varphi(t)$ and substitute in (1.1). Then (1.1) is changed to

$$\begin{gathered}
|v|^{l-2}v(|\varphi'|^{l-2}\varphi')' - \Big(\int_{-\infty}^{\infty} |v_x(x,t)|^2\,dx\Big)^{\frac{p-1}{2}} \\
\times |\varphi|^{p-1}\varphi v_{xx} - \gamma|v|^{p-1}v|\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}, \quad t > 0, \\
u(x,0) = v(x)\varphi(0) = \varphi_0 v(x), \quad x \in \mathbb{R}, \\
u_t(x,0) = v(x)\varphi_t(0) = \varphi_1 v(x), \quad x \in \mathbb{R}.
\end{gathered} \tag{2.1}$$

Here we consider the case $\varphi(t) > 0$, so that the first equation of (2.1) is equivalent to

$$\frac{\alpha^{p-1}v_{xx} + \gamma|v|^{p-1}v}{|v|^{l-2}v} = \frac{(|\varphi'|^{l-2}\varphi')'}{\varphi^p} = \lambda, \tag{2.2}$$

where $\alpha^2 = \int_{-\infty}^{\infty} |v_x(x)|^2\,dx$ and λ is a positive constant. Therefore, we obtain two problems from (2.2): First problem

$$\begin{gathered}
\alpha^{p-1}v_{xx} - \lambda|v|^{l-2}v + \gamma|v|^{p-1}v = 0, \quad x \in \mathbb{R}, \\
\alpha^2 = \int_{-\infty}^{\infty} |v_x|^2\,dx < +\infty\,,
\end{gathered} \tag{2.3}$$

and second problem

$$\begin{gathered}
(|\varphi'|^{l-2}\varphi')' = \lambda\varphi^p, \quad t \geq 0, \\
\varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1, \\
\varphi(t) \geq 0, \quad t \geq 0.
\end{gathered} \tag{2.4}$$

3. First Problem

In this section, we construct a positive solution $v(x)$ of (2.3) with $\lim_{|x|\to\infty} v(x) = 0$. For this purpose we study the problem

$$\begin{gathered}
\alpha^{p-1}v_{xx} - \lambda|v|^{l-2}v + \gamma|v|^{p-1}v = 0, \quad x \in \mathbb{R}, \\
v(0) = A \quad \text{(a positive constant)}, \\
v_x(0) = 0, \\
\lim_{|x|\to\infty} v(x) = 0, \\
\frac{\alpha^2}{2} = \int_0^{\infty} |v_x(x)|^2\,dx < +\infty,
\end{gathered} \tag{3.1}$$

where α is a fixed positive number. If $v(x)$ is a solution of (3.1), then we can solve (2.3) by setting $v(-x) = v(x)$ for $x > 0$, because of the second equation in (3.1).

Multiplying the first equation of (3.1) by $2v_x$ and integrating from 0 to x, we obtain

$$\alpha^{p-1}v_x^2 = \frac{2\lambda}{l}v^l - \frac{2\gamma}{p+1}v^{p+1} - \frac{2\lambda}{l}A^l + \frac{2\gamma}{p+1}A^{p+1}. \tag{3.2}$$

If we choose $A > 0$ such that

$$A = \Big(\frac{(p+1)\lambda}{\gamma\, l}\Big)^{\frac{1}{p+1-l}}, \tag{3.3}$$

then (3.2) implies

$$\alpha^{\frac{p-1}{2}}v_x = \mp\sqrt{\frac{2\lambda}{l}v^l - \frac{2\gamma}{p+1}v^{p+1}}. \tag{3.4}$$

Here we consider the case where v is positive and $v_x < 0$, so that we treat the following equation which is derived from (3.4):

$$\frac{v_x}{v^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}v^{p+1-l}}} = -\alpha^{-\frac{p-1}{2}}. \tag{3.5}$$

If we integrate (3.5) from c to v, then we obtain

$$\int_v^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}z^{p+1-l}}} = \alpha^{-\frac{p-1}{2}}x. \tag{3.6}$$

If there exists $x^* \in (0,\infty)$ such that $v(x^*) = 0$, then we get

$$\int_0^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}z^{p+1-l}}} = \alpha^{-\frac{p-1}{2}}x^*.$$

But one can easily show $\int_0^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}z^{p+1-l}}} = \infty$ with use of (3.3) and $\alpha^{-\frac{p-1}{2}}x^* < +\infty$, thus $v(x)$ is monotone decreasing and $v(x) > 0$. And if $\lim_{x\to\infty} v(x) = k > 0$, then from (3.6), we obtain

$$\int_k^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}z^{p+1-l}}} = \lim_{x\to\infty}\alpha^{-\frac{p-1}{2}}x \quad (=\infty).$$

However, $\int_k^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}z^{p+1-l}}}$ is finite, so we deduce that $\lim_{x\to\infty} v(x) = 0$.

By putting $y = \sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}z^{p+1-l}}$ in order to calculate the left hand side of (3.6), we see that

$$I = \int_v^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}z^{p+1-l}}} = \int_0^{\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}z^{p+1-l}}} \frac{2^{\frac{2p-l}{2(p+1-l)}}\big(\frac{\gamma}{(p+1)}\big)^{\frac{l-2}{2(p+1-l)}}}{(p+1-l)(\frac{2\lambda}{l} - y^2)^{\frac{2p-l}{2(p+1-l)}}}\,dy.$$

We suppose that $\frac{2p-l}{2(p+1-l)} = \frac{3}{2}$. We obtain

$$I = \int_0^{\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}z^{l-2}}} \frac{2^{\frac{3}{2}}(\frac{\gamma}{(p+1)})^{\frac{1}{2}}}{(l-2)(\frac{2\lambda}{l} - y^2)^{\frac{3}{2}}}\,dy = \frac{l}{\lambda(l-2)}\frac{\sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1}v^{l-2}}}{v^{\frac{l-2}{2}}}.$$

Then (3.6) becomes

$$\frac{l}{\lambda(l-2)}\frac{\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}v^{l-2}}}{v^{\frac{l-2}{2}}}=\alpha^{-\frac{p-1}{2}}x,$$

which implies

$$v(x)=\Big(\frac{l/(2\lambda)}{\big(\frac{\lambda(l-2)}{l}\big)^2\alpha^{-(p-1)}x^2+\frac{2\gamma}{p+1}}\Big)^{1/(l-2)}. \tag{3.7}$$

We put $\mu=\big(\frac{\lambda(m-2)}{m}\big)^2\alpha^{-(p-1)}$ and differentiate (3.7), to obtain

$$v_x(x)=-\frac{2}{m-2}\big(\frac{l}{2\lambda}\big)^{\frac{1}{l-2}}\mu\frac{x}{\big(\mu\ x^2+\frac{2\gamma}{p+1}\big)^{1+\frac{1}{l-2}}}.$$

Thus we have

$$\int_0^\infty|v_x(x)|^2\,dx=\Big(\frac{2}{l-2}\Big(\frac{m}{2\lambda}\Big)^{\frac{1}{l-2}}\Big)^2\mu^2\int_0^\infty\frac{x^2}{\big(\mu\ x^2+\frac{2\gamma}{p+1}\big)^{2+\frac{2}{l-2}}}\,dx.$$

When we put $H=\int_0^\infty\frac{x^2}{\big(\mu\ x^2+\frac{2\gamma}{p+1}\big)^{2+\frac{2}{l-2}}}\,dx$ and let $z=\sqrt{\mu/a}\ x$, where $a=\frac{2\gamma}{p+1}$, we have

$$\begin{aligned}H&=\frac{1}{a^{\frac12+\frac{2}{l-2}}\mu^{\frac32}}\int_0^\infty\frac{z^2}{(1+z^2)^{2+\frac{2}{l-2}}}\,dz\\&=\frac{l-2}{2l}\frac{1}{a^{\frac12+\frac{2}{l-2}}\mu^{\frac32}}\int_0^\infty\frac{1}{(1+z^2)^{1+\frac{2}{l-2}}}\,dz\\&=\frac{l-2}{2l}\frac{1}{a^{\frac12+\frac{2}{l-2}}\mu^{\frac32}}K\quad(K<+\infty).\end{aligned}$$

Therefore,

$$\int_0^\infty|v_x(x)|^2\,dx=\Big(\frac{2}{l-2}\Big(\frac{m}{2\lambda}\Big)^{\frac{1}{l-2}}\Big)^2\sqrt{\mu}\frac{l-2}{2l}\frac{1}{a^{\frac12+\frac{2}{l-2}}}K\,.$$

If we take α such as

$$\alpha=\big\{\frac{4\sqrt2}{\sqrt l}\big(\frac{l(l-1)}{2}\frac{\gamma}{\lambda}\big)^{\frac12+\frac{2}{l-2}}K\big\}^{1/l}\lambda^{\frac{3}{2l}},$$

then (3.1)(4) is satisfied. Therefore, we have the following theorem.

Theorem 3.1. *If $l>2$ and $p=2l-3$, then the solution $v(x)$ of* (3.1) *such that $v(0)=\big(\frac{2(l-2)}{l}\frac{\lambda}{\gamma}\big)^{\frac{1}{l-2}}$ and $v_x(0)=0$, is given by*

$$v(x)=\Big(\frac{1/(2\lambda)}{\big(\frac{\lambda(l-2)}{l}\big)^2\alpha^{-(p-1)}x^2+\frac{2\gamma}{p+1}}\Big)^{1/(l-2)},$$

where

$$\alpha=\big\{\frac{4\sqrt2}{\sqrt l}\big(\frac{l(l-1)}{2}\frac{\gamma}{\lambda}\big)^{\frac12+\frac{2}{l-2}}K\big\}^{1/l}\lambda^{\frac{3}{2l}},\quad K=\int_0^\infty\frac{1}{(1+z^2)^{1+\frac{2}{l-2}}}.$$

This solution can be extended to a solution of (2.3).

4. Second Problem

In this section, we consider Problem

$$
\begin{gathered}
(|\varphi'|^{l-2}\varphi')' = \lambda\varphi^p, \quad t \geq 0, \\
\varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1, \\
\varphi(t) \geq 0, \quad t \geq 0
\end{gathered} \tag{4.1}
$$

where λ is the constant from the previous section. If we multiply the equation (4.1)(1) by $\varphi_t(t)$ and integrate from 0 to t as in section 2, then we have

$$
\varphi_t(t) = \pm\Big(\frac{l\lambda}{(l-1)(p+1)}v^{p+1} + |\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1}\Big)^{1/l}, \tag{4.2}
$$

because of (4.1)(2) and (4.1)(3). In order that we construct the solution of (4.1) from (4.2), the following condition has to be satisfied

$$
G(\varphi) \equiv \frac{l\lambda}{(l-1)(p+1)}v^{p+1} + |\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} \geq 0.
$$

We consider the following three cases.

Case 1. If $\varphi_0 \geq 0$ and $\varphi_1 > 0$, then $G(\varphi(0)) = G(\varphi_0) > 0$. Hence $\varphi_t(t) = (G(\varphi(t)))^{1/l} > 0$ for sufficiently small $t > 0$ since $(G(\varphi(t)))^{1/l}$ is monotone increasing function, we see that $\varphi_t > 0$ for all $t > 0$ where $\varphi(t)$ exist. Since one can show $\int_{\varphi_0}^{\infty}(1\backslash(G(\varphi))^{1/l})\,d\varphi < +\infty$, we see that $\varphi \to +\infty$ as $t \to T^*$, where $T^* = \int_{\varphi_0}^{\infty}(1\backslash(G(\varphi))^{1/l})\,d\varphi$.

Case 2. In the case of $\varphi_0 \geq 0$ and $|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} = 0$, by solving (4.2), we obtain

$$
\varphi_\pm(t) = \Big(\varphi_0^{-\frac{p+1-l}{l}} \mp \frac{(p+1-l)\big(\frac{\lambda l}{(l-1)(p+1)}\big)^{1/l}}{l}t\Big)^{-\frac{l}{p+1-l}},
$$

with double signs in same order. Obviously φ_+ decays with the order $\mathcal{O}(t)^{-\frac{l}{p+1-l}}$ as $t \to +\infty$ and φ_- blows up at $\varphi_0^{-\frac{p+1-l}{l}}\frac{l}{p+1-l}\big(\frac{(l-1)(p+1)}{\lambda l}\big)^{1/l}$.

Case 3. If $\varphi_0 \geq 0, \varphi_1 < 0$ and $|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} > 0$, we have $\varphi_t(t) = -(G(\varphi(t)))^{1/l} < 0$ locally. Then, since $G(\varphi(t))$ is monotone increasing function, we see that $\varphi_t(t) < 0$ for all $t > 0$ where $\varphi(t)$ exist. From (4.2),

$$
\int_{\varphi_0}^{0} -\frac{d\varphi}{(G(\varphi))^{1/l}} \equiv T^{**} < +\infty.
$$

Therefore, T^{**} exists such that $\varphi(t) \to 0$ as $t \to T^{**}$. From (4.2), we can get

$$
\int_{\varphi(t)}^{0} -\frac{1}{T^{**}-t}\frac{d\varphi}{(G(\varphi))^{1/l}} = 1. \tag{4.3}
$$

Let $s = (T^{**}-t)r$ in (4.3) and let $t \to T^{**}$, we have

$$
\lim_{t\to T^{**}} \varphi(t)(T^{**}-t)^{-1} = \Big(|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1}\Big)^{1/l}. \tag{4.4}
$$

From (4.2) and (4.4), we have

$$
\lim_{t\to T^{**}} \varphi(t)\varphi'(t)(T^{**}-t)^{-1} = -\Big(|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1}\Big)^{2/l}. \tag{4.5}
$$

Then we have the following theorem.

Theorem 4.1. **(1)** *If $\varphi_0 \geq 0$ and $\varphi_1 > 0$, then by putting*

$$T^* = \int_{\varphi_0}^{\infty} (1\backslash(G(\varphi))^{1/l})\, d\varphi,$$

the solution $\varphi(t)$ of (4.1) blows up at $t = T^*$, that is $\varphi(t) \to +\infty$ as $t \to T^*$.
(2) *If $\varphi_0 \geq 0$ and $|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} = 0$, then the solution $\varphi(t)$ of (4.1) is given by*

$$\varphi_{\pm}(t) = \Big(\varphi_0^{-\frac{p+1-l}{l}} \mp \frac{(p+1-l)\big(\frac{\lambda l}{(l-1)(p+1)}\big)^{1/l}}{l} t\Big)^{-\frac{l}{p+1-l}},$$

with double signs in the same order.
(3) *If $\varphi_0 \geq 0, \varphi_1 < 0$ and $|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} > 0$, then by putting*

$$\int_{\varphi_0}^{0} -\frac{d\varphi}{(G(\varphi))^{1/l}} \equiv T^{**},$$

the solution $\varphi(t)$ vanishes at $t = T^{**}$ and satisfies (4.4) and (4.5).

Depending on the choice of the constants φ_i ($i = 0, 1$), we obtain a solution which blows up in a finite time and another case we have a solution which decays with order $O(t)^{-\frac{l}{p+1-l}}$ as $t \to \infty$.

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