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PERIODIC SOLUTIONS FOR SECOND-ORDER HAMILTONIAN SYSTEMS WITH THE P-LAPLACIAN

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ABSTRACT. In this paper, we investigate the periodic solutions of Hamiltonian system with the p-Laplacian. By using Mountain Pass Theorem the existence of at least one periodic solution is obtained, Furthermore, under suitable assumptions, we obtain the existence of infinitely many solutions via Z_2-symmetric version of the Mountain Pass Theorem.

1. INTRODUCTION

There has been published an extensive literature related to the existence of the periodic solutions of second-order differential equations (systems) recently; see for example the refeences in this article and the references cited therein. In [3, 4] the authors considered the system

$$\begin{gathered} \ddot{u}(t) + \nabla H(t, u(t)) = 0 \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{gathered} \tag{1.1}$$

and obtained multiple solutions under the following assumption on the potential H: There exist $R > 0$, $\theta \in]0, 1/2[$ such that

$$0 < H(t, u) \leq \theta \nabla_u H(t, u)u \tag{1.2}$$

for each $t \in [0, T]$, for each $u \in R^k, |u| \geq R$. When $H(t, u(t)) = b(t)V(u(t))$, $b \in C([0, T], R)$ and b changes its sign, there are many existence results of nontrivial periodic solutions for problem (1.1) (see [1, 5, 9]). All of them assumed that

$$\int_0^T b(t)dt \neq 0.$$

Ding [8] established the existence of periodic solutions for

$$\int_0^T b(t)dt = 0 \tag{1.3}$$

under the so-called global Ambrosetti-Rabinowitz condition, that is, there exists a constant $\theta \in]0, \frac{1}{2}[$ such that (1.2) holds for all $t \in [0, T]$ and $u \in \mathbb{R}^N \setminus \{0\}$. In

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the case of (1.3), Chen and Long [6] obtained the existence of one solution by the Saddle Point Theorem. Very recently, Tang and Wu [17] generalized the results in [6] and got the following theorem with the aid of generalized Mountain Pass Theorem.

Theorem 1.1. *Suppose $\mu > 2$, $b \in C([0,T], R)$ satisfying $\int_0^T b(t)dt = 0$, $b \not\equiv 0$ and $H : [0,T] \times \mathbb{R}^N \to R$, $H(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, such that*

- (A1) *$\int_0^T H(t,x)dt \geq 0$ for all $x \in \mathbb{R}^N$.*
- (A2) *There exist $g \in L^1(0,T)$, $\alpha_0 \in (0, \omega^2/2)$ and $r_0 > 0$ such that $|\nabla H(t,x)| \leq g(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.*
- (A3) *$|H(t,x)| \leq \alpha_0 |x|^2$ for all $|x| \leq r_0$ and a.e. $t \in [0,T]$, where $\omega = 2\pi/T$.*

Then the problem

$$\begin{gathered}\ddot{u}(t) + b(t)|u(t)|^{\mu-2}u(t) + \nabla H(t,u(t)) = 0 \quad \text{a.e. } t \in [0,T],\\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0,\end{gathered}$$

has at least one nonzero solution.

However, in [1-14, 16, 17], the highest order derivatives are linear. But so far, few papers discuss periodic solutions for second order system with the p-Laplacian. On the other hand, it is well known that the study of the existence for quasilinear differential equations is very important. Motivated by the above works, we consider the existence of solutions for the following second-order Hamiltonian system with p-Laplacian:

$$\begin{gathered}\frac{d}{dt}(\Phi_p(\dot{u}(t))) + B(t)\Phi_\mu(u(t)) + \nabla H(t,u(t)) = 0, \quad t \in [0,T],\\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0,\end{gathered} \tag{1.4}$$

where $p > 1$, $\Phi_p(u) := (|u_1|^{p-2}u_1, \dots, |u_N|^{p-2}u_N)$, $u = (u_1, \dots, u_N)$, $\mu > p$, $T > 0$, $H : [0,T] \times \mathbb{R}^N \to R$, $H(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, $\nabla H(t,x) = (\partial H/\partial x)(t,x)$.

$$B(t) = \begin{bmatrix} b_1(t) & & \\ & \ddots & \\ & & b_N(t) \end{bmatrix},$$

$b_i \in C([0,T], R)$, $b_i \not\equiv 0$, $i = 1, 2, \dots, N$. For $p = 2$, $\frac{d}{dt}(\Phi_p(\dot{u}(t))) = \ddot{u}(t)$. To apply critical point theory to (1.2), it is necessary to check that corresponding functional verifies the Palais-Smale condition ((PS)-condition). Taking the quasi-linear into consideration, uniformly convex of L^p and related differential inequalities have to be used (see section 2).

Using the Mountain Pass Theorem, the existence of at least one solution is obtained. Furthermore, under the hypothesis of eveness of the functional, the existence of infinitely many solutions is obtained by using Z_2 version of the Mountain Pass Theorem.

The following lemmas are taken from [16], and will be useful in the proofs of our main results.

Lemma 1.2. *Let E be a real Banach space with $E = V \oplus X$, where V is finite dimensional. Suppose $I \in C^1(E,R)$ satisfies (PS), and*

(I1) *There are constants* $\rho, \alpha > 0$ *such that* $I_{\partial B_\rho \cap X} \geq \alpha$, *and*
(I2) *There is an* $e \in \partial B_1 \cap X$ *and* $R > \rho$ *such that if* $Q \equiv (\overline{B}_R \cap V) \oplus \{re : 0 < r < R\}$, *then* $I_{\partial Q} \leq 0$.

Then I *possesses a critical value* $c \geq \alpha$ *which can be characterized as*

$$c \equiv \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)),$$

where $\Gamma = \{h \in C(\overline{Q}, E) : h = id \text{ on } \partial Q\}$, *where* id *denotes the identity operator.*

Lemma 1.3. *Let* E *be an infinite dimensional real Banach space and let* $I \in C^1(X, R)$ *be even, satisfying the (PS) condition and* $I(0) = 0$. *If* $E = V \oplus X$, *where* V *is finite dimensional, and* I *satisfies*

(J1) *there exist constants* $\rho, \alpha > 0$ *such that* $I|_{\partial B_\rho \cap X} \geq \alpha$ *and*
(J2) *for each finite dimensional subspace* $V_1 \subset E$, *the set* $\{x \in V_1 : I(x) \geq 0\}$ *is bounded.*

Then I *has an unbounded sequence of critical values.*

2. Preliminaries

In the proof of main results, we will need the following preliminary results. For convenience, let

$$W_T^{1,p}([0,T]) = \big\{u : [0,T] \mapsto \mathbb{R}^N : u \text{ is abs. cont., } u(0) = u(T),\ \dot{u} \in L^p(0,T;\mathbb{R}^N)\big\}$$

be a Sobolev space with the norm

$$\|u\|_{W_T^{1,p}} = \Big(\int_0^T |u(t)|^p + |\dot{u}(t)|^p dt\Big)^{1/p}.$$

For any $x \in W_T^{1,p}([0,T])$, define $A : W_T^{1,p}([0,T]) \to R$ by

$$A(x) = \int_0^T \sum_{i=1}^N |\dot{x}_i(t)|^p dt = \sum_{i=1}^N \|\dot{x}_i\|_{L^p}^p. \tag{2.1}$$

Let $A_i(x) = \|\dot{x}_i\|_{L^p}^p$, so $A(x) = \sum_{i=1}^N A_i(x)$. Clearly, A is convex. Now we claim that A is lower semi-continuous on $W_T^{1,p}([0,T])$. So A is weakly lower semi-continuous on $W_T^{1,p}([0,T])$ (see [14, Theorem 1.2]). In fact, we only need to show A_i, $i \in \{1, 2, \dots, N\}$ is weakly lower semi-continuous on $W_T^{1,p}([0,T])$. Let $x_n \to x$ in $W_T^{1,p}([0,T])$, from which it follows $\|\dot{x}_n - \dot{x}\|_{L^p} \to 0$. By

$$\|x + y\|_{L^p} \leq \|x\|_{L^p} + \|y\|_{L^p} \quad \forall x, y \in L^p([0,T]),$$

we have

$$A_i(x_n) - A_i(x) = \|\dot{x}_{ni}\|_{L^p}^p - \|\dot{x}_i\|_{L^p}^p \leq (\|\dot{x}_{ni} - \dot{x}_i\|_{L^p} + \|\dot{x}_i\|_{L^p})^p - \|\dot{x}_i\|_{L^p}^p \to 0$$

as $n \to \infty$, and

$$\begin{aligned} A_i(x_n) - A_i(x) &= \|\dot{x}_{ni}\|_{L^p}^p - \|\dot{x}_i\|_{L^p}^p \\ &\geq \Big(\Big|\|\frac{\dot{x}_{ni} + \dot{x}_i}{2}\|_{L^p} - \|\frac{\dot{x}_{ni} - \dot{x}_i}{2}\|_{L^p}\Big|\Big)^p \\ &\quad - \Big(\|\frac{\dot{x}_{ni} + \dot{x}_i}{2}\|_{L^p} - \|\frac{-\dot{x}_{ni} + \dot{x}_i}{2}\|_{L^p}\Big)^p \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore,

$$\lim_{n \to \infty} A_i(x_n) = A_i(x).$$

Lemma 2.1 ([7]). *For the space $L^p([0,T])$, the following inequalities between the norms of two arbitrary elements x and y of the space are valid (here q is the conjugate index, $q = p/(p-1)$):*

$$\|\frac{x+y}{2}\|^p_{L^p} + \|\frac{x-y}{2}\|^p_{L^p} \le \frac{1}{2}(\|x\|^p_{L^p} + \|y\|^p_{L^p}) \quad \text{for } p \ge 2, \tag{2.2}$$

$$\|\frac{x+y}{2}\|^q_{L^p} + \|\frac{x-y}{2}\|^q_{L^p} \le \big[\frac{1}{2}(\|x\|^p_{L^p} + \|y\|^p_{L^p})\big]^{q-1} \quad \text{for } 1 < p < 2. \tag{2.3}$$

Proposition 2.2 ([15]). *Let A be defined as in (2.1). Suppose that $(x_n)_{n\in N}$ is a sequence in $W^{1,p}_T([0,T])$ satisfying $x_n \rightharpoonup x$ and the following inequality*

$$\limsup_{n\to\infty}\langle DA(x_n), x_n - x\rangle \le 0. \tag{2.4}$$

Then $x_n \to x$ strongly in $W^{1,p}_T([0,T])$.

Proof. Let $x_n \rightharpoonup x$ in $W^{1,p}_T([0,T])$ and satisfy (2.4). So (x_n) is bounded: $\|x\|_{W^{1,p}_T} \le M$, clearly, $A_i(x_n)$ is bounded. For a subsequence, $A_i(x_n) \to c_i$. So $A(x_n) \to c = \sum_{i=1}^N c_i$. By weakly lower semi-continuous of A:

$$A(x) \le \liminf_{n\to\infty} A(x_n) = c.$$

On the other hand, since A is convex and lower semi-continuous, its graphic lies over the tangent hyper-plane at x_n,

$$A(x) \ge A(x_n) + \langle DA(x_n), x - x_n\rangle.$$

From (2.4), we deduce that $A(x) \ge c$, then $A(x) = c$.

Also we have $\frac{x+x_n}{2} \rightharpoonup x$, and again by weakly lower semi-continuous of A, A_i,

$$c = A(x) \le \liminf_{n\to\infty} A\left(\frac{x+x_n}{2}\right), \quad c_i = A(x_i) \le \liminf_{n\to\infty} A_i\left(\frac{x+x_n}{2}\right). \tag{2.5}$$

If we suppose that (x_n) does not converge strongly to x, then there exist $\varepsilon > 0$ and subsequence (x_{nk}) that satisfy $\|x_{nk} - x\|_{W^{1,p}_T} \ge \varepsilon$. Since (x_{n_k}) converges uniformly to x in $C([0,T])$, $\|\dot{x}_{nk} - \dot{x}\|_{L^p} \ge \varepsilon$. So there exists $j \in \{1, 2, \dots, N\}$ and $\widetilde{\varepsilon} > 0$, $\|\dot{x}_{nkj} - \dot{x}_j\|_{L^p} \ge \widetilde{\varepsilon}$. From (2.2),

$$\begin{aligned}
\limsup_{n_k\to\infty} A\big(\frac{x+x_{nk}}{2}\big) &\le \limsup_{n_k\to\infty} \frac{1}{2}A(x) + \frac{1}{2}A(x_{nk}) - A\big(\frac{x-x_{nk}}{2}\big)\\
&\le c - \limsup_{n_k\to\infty}\sum_{i=1}^N \|\frac{\dot{x}_{nki} - \dot{x}_i}{2}\|^p_{L^p}\\
&\le c - \frac{1}{2^p}\widetilde{\varepsilon}^p,
\end{aligned}$$

which contradicts (2.5). For $1 < p < 2$, by (2.3) we have

$$\begin{aligned}
\limsup_{n_k\to\infty}\|\frac{\dot{x}_j + \dot{x}_{nkj}}{2}\|^q_{L^p} &\le \limsup_{n_k\to\infty}\Big[\frac{1}{2}\|\dot{x}_j\|^p_{L^p} + \frac{1}{2}\|\dot{x}_{nkj}\|^p_{L^p}\Big]^{q-1} - \|\frac{\dot{x}_j - \dot{x}_{nkj}}{2}\|^q_{L^p}\\
&\le c_j^{q-1} - \frac{\widetilde{\varepsilon}^q}{2^q}.
\end{aligned} \tag{2.6}$$

By (2.5) and (2.6), we have

$$\frac{\widetilde{\varepsilon}^q}{2^q} + c_j^{\frac{q}{p}} \le c_j^{q-1},$$

that is

$$\frac{\widetilde{\varepsilon}^{pq}}{2^{pq}} + c_j^q \le \Big(\frac{\widetilde{\varepsilon}^q}{2^q} + c_j^{\frac{q}{p}}\Big)^p \le c_j^q,$$

a contraction. The proof is complete. □

3. Main results

It is well known that u is a T-periodic solution of system (1.4) if and only if u is a critical point in $W_T^{1,p}([0,T])$ of functional φ, where

$$\varphi(u) = \frac{1}{p}\int_0^T \sum_{i=1}^N |\dot{u}_i(t)|^p dt - \frac{1}{\mu}\int_0^T \sum_{i=1}^N b_i(t)|u_i(t)|^\mu dt - \int_0^T H(t,u(t))dt \tag{3.1}$$

for $u \in W_T^{1,p}([0,T])$. $\varphi : W_T^{1,p} \to R$ is well defined and C^1. Its derivative is given by

$$\begin{aligned}\langle \varphi'(u), v\rangle = &\int_0^T \sum_{i=1}^N \Phi_p(\dot{u}_i(t))\dot{v}_i(t)dt - \int_0^T \sum_{i=1}^N b_i(t)\Phi_\mu(u_i(t))v_i(t)dt \\ &- \int_0^T (\nabla H(t,u(t)), v(t))dt,\end{aligned} \tag{3.2}$$

where $(\cdot,\cdot)$ is the usual inner product of $\mathbb{R}^N$. It follows from Sobolev's inequality that

$$\|u\|_\infty \le C\|u\|_{W_T^{1,p}} \tag{3.3}$$

for all $u \in W_T^{1,p}([0,T])$, where $\|u\|_\infty = \max_{0\le t\le T}|u(t)|$.

Theorem 3.1. *Suppose $p \ge 2, b_i \in C([0,T],R)$ and $\int_0^T b_i(t)dt = 0$, $b_i \not\equiv 0$, $i = 1,2,\dots,N$. $\int_0^T H(t,x)dt \ge 0$ for all $x \in \mathbb{R}^N$. Assume that there exist*

$$g, h \in L^1([0,T]), \quad \|h\|_{L^1} \le [pT^{\frac{p}{q}} \max\{N^{\frac{p}{2}-1}, 1\}]^{-1}, \quad \theta \in [0, p-1), \quad r > 0$$

such that

$$|\nabla H(t,x)| \le g(t)|x|^\theta \tag{3.4}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, and

$$|H(t,x)| \le h(t)|x|^p \tag{3.5}$$

for $|x| \le r$ and a.e. $t \in [0,T]$. Then system (1.4) has at least one nonzero solution.

Proof. The proof is divided into three steps.
Step 1. We claim that the functional φ satisfies the Palais-Smale condition, that is, (u_n) has a convergent subsequence whenever it satisfies $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded.

First we prove that (u_n) is a bounded sequence in $W_T^{1,p}([0,T])$. Suppose that (u_n) is unbounded. Passing to a subsequence, we may assume if necessary, that $\|u_n\|_{W_T^{1,p}} \to \infty$ as $n \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|_{W_T^{1,p}}}$. Then (v_n) is bounded so that it has a subsequence, say (v_n), which weakly converges to v_0. By [14, Proposition

1.2], (v_n) converges to v_0 uniformly on $[0,T]$. Hence one has $\overline{v}_n \to \overline{v}_0$, where $\overline{v} = \frac{1}{T}\int_0^T v(s)ds$. It follows from (3.3) and (3.4) that

$$\begin{aligned}
&\mu \int_0^T H(t,u_n(t))dt - \int_0^T (\nabla H(t,u_n(t)),u_n(t))dt \\
&\le \mu \int_0^T \Big(\int_0^1 (\nabla H(t,su_n(t)),u_n(t))\,ds\Big)dt + \mu\int_0^T H(t,0)dt \\
&\quad + \int_0^T |\nabla H(t,u_n(t))||u_n(t)|dt \\
&\le \frac{\mu}{\theta+1}\int_0^T g(t)|u_n(t)|^{\theta+1}dt + \mu\int_0^T H(t,0)dt + \int_0^T g(t)|u_n(t)|^{\theta+1}dt \\
&\le \big(\frac{\mu}{\theta+1}+1\big)\|g\|_{L^1}C^{\theta+1}\|u_n\|^{\theta+1}_{W_T^{1,p}} + \mu\int_0^T H(t,0)dt.
\end{aligned} \tag{3.6}$$

Then by (3.1), (3.2), (3.6), we have

$$\begin{aligned}
&\big(\frac{\mu}{p}-1\big)\int_0^T \sum_{i=1}^N |\dot{u}_{ni}(t)|^p dt \\
&= \mu\varphi(u_n) - \langle\varphi'(u_n),u_n\rangle + \int_0^T \mu H(t,u_n)dt - \int_0^T (\nabla H(t,u_n),u_n)dt \\
&\le \mu\varphi(u_n) - \langle\varphi'(u_n),u_n\rangle + \big(\frac{\mu}{\theta+1}+1\big)\|g\|_{L^1}C^{\theta+1}\|u_n\|^{\theta+1}_{W_T^{1,p}} + \mu\int_0^T H(t,0)dt,
\end{aligned}$$

which implies $\|\dot{v}_{ni}\|_{L^p} \to 0$ as $n\to\infty$, $i=1,2,\dots,N$. So $\widetilde{v}_n \to 0$ in $W_T^{1,p}([0,T])$ as $n\to\infty$, thus $v_n \to \overline{v}_0$ as $n\to\infty$, where $\widetilde{v} = v - \overline{v}$. Hence $v_0 = \overline{v}_0$ and $\|v_0\|_{W_T^{1,p}} = 1$. On the other hand, from (3.2),

$$\begin{aligned}
&\Big|\int_0^T (B(t)\Phi_\mu(v_n(t)),v(t))dt\Big| \\
&= \Big|\int_0^T \sum_{i=1}^N b_i(t)|v_{ni}(t)|^{\mu-2}v_{ni}(t)v_i(t)dt\Big| \\
&\le \|u_n\|^{1-\mu}_{W_T^{1,p}}|\langle\varphi'(u_n),v\rangle| + \|u_n\|^{p-\mu}_{W_T^{1,p}}\Big|\int_0^T \sum_{i=1}^N |\dot{v}_{ni}(t)|^{p-2}\dot{v}_{ni}(t)\dot{v}_i(t)dt\Big| \\
&\quad + \|u_n\|^{1-\mu}_{W_T^{1,p}}\Big|\int_0^T (\nabla H(t,u_n(t)),v(t))dt\Big| \\
&\le \|u_n\|^{1-\mu}_{W_T^{1,p}}\|\varphi'(u_n)\|\|v\| + \|u_n\|^{p-\mu}_{W_T^{1,p}}\sum_{i=1}^N \|\dot{v}_{ni}\|^{p-1}_{L^p}\|\dot{v}_i\|_{L^p} \\
&\quad + C^\theta\|u_n\|^{1-\mu+\theta}_{W_T^{1,p}}\|g\|_{L^1}\|v\|_\infty,
\end{aligned}$$

as $n\to\infty$, which implies that

$$\Big|\int_0^T (B(t)\Phi_\mu(v_0(t)),v(t))dt\Big| = 0$$

for all $v \in W_T^{1,p}([0,T])$. By the arbitrariness of v, one has

$$B(t)\Phi_\mu(v_0(t)) = 0$$

for a.e. $t \in [0,T]$. Because $v_0 = \overline{v}_0 \neq 0$, we have $b_i(t) = 0$ for a.e. $t \in [0,T]$. It follows from the continuity of b_i that $b_i = 0$ for $t \in [0,T]$, which contradicts the condition $b_i \not\equiv 0$. Hence (u_n) is a bounded sequence.

From the reflexivity of $W_T^{1,p}([0,T])$, we extract a weakly convergent subsequence, that for simplicity, we call $(u_n), u_n \rightharpoonup u$. Following we will show that (u_n) converges strongly to u. To this end, we note that $A(u) = \sum_{i=1}^N \int_0^T |\dot{u}_i(t)|^p dt$. From Proposition 2.2 it is enough to prove

$$\limsup_{n\to\infty} \int_0^T (DA(u_n), u_n - u)dt = \limsup_{n\to\infty} \int_0^T \sum_{i=1}^N \Phi_p(\dot{u}_{ni}(t))(\dot{u}_{ni}(t) - \dot{u}_i(t))dt \le 0.$$

From (3.2), it follows

$$\begin{aligned}
&\int_0^T \sum_{i=1}^N \Phi_p(\dot{u}_{ni}(t))(\dot{u}_{ni}(t) - \dot{u}_i(t))dt \\
&= \langle \varphi'(u_n), u_n - u\rangle + \int_0^T \sum_{i=1}^N b_i(t)\Phi_\mu(u_{ni}(t))(u_{ni}(t) - u_i(t))dt \\
&\quad + \int_0^T (\nabla H(t, u_n(t)), u_n(t) - u(t))dt.
\end{aligned}$$

Now since the Sobolev embedding $W_T^{1,p}([0,T]) \hookrightarrow C([0,T])$ is compact, we get (for a subsequence) that $u_n \to u$ uniformly in $C([0,T])$. Since $\varphi'(u_n) \to 0$ by assumption, and $u_n - u$ is bounded in $W_T^{1,p}([0,T])$, we deduce that $\langle \varphi'(u_n), u_n - u\rangle \to 0$. Moreover,

$$\begin{aligned}
&\int_0^T \sum_{i=1}^N b_i(t)\Phi_\mu(u_{ni}(t))(u_{ni}(t) - u_i(t))dt \\
&\le T \sum_{i=1}^N \max_{t\in[0,T]} b_i(t)\Phi_\mu(\|u_{ni}\|_\infty)\|u_{ni} - u_i\|_\infty \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T (\nabla H(t, u_n), u_n - u)dt \\
&\le \int_0^T g(t)|u_n|^\theta (u_n - u)dt \le \|g\|_{L^1}\|u_n\|_\infty^\theta \|u_n - u\|_\infty \to 0.
\end{aligned}$$

So $\limsup_{n\to\infty} \int_0^T \sum_{i=1}^N \Phi_p(\dot{u}_{ni})(\dot{u}_{ni} - \dot{u}_i)dt = 0$ and from Proposition 2.2, $u_n \to u$ strongly in $W_T^{1,p}([0,T])$.

Step 2. Let $W_T^{1,p} = \mathbb{R}^N \oplus \widetilde{W}_T^{1,p}$, where

$$\widetilde{W}_T^{1,p}([0,T]) = \{u \in \widetilde{W}_T^{1,p}([0,T]) : \int_0^T u(t)dt = 0\}.$$

We claim there exist $\rho > 0, \alpha > 0$ such that $\varphi(u) \ge \alpha$ for all $u \in S = \{u \in \widetilde{W}_T^{1,p}([0,T]) : \|u\|_{W_T^{1,p}} = \rho\}$.

By (3.5), for $u \in \widetilde{W}_T^{1,p}([0,T])$, we have

$$\begin{aligned}
\int_0^T H(t,u)dt &\le \int_0^T |h(t)|dt\|u\|_\infty^p = \|h\|_{L^1}\Big(\sum_{i=1}^N \|u_i\|_\infty^2\Big)^{p/2} \\
&\le \|h\|_{L^1}\Big[\sum_{i=1}^N \Big(\int_0^T |\dot u_i(t)|dt\Big)^2\Big]^{p/2} \\
&\le \|h\|_{L^1}\max\{N^{\frac{p}{2}-1},1\}\sum_{i=1}^N \left(\int_0^T |\dot u_i(t)|dt\right)^p \\
&\le T^{\frac{p}{q}}\|h\|_{L^1}\max\{N^{\frac{p}{2}-1},1\}\sum_{i=1}^N \|\dot u_i\|_{L^p}^p.
\end{aligned} \tag{3.7}$$

From (3.5) and (3.7) it follows that

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\int_0^T \sum_{i=1}^N |\dot u_i(t)|^p dt - \frac{1}{\mu}\int_0^T \sum_{i=1}^N b_i(t)|u_i(t)|^\mu dt - \int_0^T H(t,u(t))dt \\
&\ge \Big(\frac{1}{p} - \|h\|_{L^1}T^{\frac{p}{q}}\max\{N^{\frac{p}{2}-1},1\}\Big)\sum_{i=1}^N \|\dot u_i\|_{L^p}^p - \frac{T^{1+\mu/q}}{\mu}\sum_{i=1}^N \|b_i\|_\infty\|\dot u_i\|_{L^p}^\mu
\end{aligned}$$

for all $u \in \widetilde{W}_T^{1,p}([0,T])$. Since $\mu > p$, we can choose $\rho > 0$ small enough such that $\varphi(u) \ge \alpha > 0$.

Step 3. By $\int_0^T b_i(t)dt = 0$, there exist $t_{i1}, t_{i2} \in [0,T]$ such that $b_i(t) \ge 0$ for $t \in [t_{i1}, t_{i2}]$. Choose $e_i \in \widetilde{W}_T^{1,p}([0,T])$ such that $e_i(t) \equiv 0$ for all $t \in [0,t_{i1}] \cup [t_{i2},T]$, $e_i \not\equiv 0$ and $\int_0^T b_i(t)e_i(t)dt = 0, i = 1,2,\dots,N$. Now $e = (e_1,e_2,\dots,e_N) \in \widetilde{W}_T^{1,p}([0,T])$. Since $\mathbb{R}^N$ is definite dimensional, we will show that there exists $R > \rho > 0$ such that

$$\varphi|_{\partial Q} \le 0, \quad Q \equiv \{re : r \in [0,R]\} \oplus (B_R \cap \mathbb{R}^N).$$

To this end, we denote

$$\varphi_1(u) = \frac{1}{p}\int_0^T \sum_{i=1}^N |\dot u_i(t)|^p dt, \quad \varphi_2(u) = -\int_0^T \sum_{i=1}^N b_i(t)|u_i(t)|^\mu dt,$$
$$\varphi_3(u) = -\int_0^T H(t,u)dt.$$

Then $\varphi = \varphi_1 + \varphi_2 + \varphi_3$. By $\sigma \in \mathbb{R}^N$, $\int_0^T b_i(t)dt = 0$ and $\int_0^T b_i(t)e_i(t)dt = 0$ we have

$$\begin{aligned}
\int_{t_{i1}}^{t_{i2}} b_i(t)(|\sigma_i|^2 + |re_i|^2)dt &= \int_{t_{i1}}^{t_{i2}} b_i(t)|\sigma_i + re_i|^2 dt \\
&\le \Big(\int_{t_{i1}}^{t_{i2}} b_i(t)|\sigma_i + re_i|^\mu dt\Big)^{2/\mu}\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{\frac{\mu-2}{\mu}}.
\end{aligned} \tag{3.8}$$

Then one has

$$\begin{aligned}
&\varphi_2(\sigma+re)\\
&=-\sum_{i=1}^{N}\int_{t_{i1}}^{t_{i2}} b_i(t)|\sigma_i+re_i(t)|^{\mu}dt+\sum_{i=1}^{N}\int_{t_{i1}}^{t_{i2}} b_i(t)|\sigma_i|^{\mu}dt\\
&\le-\sum_{i=1}^{N}\Big[\int_{t_{i1}}^{t_{i2}} b_i(t)(|\sigma_i|^2+r^2|e_i(t)|^2)dt\Big]^{\mu/2}\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{\frac{2-\mu}{2}}\\
&\quad+\sum_{i=1}^{N}|\sigma_i|^{\mu}\int_{t_{i1}}^{t_{i2}} b_i(t)dt\\
&=-\sum_{i=1}^{N}\Big[\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{2/\mu}|\sigma_i|^2+r^2\int_{t_{i1}}^{t_{i2}} b_i(t)|e_i(t)|^2dt\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{\frac{2-\mu}{\mu}}\Big]^{\mu/2}\\
&\quad+\sum_{i=1}^{N}|\sigma_i|^{\mu}\int_{t_{i1}}^{t_{i2}} b_i(t)dt
\end{aligned}\tag{3.9}$$

for $\sigma\in\mathbb{R}^N, e\in\widetilde{W}_T^{1,p}$. Because $\int_0^T H(t,u)dt>0$ for all $u\in\mathbb{R}^N$, we have

$$\begin{aligned}
\varphi_3(\sigma+re)&=-\int_0^T H(t,\sigma+re(t))dt\\
&\le-\min\Big\{\int_0^T H(t,q)dt:q\in\mathbb{R}^N\Big\}\\
&\le 0.
\end{aligned}\tag{3.10}$$

Therefore, by (3.9) and (3.10), there exist $0<L,m_i,N<\infty$ such that

$$\begin{aligned}
&\varphi(\sigma+re)\\
&=\varphi_1(\sigma+re)+\varphi_2(\sigma+re)+\varphi_3(\varrho+re)\\
&\le\frac{r^p}{p}\sum_{i=1}^{N}\int_0^T|\dot e_i(t)|^pdt+\sum_{i=1}^{N}|\sigma_i|^{\mu}\int_{t_{i1}}^{t_{i2}} b_i(t)dt\\
&\quad-\sum_{i=1}^{N}\Big[\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{2/\mu}|\sigma_i|^2+r^2\int_{t_{i1}}^{t_{i2}} b_i(t)|e_i(t)|^2dt\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{\frac{2-\mu}{\mu}}\Big]^{\mu/2}\\
&=Lr^p+\sum_{i=1}^{N}m_i|\sigma_i|^{\mu}-\sum_{i=1}^{N}\Big[m_i^{2/\mu}|\sigma_i|^2+r^2N\Big]^{\mu/2}
\end{aligned}\tag{3.11}$$

for all $\sigma\in\mathbb{R}^N, r\ge 0$, which implies that there exists $R_1>\rho$ large enough such that

$$\varphi(\sigma+re)<0\quad\text{for }\|\sigma\|=R,r\in[0,R],R>R_1.\tag{3.12}$$

On the other hand, for $\sigma \in \mathbb{R}^N, r = R$, by (3.9),

$$\begin{aligned}
&-\sum_{i=1}^{N}\Big[\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{2/\mu}|\sigma_i|^2 + R^2\int_{t_{i1}}^{t_{i2}} b_i(t)|e_i(t)|^2 dt\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{\frac{2-\mu}{\mu}}\Big]^{\mu/2}\\
&+\sum_{i=1}^{N}|\sigma_i|^{\mu}\int_{t_{i1}}^{t_{i2}} b_i(t)dt\\
&\le -R^{\mu}\sum_{i=1}^{N}\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)|e_i(t)|^2 dt\Big)^{\mu/2}\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{\frac{2-\mu}{2}}.
\end{aligned} \tag{3.13}$$

Then by (3.11) and (3.13),

$$\begin{aligned}
&\varphi(\sigma+Re)\\
&\le \frac{R^p}{p}\sum_{i=1}^{N}\int_0^T|\dot{e}_i(t)|^p dt - R^{\mu}\sum_{i=1}^{N}\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)|e_i(t)|^2dt\Big)^{\mu/2}\Big(\int_{t_{i1}}^{t_{i2}} b_i(t)dt\Big)^{\frac{2-\mu}{2}},
\end{aligned}$$

which implies that there exists $R_2 > R_1$ such that

$$\varphi(\sigma+Re) \le 0 \quad \text{for all } \sigma \in \mathbb{R}^N,\ R > R_2. \tag{3.14}$$

Therefore, (3.12) and (3.14) give that $\varphi|_{\partial Q} \le 0$. Hence Theorem 3.1 is proved by Lemma 1.2. $\square$

Remark 3.2. It is clear that our theorem generalizes the results in [1-14, 16, 17] since $p \ge 2$. Even for $p = 2$, Theorem 3.1 generalizes Theorem 1.1 since $\nabla H(t,x)$ has more freedom in (3.4) than in (A2). There are functions satisfying Theorem 3.1 and not satisfying the corresponding results in [1, 5, 6, 8, 9, 17]. For example, let

$$b_i(t) = \sin\frac{2\pi t}{T}, \quad i = 1, 2\dots, N,$$

$$H(t,u) = \begin{cases} \frac{\pi}{4T^2}\left(\sin\frac{2\pi t}{T}\right)\left(\frac{2}{3}|u|^{\frac{3}{2}} + \sin(|u|-1) + \frac{\pi}{6T}\right), & |u| \ge 1,\\ \frac{\pi}{4T^2}\left(\sin\frac{2\pi t}{T}\right)|u|^2, & |u| < 1.\end{cases}$$

A straight forward computation shows that $H(t,u)$ satisfies our Theorem 3.1 and neither satisfies assumptions in Theorem 1.1, nor (1.2), hence $H(t,u)$ does not satisfy the corresponding results in [3, 4]. Moreover, $F(t,x) = \frac{B(t)}{\mu}|u|^{\mu} + H(t,u)$ does not satisfy the conditions of the results in [1, 5, 9] because that $\int_0^T b_i(t)dt = 0$; $\nabla F(t,x) \to 0$ as $|x| \to \infty$, it does not satisfy [6].

Remark 3.3. For $1 < p < 2$, inequalities (3.11) and (3.13) do not hold.

Theorem 3.4. *Suppose that $p > 1$, $H : [0,T]\times\mathbb{R}^N \to R$ is even with respect to the second argument and $H(t,0) = 0$. Suppose $b_i \in C(0,T;R^+)$ and there exist $g,h \in L^1([0,T])$, with*

$$\|h\|_{L^1} \le [pT^{\frac{p}{q}}\max\{N^{\frac{p}{2}-1},1\}]^{-1}, \quad \theta \in [0,p-1), \quad r > 0$$

such that (3.4) (3.5) *hold. Then system* (1.4) *has infinitely many solutions.*

Proof. Since H is even in the second argument, the functional φ is even and satisfying $\varphi(0) = 0$. By Step 1 in the proof of Theorem 3.1, φ satisfies (PS) condition. Let $W_T^{1,p} = \mathbb{R}^N \oplus \widetilde{W}_T^{1,p}$. From Step 2 in the proof of Theorem 3.1, there exist $\alpha, \rho > 0$

such that $\varphi(u) \geq \alpha$ if $\|u\| = \rho$, $u \in \widetilde{W}_T^{1,p}([0,T])$. Now we will verify condition (J2) in Lemma 1.3.

$$
\begin{aligned}
&\varphi(u)\\
&= \frac{1}{p}\int_0^T \sum_{i=1}^N |\dot{u}_i(t)|^p ddt - \frac{1}{\mu}\int_0^T \sum_{i=1}^N b_i(t)|u_i(t)|^\mu dt - \int_0^T H(t,u(t))dt\\
&\leq \frac{1}{p}\int_0^T \sum_{i=1}^N |\dot{u}_i(t)|^p dt - \frac{1}{\mu}\int_0^T \sum_{i=1}^N b_i(t)|u_i(t)|^\mu dt - \int_0^T \Big(\int_0^1 (\nabla H(t,su),u)ds\Big)dt\\
&\leq \frac{1}{p}\int_0^T \sum_{i=1}^N |\dot{u}_i(t)|^p dt - \frac{1}{\mu}\int_0^T \sum_{i=1}^N b_i(t)|u_i(t)|^\mu dt + \int_0^T \frac{g(t)|u|^{\theta+1}}{\theta+1}dt.
\end{aligned}
\tag{3.15}
$$

From (3.3), we have

$$
\begin{aligned}
\|u\|_\infty^{\theta+1} &\leq \Big(\sum_{i=1}^N \|u_i\|_\infty^2\Big)^{\frac{\theta+1}{2}}\\
&\leq C^{\theta+1}\Big(\sum_{i=1}^N \|u_i\|_{W_T^{1,p}}^2\Big)^{\frac{\theta+1}{2}}\\
&\leq C^{\theta+1}\max\{N^{\frac{\theta-1}{2}},1\}\sum_{i=1}^N \|u_i\|_{W_T^{1,p}}^{\theta+1}.
\end{aligned}
\tag{3.16}
$$

For any finite dimensional subspace $V_1 \subset W_T^{1,p}([0,T])$, the norm $\|\cdot\|_{W_T^{1,p}}$ and $\|\cdot\|_{L^\theta}$. So there exists $\overline{c} > 0$ such that

$$
\|u\|_{W_T^{1,p}} \leq \overline{c}\Big(\int_0^T |u(t)|^\mu dt\Big)^{\frac{1}{\mu}} \quad \text{for } u \in V_1. \tag{3.17}
$$

Moreover by Hölder inequality, there exists a positive constant $\widetilde{c}$ such that

$$
\|u\|_{L^{\theta+1}} \leq \widetilde{c}\|u\|_{W_T^{1,p}}^{\theta+1} \tag{3.18}
$$

holds. Thus (3.16) (3.17) and (3.18) give that there exist $0 < \Gamma_i < \infty$, $i = 1,2,3$ such that

$$
\varphi(u) \leq \Gamma_1 \sum_{i=1}^N \|u_i\|_{W_T^{1,p}}^p - \Gamma_2 \sum_{i=1}^N \|u_i\|_{W_T^{1,p}}^\mu + \Gamma_3 \sum_{i=1}^N \|u_i\|_{W_T^{1,p}}^{\theta+1},
$$

which implies that $\{x \in X_1 : \varphi(x) \geq 0\}$ is bounded. Then Lemma 1.3 can be applied to the functional φ. The proof is completed. $\square$

Remark 3.5. It is clear that our theorem generalizes the results in [1-14, 16, 17] since $p > 1$. Even for $p = 2$, the conditions of Theorem 3.1 are different from the conditions in [1, 3, 4] since in [1] there exist $\beta > 2, \alpha > 0$ such that $H(u) \geq \alpha|u|^\beta$ for all $u \in \mathbb{R}^N$ and in [3, 4] there exists $\mu > 2$ such that (1.2) holds for $t \in [0,T], |x| \geq R$. There are functions satisfying Theorem 3.4 but not satisfying [1, 3, 4]. For example, let

$$b_i(t) = t, i = 1, 2, \dots, N, \quad \mu = 3,$$

$$H(t,u) = \begin{cases} \frac{t}{2T^3}\left(\frac{7}{4}|u|^{\frac{7}{4}} + \sin(|u|-1) + \frac{3}{7}\right), & |u| \geq 1, \\ \frac{|u|^2 t}{2T^3}, & |u| < 1. \end{cases}$$

Then $F(t,u) = \frac{B(t)}{3}|u|^3 + H(t,u)$ satisfies Theorem 3.4, but does not satisfies the conditions in [1, 3, 4].

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