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A PROPERTY OF THE H-CONVERGENCE FOR ELASTICITY IN PERFORATED DOMAINS

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Abstract. In this article, we obtain the H_e^0-convergence as a limit case of the H_e-convergence. More precisely, if Ω_ε is a perforated domain with (admissible) holes T_ε and χ_ε denote its characteristic function and if $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$, we show how the behavior as $(\varepsilon, \delta) \to (0,0)$ of the double sequence of tensors $A_\delta^\varepsilon = (\chi_\varepsilon + \delta(1-\chi_\varepsilon))A^\varepsilon$ is connected to A^0. These results extend those given by Cioranescu, Damlamian, Donato and Mascarenhas in [3] for the H-convergence of the scalar second elliptic operators to the linearized elasticity systems.

1. Introduction

The notion of H-convergence was introduced by Murat and Tartar [7, 8, 9] for the second-order elliptic operators (non necessary symmetric) and extended to the case of holes by Briane, Damlamian and Donato in [2] and called H^0-convergence. Cioranescu, Damlamian, Donato and Mascarenhas [3] obtain the H^0-convergence as a limit case of the H-convergence with a vanishing coercivity constant in the holes.

In this work, we show that a similar property holds for the linearized elasticity systems, namely between the H_e-convergence studied by Francfort and Murat in [6] and its generalization to the case of holes, denoted by H_e^0-convergence, which has been developed by Donato and El Hajji in [5]. The H_e-convergence deals with the convergence of the solutions of a system of linearized elasticity whose tensor coefficients $\{A^\varepsilon\}$ are equibounded and uniformly definite positive. The H_e^0-convergence treat the same problem in a perforated domain Ω_ε with a traction condition on the holes for which uniform Korn estimates hold.

Let us briefly describe here the main results of this paper. Let Ω a bounded open subset of $\mathbb{R}^n$, $\{T_\varepsilon\}$ a sequence of (admissible) holes, denote $\Omega_\varepsilon = \Omega \setminus T_\varepsilon$ the perforated domain and χ^ε the characteristic function of Ω_ε. Let also $\{A^\varepsilon\}$ a sequence of linearized elasticity tensors on Ω such that $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$. We prove (Theorem 4.1) that if we set for every $\delta > 0$

$$A_\delta^\varepsilon = (\chi_\varepsilon + \delta(1-\chi_\varepsilon))A^\varepsilon \quad \text{a.e. in } \Omega$$

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and if A^ε_δ H_e-converges to a tensor A_δ (for some subsequence), then $A_\delta \to A^0$ strongly in $L^p(\Omega)$ for any $p \geq 1$, and weakly $\star$ in $L^\infty(\Omega)$. Moreover, under suitable assumption (see (4.3) below), we have also (Theorem 4.2)

$$(A^\varepsilon_\delta \overset{\delta\to 0}{\longrightarrow} (A^\varepsilon, T_\varepsilon)) \text{ in the sense } \begin{cases} u^\varepsilon_\delta \to u^\varepsilon \text{ strongly in } H^1_0(\Omega_\varepsilon)^n, \\ A^\varepsilon_\delta e(u^\varepsilon_\delta) \to A^\varepsilon \widetilde{e(u^\varepsilon)} \text{ strongly in } L^2(\Omega)^{n\times n} \end{cases}$$

and (Theorem 4.3)

$$(A^\varepsilon_\delta \overset{(\varepsilon,\delta)\to(0,0)}{\longrightarrow} A^0) \text{ in the sense } \begin{cases} u^\varepsilon_\delta \to u^0 \text{ weakly in } H^1_0(\Omega)^n, \\ A^\varepsilon_\delta e(u^\varepsilon_\delta) \to A^0 e(u^0) \text{ weakly in } L^2(\Omega)^{n\times n}, \end{cases}$$

where u^ε, u^ε_δ and u^0 are the solutions of (2.2), (3.5) and (2.4) respectively. This results can be resumed by the following commutative schema:

$$\begin{array}{ccc} A^\varepsilon_\delta & \overset{H_e}{\rightarrow} & A_\delta \\ \downarrow & \searrow & \downarrow \\ (A^\varepsilon, T_\varepsilon) & \overset{H^0_e}{\rightarrow} & A^0. \end{array}$$

The definition and the main properties of the H^0_e-convergence are recalled in Section 2. In Section 3 we give some preliminary results and in Section 4 we state and prove the main results.

2. The H^0_e-convergence

We use the following notation:

• If $A = (A_{ijkl})_{1\leq i,j,k,l\leq n}$ is a forth order tensors and $\Lambda \in \mathbb{R}^{n\times n}$, we set

$$A\Lambda = \sum_{1\leq i,j,k,l\leq n} A_{ijkl}\Lambda_{pq},$$
$$\Lambda.\Upsilon = \sum_{1\leq i,j\leq n} \Lambda_{ij}\Upsilon_{ij},$$
$$|\Lambda| = (\sum_{1\leq i,j\leq n} |\Lambda_{ij}|^2)^{\frac{1}{2}},$$

• Ω is a domain of $\mathbb{R}^n$,
• if F is a set of matrices fields, $F_S = \{M \in F \ s.\ t.\ M \ is\ symmetric\}$,
• $\{\varepsilon\}$ and $\{\delta\}$ denote a two strictly decreasing sequence converging to zero,
• if $v = (v_1,\dots, v_n)$ is a vector valued function and $\zeta = (\zeta_{ij})_{1\leq i,j\leq n}$ is a second order tensor of variable $x = (x_1,\dots, x_n)$, we set

$$(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j} \equiv D_{x_j} v_i,$$
$$e(v) = \frac{1}{2}(\nabla v +^t \nabla v),$$
$$(\operatorname{div} \xi)_i = \frac{\partial \xi_{ij}}{\partial x_j};$$

• for two real numbers α and β such that $0 < \alpha < \beta$, $M_e(\alpha, \beta, \Omega)$ is the set of the tensors $A = (A_{ijkl})_{1\leq i,j,k,l\leq n}$ defined on Ω, such that a.e. on Ω we have

(i) $A_{ijkl} \in L^\infty(\Omega)$, for any $i, j, k, l = 1,\dots, n$,
(ii) $A_{ijkl} = A_{jikl} = A_{klij}$, for any $i, j, k, l = 1,\dots, n$,
(iii) $\alpha|\Lambda|^2 \leq A\Lambda.\Lambda$, for any symmetric matrix Λ,

(iv) $|A\Lambda| \le \beta|\Lambda|$, for any matrix Λ,

- if $f \in H^{-1}(\Omega)^n$ and $u \in H_0^1(\Omega)^n$, we set $\langle f, u\rangle = \langle f, u\rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n}$.

Let us recall first the main results concerning the H_e^0-convergence introduced by Donato and El Hajji [5]. We introduce the perforated domain

$$\Omega_\varepsilon = \Omega \backslash T_\varepsilon,$$

where T_ε is a sequence of compact subsets of Ω and set

$$V_\varepsilon = \{v \in H^1(\Omega_\varepsilon)^n \text{ s. t. } v = 0 \text{ on } \partial\Omega\}.$$

In the following, we denote by ν the outward normal unit vector on the boundary of Ω_ε and $\tilde{\cdot}$ the extension by 0 from Ω_ε to Ω and set $\chi^\varepsilon = \chi_{\Omega_\varepsilon}$.

Definition 2.1 ([5])**.** The set T_ε is said to be admissible (in Ω) for the linearized elasticity (or e-admissible), if and only if:

(i) Every L^∞ weak $*$-limit point of $\{\chi_{\Omega_\varepsilon}\}_\varepsilon$ is positive a.e. in Ω;

(ii) there exists a positive real C, independent of ε, and a sequence $\{P_\varepsilon\}_\varepsilon$ of linear extension operators such that for each ε

$$\begin{gathered} P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)^n), \\ (P_\varepsilon v)|_{\Omega_\varepsilon} = v, \quad \forall v \in V_\varepsilon, \\ \|e(P_\varepsilon v)\|_{0,\Omega} \le C\|e(v)\|_{0,\Omega_\varepsilon}, \quad \forall v \in V_\varepsilon. \end{gathered} \tag{2.1}$$

We denote by $P_\varepsilon^\star$ the adjoint operator of P_ε, which is defined from $H^{-1}(\Omega)^n$ to V_ε' with $P_\varepsilon^\star$ given for every $f \in H^{-1}(\Omega)^n$ by

$$\forall v \in V_\varepsilon,\ \langle P_\varepsilon^\star f, v\rangle_{V_\varepsilon', V_\varepsilon} = \langle f, P_\varepsilon v\rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n}.$$

Definition 2.2 ([5])**.** Let $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$, T_ε be e-admissible in Ω. The pair $(A^\varepsilon, T_\varepsilon)$ is said H_e^0-converge to the tensor $A^0 \in M_e(\alpha', \beta', \Omega)$ and denoted by $(A^\varepsilon, T_\varepsilon) \overset{H_e^0}{\rightharpoonup} A^0$ if and only if for each function $f^\varepsilon \in H^{-1}(\Omega)^n$ such that $f^\varepsilon \to f$ strongly in $H^{-1}(\Omega)^n$, the solution u^ε of

$$\begin{gathered} -\operatorname{div}\,(A^\varepsilon e(u^\varepsilon)) = P_\varepsilon^\star f^\varepsilon \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon e(u^\varepsilon))\nu = 0 \quad \text{on } \partial T_\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

satisfies the weak convergence

$$\begin{gathered} P_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega)^n, \\ A^\varepsilon \widetilde{e(u^\varepsilon)} \rightharpoonup A^0 e(u^0) \quad \text{weakly in } L^2(\Omega)^{n\times n}, \end{gathered} \tag{2.3}$$

where u^0 is the unique solution of the problem

$$\begin{gathered} -\operatorname{div}(A^0 e(u^0)) = f \quad \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.4}$$

Remark 2.3. (1) In [5] the definition is given for fixed $f_\varepsilon \doteq f$. The two definitions are equivalent in view of [5, Proposition 2].
(2) In the case where $T_\varepsilon = \emptyset$, this definition reduces to the definition of the H_e-convergence [6].

This notion of convergence makes sense in view of the following compactness theorem:

Theorem 2.4 ([5]). *Let $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$ and T_ε be e-admissible in Ω. Then there exists a subsequence, still denoted by $\{\varepsilon\}$, and a tensor $A^0 \in M_e(\frac{\alpha}{C^2},\beta,\Omega)$, such that the sequence $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H_e^0-converge to A^0.*

Remark 2.5. The fact that A^0 belongs to $M_e(\frac{\alpha}{C^2},\beta,\Omega)$, does not appears explicitly in the statement given in [5], but can be easily deduced with the same arguments as that used in the non perforated case.

Let us recall also a property proved recently in [4].

Theorem 2.6 ([4]). *Let $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha,\beta,\Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha',\beta',\Omega)$ such that $A^\varepsilon \overset{H_e}{\rightharpoonup} A^0$ and $B^\varepsilon \overset{H_e}{\rightharpoonup} B^0$. Assume that there exists two functions $h^\varepsilon,\ h^0 \in L^1(\Omega)$ such that*

$$|A^\varepsilon - B^\varepsilon| \le h^\varepsilon \longrightarrow h^0 \quad \textit{strongly in } L^1(\Omega).$$

Then

$$|A^0(x) - B^0(x)| \le \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, h^0(x) \quad \textit{a.e. in } \Omega.$$

The following proposition completes a result given in [5]:

Proposition 2.7. *One has*

(1) *If $\{v^\varepsilon\}$ is a bounded sequence in $H_0^1(\Omega)$, then*

$$(v^\varepsilon \rightharpoonup v \textit{ weakly in } H_0^1(\Omega)^n) \Leftrightarrow \Big(P_\varepsilon(v^\varepsilon|_{\Omega_\varepsilon}) \rightharpoonup v \textit{ weakly in } H_0^1(\Omega)^n\Big).$$

(2) *If (ε,δ) is a sequence of $\mathbb{R}_+^\star \times \mathbb{R}_+^\star$ such that $(\varepsilon,\delta) \to (0,0)$ and $\{v_\delta^\varepsilon\}$ is a sequence in $H_0^1(\Omega)$ bounded independently of ε and δ, then*

$$(v_\delta^\varepsilon \rightharpoonup v \textit{ weakly in } H_0^1(\Omega)^n) \Leftrightarrow \Big(P_\varepsilon(v_\delta^\varepsilon|_{\Omega_\varepsilon}) \rightharpoonup v \textit{ weakly in } H_0^1(\Omega)^n\Big).$$

Proof. Suppose that

$$P_\varepsilon(v^\varepsilon|_{\Omega_\varepsilon}) \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega)^n. \tag{2.5}$$

Observe first that

$$v^\varepsilon\chi^\varepsilon = P_\varepsilon(v^\varepsilon{}_{|\Omega_\varepsilon})\chi^\varepsilon. \tag{2.6}$$

On the other hand, since $\{v^\varepsilon\}$ is a bounded sequence in $H_0^1(\Omega)$, there exists a $\{\varepsilon'\} \subset \{\varepsilon\}$ and $w \in H_0^1(\Omega)^n$ such that

$$v^{\varepsilon'} \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega)^n. \tag{2.7}$$

But $|\chi^{\varepsilon'}| \le 1$, hence there exists $\chi^0 \in L^\infty(\Omega)$ and $\{\varepsilon''\} \subset \{\varepsilon'\}$ such that

$$\chi^{\varepsilon''} \rightharpoonup \chi^0 \quad \text{weakly} \star \text{ in } L^\infty(\Omega). \tag{2.8}$$

Passing to the limit (in $D'(\Omega)$) in (2.6) by using (2.5), (2.7) and (2.8), we find

$$\chi^0 w = \chi^0 v.$$

Taking now into account the fact that (in view of Definition 2.1) $\chi^0 > 0$, we obtain $w = v$. This, together with (2.7), implies that the whole sequence $P_\varepsilon(v^\varepsilon_{\delta|_{\Omega_\varepsilon}})$ converge weakly to v. We refer to [5] for the converse implication. The proof of (2) follows by the same arguments. $\square$

3. Preliminary results

In this paper, $\{A^\varepsilon\}$ is a sequence of fourth-order tensors of $\mathcal{M}_e(\alpha,\beta,\Omega)$ and $\{T_\varepsilon\}$ is a sequence of holes e-admissible in Ω such that

$$A^\varepsilon \overset{H_e^0}{\rightharpoonup} A^0. \tag{3.1}$$

Set, for every $\delta > 0$,

$$A_\delta^\varepsilon = (\chi_\varepsilon + \delta(1-\chi_\varepsilon))A^\varepsilon \quad \text{a.e. in } \Omega. \tag{3.2}$$

Since, for fixed $\delta > 0$, $A_\delta^\varepsilon \in \mathcal{M}_e(\min(1,\delta)\alpha, \max(1,\delta)\beta,\Omega)$, in view of the compactness properties of H_e-convergence, there exists a subsequence $\{\varepsilon_m\}$ of $\{\varepsilon\}$ and $A_\delta \in \mathcal{M}_e(\min(1,\delta)\alpha, \max(1,\delta)\beta,\Omega)$ such that $A_\delta^{\varepsilon_m} \overset{H_e}{\rightharpoonup} A_\delta$ as $\varepsilon_m \to 0$. Hence, for every $\delta > 0$, the set

$$W_\delta = \{A_\delta;\ \exists \{\varepsilon_m\}_{m\in\mathbb{N}} \subset \{\varepsilon\} \text{ s. t. } A_\delta^{\varepsilon_m} \overset{H_e}{\rightharpoonup} A_\delta\} \tag{3.3}$$

is not empty. Let $\{f^\varepsilon\}$ be a sequence in $H^{-1}(\Omega)^n$ such that

$$f^\varepsilon \to f quad \text{strongly in } H^{-1}(\Omega)^n \tag{3.4}$$

and let A_δ be in W_δ. Let u_δ^ε and u_δ the solutions of

$$\begin{gathered} -\operatorname{div}(A_\delta^\varepsilon e(u_\delta^\varepsilon)) = f^\varepsilon \quad \text{in } \Omega, \\ u_\delta^\varepsilon = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.5}$$

and

$$\begin{gathered} -\operatorname{div}(A_\delta e(u_\delta)) = f \quad \text{in } \Omega, \\ u_\delta = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.6}$$

respectively. We consider now the following sets:

$$\begin{gathered} U_\delta = \{u_\delta : u_\delta \text{ is the solution of (3.6) for some } A_\delta \in W_\delta\}, \\ V_\delta = \{\text{The set of weak limit points of } A_\delta^\varepsilon e(u_\delta^\varepsilon) \text{ in } L^2(\Omega)^n \text{ as } \varepsilon \to 0\}. \end{gathered} \tag{3.7}$$

One has the following result:

Lemma 3.1. *One has*

$$V_\delta = \{A_\delta e(u_\delta) : A_\delta \in W_\delta \text{ and } u_\delta \text{ is the solution of (3.6)}\}.$$

Proof. It is clear that, if $A_\delta \in W_\delta$ and u_δ is the solution of (3.6), then $A_\delta e(u_\delta)$ belongs to V_δ. On the other hand, let $v \in V_\delta$. Then, there exists a subsequence $\{\varepsilon_m\}$ of ε such that

$$A_\delta^{\varepsilon_m} e(u_\delta^{\varepsilon_m}) \rightharpoonup v \quad \text{weakly in } L^2(\Omega)^n, \tag{3.8}$$

as $\varepsilon_m \to 0$. But the compactness property of the H_e-convergence shows that there exists a subsequence $\{\varepsilon_m'\}$ of $\{\varepsilon_m\}$ and a forth-order tensor A_δ such that

$$A_\delta^{\varepsilon_m'} \overset{H_e}{\rightharpoonup} A_\delta.$$

This implies in particular

$$A_\delta^{\varepsilon_m'} e(u_\delta^{\varepsilon_m'}) \rightharpoonup A_\delta e(u_\delta) \quad \text{weakly in } L^2(\Omega)^n.$$

This, together with (3.8), gives $v = A_\delta e(u_\delta)$. □

Remark 3.2. Let us show that in view of Theorem 2.6, there exists $\{\varepsilon_m\} \subset \{\varepsilon\}$ and for all $\delta > 0$ a tensor $\widehat{A_\delta}$ such that

$$A_\delta^{\varepsilon_m} \xrightarrow{H_e} \widehat{A_\delta}. \tag{3.9}$$

Let us also point out that in Theorem 4.1 we will consider a more general situation, where for every $\delta > 0$, there exists $\{\varepsilon_\delta\}$ and a tensor A_δ such that $A_\delta^{\varepsilon_\delta} \xrightarrow{H_e} A_\delta$.

Let us prove (3.9). Using the diagonal subsequence procedure and the compactness property of the H_e-convergence, one extracts a subsequence $\{\varepsilon_m\}$ of $\{\varepsilon\}$ such that, for every $\delta \in \mathbb{Q}_+^\star$, one has

$$A_\delta^{\varepsilon_m} \ H_e\text{-convergences to a limit } A_\delta. \tag{3.10}$$

Since a.e in Ω one has

$$\begin{gathered}
|A_{\delta_1}^{\varepsilon_m} - A_{\delta_2}^{\varepsilon_m}| \le \beta|\delta_1 - \delta_2|, \quad \forall \delta_1,\ \delta_2 \in \mathbb{Q}_+^\star, \\
A_{\delta_1}^{\varepsilon_m} \in \mathcal{M}_e(\min(1,\delta_1)\alpha, \max(1,\delta_1)\beta, \Omega), \\
A_{\delta_2}^{\varepsilon_m} \in \mathcal{M}_e(\min(1,\delta_2)\alpha, \max(1,\delta_2)\beta, \Omega).
\end{gathered}$$

Then, from Theorem 2.6, it follows

$$|A_{\delta_1} - A_{\delta_2}| \le \frac{\beta^2}{\alpha} \sqrt{\frac{\max(1,\delta_1)\max(1,\delta_2)}{\min(1,\delta_1)\min(1,\delta_2)}}\ |\delta_1 - \delta_2|.$$

This implies that the mapping $\delta \in \mathbb{Q}_+^\star \mapsto A_\delta \in \mathbb{L}^\infty(\Omega)$ is uniformly continuous. Hence, it can be extended to a mapping (still denoted by $\delta \mapsto A_\delta$) defined and uniformly continuous on all $\mathbb{R}_+^\star$ (since $\mathbb{Q}_+^\star$ is dense in $\mathbb{R}_+^\star$).

Let now δ be a strictly positive real and $\{\delta_s\}$ be a sequence of $\mathbb{Q}_+^\star$ which converges to δ as $s \to \infty$. Then, there exists a sub-subsequence $\{\varepsilon'_m\}$ of $\{\varepsilon_m\}$ such that

$$A_\delta^{\varepsilon'_m} \ H_e\text{-converges to some } A. \tag{3.11}$$

In view of Theorem 2.6 this give, together with (3.10) and the fact that $|A_\delta^{\varepsilon'_m} - A_{\delta_s}^{\varepsilon'_m}| \le \beta|\delta - \delta_s|$, the following inequality:

$$|A - A_{\delta_s}| \le \frac{\beta^2}{\alpha} \sqrt{\frac{\max(1,\delta)\max(1,\delta_s)}{\min(1,\delta)\min(1,\delta_s)}}\ |\delta - \delta_s| \quad \text{a.e. in } \Omega.$$

Using the continuity of the mapping $\delta \mapsto A_\delta$ on $\mathbb{R}_+^\star$ and passing to the limit in this inequality as $s \to \infty$, one finds

$$A = A_\delta, \quad \text{a.e. in } \Omega.$$

The uniqueness of the limit implies then that the whole subsequence $A_\delta^{\varepsilon_m}$ H_e-converges to A_δ, for every $\delta > 0$.

The following results state some a priori estimates that we will need in the following:

Proposition 3.3. *Let u^ε and u_δ^ε the solutions of* (2.2) *and* (3.5) *respectively. Then, there exists $c > 0$ independent of ε and δ such that*

$$\begin{gathered}
\|P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - P^\varepsilon u^\varepsilon\|_{H_0^1(\Omega)^n} \le c(\delta^{1/2} + |\langle f_\varepsilon, P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u_\delta^\varepsilon\rangle|^{1/2}), \\
\|e(u_\delta^\varepsilon)\|_{L^2(T_\varepsilon)^{n\times n}} \le c(1 + \delta^{-\frac{1}{2}}|\langle f_\varepsilon, P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u_\delta^\varepsilon\rangle|^{1/2}), \\
\|A_\delta^\varepsilon e(u_\delta^\varepsilon) - A^\varepsilon \widetilde{e(u^\varepsilon)}\ \|_{L^2(\Omega)^{n\times n}} \le c(\delta^{1/2} + |\langle f_\varepsilon, P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u_\delta^\varepsilon\rangle|^{1/2}).
\end{gathered} \tag{3.12}$$

Proof. Observe that the variational formulations of problems (2.2) and (3.5) are

$$\forall w \in H_0^1(\Omega)^n, \quad \int_{\Omega_\varepsilon} A^\varepsilon e(u^\varepsilon).e(w)dx = \langle f_\varepsilon, P^\varepsilon(w_{|\Omega_\varepsilon})\rangle \tag{3.13}$$

and

$$\forall w \in H_0^1(\Omega)^n, \int_{\Omega_\varepsilon} A^\varepsilon e(u_\delta^\varepsilon).e(w)dx + \delta\int_{T_\varepsilon} A^\varepsilon e(u_\delta^\varepsilon).e(w)dx = \langle f_\varepsilon, w\rangle$$

respectively. Then, for every $w \in H_0^1(\Omega)^n$, one has

$$\int_{\Omega_\varepsilon} A^\varepsilon(e(u_\delta^\varepsilon) - e(u^\varepsilon)).e(w)dx + \delta\int_{T_\varepsilon} A^\varepsilon e(u_\delta^\varepsilon).e(w)dx = -\langle f_\varepsilon, P^\varepsilon(w_{|\Omega_\varepsilon}) - w\rangle.$$

In particular, for $w = u_\delta^\varepsilon - P^\varepsilon u^\varepsilon$, this gives

$$\begin{aligned}&\int_{\Omega_\varepsilon} A^\varepsilon(e(u_\delta^\varepsilon) - e(u^\varepsilon)).(e(u_\delta^\varepsilon) - e(P^\varepsilon u^\varepsilon))dx + \delta\int_{T_\varepsilon} A^\varepsilon e(u_\delta^\varepsilon).(e(u_\delta^\varepsilon) - e(P^\varepsilon u^\varepsilon))dx\\ &= -\langle f_\varepsilon, P^\varepsilon((u_\delta^\varepsilon - P^\varepsilon u^\varepsilon)_{|\Omega_\varepsilon}) - u_\delta^\varepsilon - P^\varepsilon u^\varepsilon\rangle.\end{aligned}$$

Using that $P^\varepsilon u^\varepsilon{}_{|\Omega_\varepsilon} = u^\varepsilon$, one deduces

$$\begin{aligned}&\int_{\Omega_\varepsilon} A^\varepsilon(e(u_\delta^\varepsilon) - e(u^\varepsilon)).(e(u_\delta^\varepsilon) - e(u^\varepsilon))dx + \delta\int_{T_\varepsilon} A^\varepsilon e(u_\delta^\varepsilon).e(u_\delta^\varepsilon)dx\\ &= \delta\int_{T_\varepsilon} A^\varepsilon e(u_\delta^\varepsilon).e(P^\varepsilon u^\varepsilon)dx - \langle f_\varepsilon, P^\varepsilon(u_\delta^\varepsilon)_{|\Omega_\varepsilon} - u_\delta^\varepsilon\rangle.\end{aligned}$$

In view of the fact that $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$, this gives

$$\begin{aligned}&\alpha\int_{\Omega_\varepsilon} |e(u_\delta^\varepsilon) - e(u^\varepsilon))|^2dx + \alpha\delta\int_{T_\varepsilon} |e(u_\delta^\varepsilon)|^2dx\\ &\le \delta|\int_{T_\varepsilon} A^\varepsilon e(u_\delta^\varepsilon).e(P^\varepsilon u^\varepsilon)dx| + |\langle f_\varepsilon, P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u_\delta^\varepsilon\rangle|.\end{aligned} \tag{3.14}$$

Using the Young's inequality, one obtains

$$\begin{aligned}\Big|\int_{T_\varepsilon} A^\varepsilon e(u_\delta^\varepsilon).e(P^\varepsilon u^\varepsilon)dx\Big| &\le \beta\int_{T_\varepsilon} |e(u_\delta^\varepsilon)||e(P^\varepsilon u^\varepsilon)|dx\\ &\le \frac{\alpha}{2}\int_{T_\varepsilon} |e(u_\delta^\varepsilon)|^2dx + \frac{\beta^2}{2\alpha}\int_{T_\varepsilon} |e(P^\varepsilon u^\varepsilon)|^2dx\\ &\le \frac{\alpha}{2}\int_{T_\varepsilon} |e(u_\delta^\varepsilon)|^2dx + \frac{\beta^2}{2\alpha}\int_{\Omega} |e(P^\varepsilon u^\varepsilon)|^2dx.\end{aligned}$$

But, taking $w = P^\varepsilon u^\varepsilon$ in (3.13), one finds

$$\int_\Omega |e(P^\varepsilon u^\varepsilon)|dx \le c_1,$$

with $c_1 > 0$ independent of ε and δ. Then

$$\Big|\int_{T_\varepsilon} A^\varepsilon e(u_\delta^\varepsilon).e(P^\varepsilon u^\varepsilon)dx\Big| \le \frac{\alpha}{2}\int_{T_\varepsilon} |e(u_\delta^\varepsilon)|^2dx + c_2,$$

where $c_2 > 0$ independent of ε and δ. This, together with (3.14), gives

$$\alpha\int_{\Omega_\varepsilon} |e(u_\delta^\varepsilon) - e(u^\varepsilon))|^2dx + \frac{\alpha\delta}{2}\int_{T_\varepsilon} |e(u_\delta^\varepsilon)|^2dx \le c_3(\delta + |\langle f_\varepsilon, P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u_\delta^\varepsilon\rangle|). \tag{3.15}$$

with $c_3 > 0$ independent of ε and δ. From this, (3.12)ii) follows immediately. Moreover, since $u^\varepsilon_{\delta|\Omega_\varepsilon} - u^\varepsilon \in H^1_0(\Omega_\varepsilon)^n$, Definition 2.1 shows that

$$\|e(P^\varepsilon(u^\varepsilon_\delta|_{\Omega_\varepsilon} - u^\varepsilon))\|_{0,\Omega} \leq C\|e(u^\varepsilon_\delta - u^\varepsilon)\|_{0,\Omega_\varepsilon}.$$

Hence, by virtue of the Korn inequality, (3.15) gives also (3.12)i).

On the other hand, since $A^\varepsilon_\delta = A^\varepsilon$ a.e. in Ω_ε and $A^\varepsilon_\delta = \delta A^\varepsilon$ a.e. in T_ε, one has

$$\begin{aligned}\int_\Omega |A^\varepsilon_\delta e(u^\varepsilon_\delta) - A^\varepsilon \widetilde{e(u^\varepsilon)}|^2 dx &\leq \int_{\Omega_\varepsilon} |A^\varepsilon e(u^\varepsilon_\delta) - A^\varepsilon e(u^\varepsilon)|^2 dx + \delta^2 \int_{T_\varepsilon} |A^\varepsilon e(u^\varepsilon_\delta)|^2 dx \\ &\leq \beta^2 \int_{\Omega_\varepsilon} |e(u^\varepsilon_\delta) - e(u^\varepsilon)|^2 dx + \beta^2\delta^2 \int_{T_\varepsilon} |e(u^\varepsilon_\delta)|^2 dx,\end{aligned}$$

which, together with (3.12)(i) and (3.12)(ii), gives (3.12)(iii). □

Proposition 3.4. *Let* u^0 *the solution of* (2.4)*. Then*

$$\begin{gathered}\sup_{u\in U_\delta} \|u - u^0\|_{H^1_0(\Omega)^n} \leq c\, \delta^{1/2}, \\ \sup_{v\in V_\delta} \|v - A^0 e(u^0)\|_{L^2(\Omega)^{n\times n}} \leq c\, \delta^{1/2}.\end{gathered} \tag{3.16}$$

Proof. Let be u_δ in U_δ. This means that there exists $A_\delta \in W_\delta$ such that u_δ is the solution of (3.6). But the fact that A_δ is in W_δ implies that there exists a subsequence $\{\varepsilon_m\}$ of ε such that $A^{\varepsilon_m}_\delta$ H_e-converges to A_δ. Hence, the solution u^ε_δ of

$$\begin{gathered}-\operatorname{div}(A^{\varepsilon_m}_\delta e(u^{\varepsilon_m}_\delta)) = f^{\varepsilon_m} \quad \text{in } \Omega, \\ u^{\varepsilon_m}_\delta = 0 \quad \text{on } \partial\Omega\end{gathered} \tag{3.17}$$

satisfies as $\varepsilon_m \to 0$

$$\begin{gathered}u^{\varepsilon_m}_\delta \rightharpoonup u_\delta \quad \text{weakly in } H^1_0(\Omega)^n, \\ A^{\varepsilon_m}_\delta e(u^{\varepsilon_m}_\delta) \rightharpoonup A_\delta e(u_\delta) \quad \text{weakly in } L^2(\Omega)^{n\times n}.\end{gathered} \tag{3.18}$$

Estimate (3.16)**(i):** By Lemma 2.7, (3.18)i) implies that, for every fixed $\delta > 0$,

$$P_{\varepsilon_m}(u^{\varepsilon_m}_\delta|_{\Omega_{\varepsilon_m}}) \rightharpoonup u_\delta \quad \text{weakly in } H^1_0(\Omega)^n. \tag{3.19}$$

Hence, by (3.4), one has

$$\lim_{\varepsilon_m\to 0} \langle f_{\varepsilon_m}, P^{\varepsilon_m}(u^{\varepsilon_m}_{\delta\ |\Omega_{\varepsilon_m}}) - u_\delta\rangle = 0. \tag{3.20}$$

From this and (3.12)(i), it comes

$$\lim_{\varepsilon_m\to 0} \|P^{\varepsilon_m}(u^{\varepsilon_m}_{\delta\ |\Omega_{\varepsilon_m}}) - P^{\varepsilon_m} u^{\varepsilon_m}\|_{H^1_0(\Omega)^n} \leq c\delta^{1/2}. \tag{3.21}$$

But, (3.19) and (2.3)i) imply

$$P_{\varepsilon_m}(u^{\varepsilon_m}_{\delta\ |\Omega_\varepsilon}) - P_{\varepsilon_m}(u^{\varepsilon_m}) \rightharpoonup u_\delta - u^0 \quad \text{weakly in } H^1_0(\Omega)^n.$$

This gives, by using the weak lower semi-continuity of the H^1_0-norm,

$$\|u_\delta - u^0\|_{H^1_0(\Omega)^n} \leq \lim_{\varepsilon_m\to 0} \|P^{\varepsilon_m}(u^{\varepsilon_m}_{\delta\ |\Omega_{\varepsilon_m}}) - P^{\varepsilon_m} u^{\varepsilon_m}\|_{H^1_0(\Omega)^n},$$

where u^0 is the solution of (2.4). Hence, (3.21) gives

$$\|u_\delta - u^0\|_{H^1_0(\Omega)^n} \leq c\delta^{1/2}.$$

This is still valid for every $u_\delta \in U_\delta$, which implies (3.16)(i).

Estimate 3.16(ii): From (3.12)(iii) and (3.20), it comes

$$\lim_{\varepsilon_m\to 0}\|A_\delta^{\varepsilon_m}e(u_\delta^{\varepsilon_m})-A^{\varepsilon_m}\widetilde{e(u^{\varepsilon_m})}\,\|_{H_0^1(\Omega)^{n\times n}}\le c\delta^{1/2}.$$

But (2.3)(ii) and (3.18)(ii) imply, as $\varepsilon_m\to 0$, that

$$(A_\delta^{\varepsilon_m}e(u_\delta^{\varepsilon_m})-A^{\varepsilon_m}\widetilde{e(u^{\varepsilon_m})})\rightharpoonup(A_\delta e(u_\delta)-A^0e(u^0))\quad\text{weakly in } L^2(\Omega)^{n\times n}.$$

In view of the weak lower semi-continuity of the L^2-norm, these two last relations give

$$\|(A_\delta e(u_\delta)-A^0e(u^0))\|_{L^2(\Omega)^{n\times n}}\le c\delta^{1/2},$$

for every $u_\delta\in U_\delta$. Hence,

$$\sup_{u_\delta\in U_\delta}\|(A_\delta e(u_\delta)-A^0e(u^0))\|_{L^2(\Omega)^{n\times n}}\le c\delta^{1/2},$$

which, together with Lemma 3.1, gives the claimed result. □

4. Main results

Theorem 4.1. *Let $A_\delta\in W_\delta$. Then, the solution u_δ of (3.6) satisfies as $\delta\to 0$*

$$\begin{gathered}u_\delta\to u^0\quad\text{strongly in } H_0^1(\Omega)^n,\\ A_\delta e(u_\delta)\to A^0e(u^0)\quad\text{strongly in } L^2(\Omega)^{n\times n},\end{gathered}\tag{4.1}$$

where u^0 is the solution of (2.4). Moreover, one has the convergence:

$$\forall p\in[1,\infty[,\quad A_\delta\to A^0,\tag{4.2}$$

strongly in $L^p(\Omega)$ and weakly $\star$ in $L^\infty(\Omega)$.

Theorem 4.2. *Let f^ε, f be in $H^{-1}(\Omega)^n$ satisfying (3.4). Suppose that*

$$\forall\varepsilon>0,\ \ \langle f_\varepsilon,v\rangle=0,\quad\forall v\in H_0^1(\Omega)^n,\ v=0\ \text{on }\Omega_\varepsilon.\tag{4.3}$$

Then, as $\delta\to 0$,

$$\begin{gathered}u_\delta^\varepsilon\to u^\varepsilon\quad\text{strongly in } H_0^1(\Omega_\varepsilon)^n,\\ A_\delta^\varepsilon e(u_\delta^\varepsilon)\to A^\varepsilon\widetilde{e(u^\varepsilon)}\quad\text{strongly in } L^2(\Omega)^{n\times n},\end{gathered}\tag{4.4}$$

where u^ε and u_δ^ε are the solutions of (2.2) and (3.5) respectively.

Theorem 4.3. *Let f^ε, $f\in H^{-1}(\Omega)^n$ satisfying (3.4) and (4.3). Then, as $(\varepsilon,\delta)\to(0,0)$*

$$\begin{gathered}u_\delta^\varepsilon\to u^0\quad\text{weakly in } H_0^1(\Omega)^n,\\ A_\delta^\varepsilon e(u_\delta^\varepsilon)\to A^0e(u^0)\quad\text{weakly in } L^2(\Omega)^{n\times n},\end{gathered}\tag{4.5}$$

where u^0 and u_δ^ε are the solutions of (2.4) and (3.5) respectively.

To prove these results we use similar arguments as those used in [3]. Before giving these proofs, we recall the following lemma:

Lemma 4.4 ([3])**.** *Let $\{\psi_m\}$ be a sequence of $L^2(\Omega)$. Suppose that there exists ψ, $\phi\in L^2(\Omega)$ such that*

$$\begin{gathered}\psi_m\to\psi\quad\text{strongly in } L^2_{loc}(\Omega),\\ \forall m\in\mathbb{N},\ |\psi_m|\le\phi\quad\text{a.e. in }\Omega.\end{gathered}$$

Then, $\psi_m\to\psi$ strongly in $L^2(\Omega)$.

Proof of Theorem 4.1. Observe first that (4.1) follows immediately from Proposition 3.4. On the other hand, taking in (2.2) and (3.5), $f^\varepsilon = f \doteq -\operatorname{div}(A^0 e(\varphi \Lambda x))$ with $\varphi \in \mathcal{D}(\Omega)$ and $\Lambda \in \mathbb{R}_S^{n\times n}$, then (2.4) reads

$$\begin{gathered}-\operatorname{div}\,(A^0 e(u^0)) = -\operatorname{div}(A^0 e(\varphi\Lambda x)) \quad \text{in } \Omega,\\ u^0 = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

This implies, in view of the fact that $A^0 \in \mathcal{M}_e(\frac{\alpha}{C^2}, \beta, \Omega)$, that $u^0 = \varphi\Lambda x$. This, together with (4.1), gives

$$\begin{gathered}u_\delta \to \varphi\Lambda x \quad \text{strongly in } H_0^1(\Omega)^n,\\ A_\delta e(u_\delta) \to A^0 e(\varphi\Lambda x) \quad \text{strongly in } L^2(\Omega)^{n\times n}.\end{gathered}$$

Taking now $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi = 1$ on ω and where $\omega \subset\subset \Omega$, one obtains

$$\begin{gathered}u_\delta \to \Lambda \quad \text{strongly in } H^1(\omega)^n,\\ A_\delta e(u_\delta) \to A^0\Lambda \quad \text{strongly in } L^2(\omega)^{n\times n}.\end{gathered} \tag{4.6}$$

On the other hand, one has almost everywhere in ω,

$$\begin{aligned}|A_\delta\Lambda - A^0\Lambda| &\le |A_\delta\Lambda - A_\delta e(u_\delta)| + |A_\delta e(u_\delta) - A^0\Lambda|\\ &\le \beta|\Lambda - e(u_\delta)| + |A_\delta e(u_\delta) - A^0\Lambda|.\end{aligned}$$

Then, using (4.6), one gets

$$A_\delta\Lambda \to A^0\Lambda \quad \text{strongly in } L^2(\omega)^{n\times n},$$

for every $\omega \subset\subset \Omega$. Since $|A_{\delta_m}\Lambda| \le \beta|\Lambda|$, this gives by Lemma 4.4 written for $\psi_m = A_{\delta_m}\Lambda$ (with $\delta_m \to 0$),

$$A_\delta\Lambda \to A^0\Lambda \quad \text{strongly in } L^2(\Omega)^{n\times n}.$$

By the symmetric properties of A_δ and A^0, this convergence is still valid for every matrix $\Lambda \in \mathbb{R}^{n\times n}$. Thus

$$A_\delta \to A^0 \quad \text{strongly in } L^2(\Omega)^{n\times n}.$$

From this convergence and the fact that $\|A_\delta\|_{L^\infty(\Omega)} \le \beta$, one obtains convergence (4.2). □

Proof of Theorem 4.2. From hypothesis (4.3), Proposition 3.3 and the fact that $P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u^\varepsilon_\delta = 0$ in Ω_ε, it follows that

$$\begin{gathered}\|u^\varepsilon_{\delta|\Omega_\varepsilon} - u^\varepsilon\|_{H_0^1(\Omega_\varepsilon)^n} \le \|P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - P^\varepsilon u^\varepsilon\|_{H_0^1(\Omega)^n} \le c\delta^{1/2},\\ \|A^\varepsilon_\delta e(u^\varepsilon_\delta) - A^\varepsilon\widetilde{e(u^\varepsilon)}\,\|_{L^2(\Omega)^{n\times n}} \le c\delta^{1/2}.\end{gathered}$$

Passing to the limit as $\delta \to 0$, one obtains (4.4). □

Proof of Theorem 4.3. **(i)** Under hypothesis (4.3), Proposition 3.3 gives

$$\lim_{(\varepsilon,\delta)\to(0,0)} \|P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - P^\varepsilon u^\varepsilon\|_{H_0^1(\Omega)^n} = 0$$

and the fact that $A^\varepsilon \xrightarrow{H_e} A^0$ implies

$$P_\varepsilon u^\varepsilon - u^0 \rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega).$$

Hence, by passing to the weak limit in $H_0^1(\Omega)$ as $(\varepsilon, \delta) \to (0, 0)$ in the following equality:

$$P_\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u^0 = (P_\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - P_\varepsilon u^\varepsilon) + (P_\varepsilon u^\varepsilon - u^0),$$

one deduces

$$P_\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega)^n. \tag{4.7}$$

On the other hand, using (4.3) and Proposition 3.3, one gets

$$\|P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - P^\varepsilon u^\varepsilon\|_{H_0^1(\Omega)^n} \le c\delta^{1/2},$$
$$\|e(u^\varepsilon_\delta)\|_{L^2(T_\varepsilon)^{n\times n}} \le c.$$

This implies

$$\|u^\varepsilon_\delta\|_{H_0^1(\Omega_\varepsilon)^n} \le \|P^\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon})\|_{H_0^1(\Omega)^n} \le c\delta^{1/2} + \|P^\varepsilon u^\varepsilon\|_{H_0^1(\Omega)^n},$$
$$\|e(u^\varepsilon_\delta)\|_{L^2(T_\varepsilon)^{n\times n}} \le c.$$

Since $P^\varepsilon u^\varepsilon$ is bounded independently of ε in $H_0^1(\Omega)^n$, one deduces that u^ε_δ is bounded independently of ε and δ in $H_0^1(\Omega)^n$. This, together with (4.7) and Proposition 2.7, gives (4.5)i).

(ii) Using Proposition 3.3, the fact that $A^\varepsilon \xrightharpoonup{H_\varepsilon} A^0$ and

$$A^\varepsilon_\delta e(u^\varepsilon_\delta) - A^0 e(u^0) = (A^\varepsilon_\delta e(u^\varepsilon_\delta) - A^\varepsilon \widetilde{e(u^\varepsilon)}) + (A^\varepsilon \widetilde{e(u^\varepsilon)} - A^0 e(u^0)),$$

one obtains the convergence 4.5(ii). □

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