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ON ASYMPTOTIC BEHAVIOR OF GLOBAL SOLUTIONS FOR HYPERBOLIC HEMIVARIATIONAL INEQUALITIES

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Abstract. In this paper we study the existence of global weak solutions for a hyperbolic differential inclusion with a discontinuous and nonlinear multi-valued term. Also we investigate the asymptotic behavior of solutions.

1. Introduction

The main purpose of this paper is to investigate the initial boundary value problem for the hyperbolic differential inclusion

$$u'' + A^2u + M(\|A^{1/2}u\|^2)Au + \varphi(u') \ni 0 \quad \text{in } (0,\infty)\times\Omega, \tag{1.1}$$

$$u(0) = u_0, u'(0) = u_1 \quad \text{in } \Omega, \tag{1.2}$$

where φ is a discontinuous and nonlinear set-valued mapping by filling in jumps a function $b \in L^\infty_{\rm loc}(\mathbb{R})$. The precise hypothesis on the above system will be given in the next section.

Recently, a class of nonlinear Cauchy problems are studied by many authors [2, 3, 6, 7, 9] Medeiros [3] studied the equation

$$u'' + A^2u + M(\|A^{1/2}u\|^2)Au = 0,$$

where A is a linear operator in a Hilbert space H and M is a real function. Rivera [9] investigated the equation

$$u'' + A^2u + M(\|A^{1/2}u\|^2)Au + g(u') = 0, \tag{1.3}$$

when the damping term is linear, i.e., $g(x) = \delta x$ and Patcheu [7] studied the existence and asymptotic behavior of the solutions of (1.3) when g is a nonlinear and nondecreasing continuous functions. Motivated by works of Patcheu [7], we consider more generalized problem (1.1) with a discontinuous and nonlinear multi-valued term φ. Thus, in this paper we shall deal with the existence and asymptotic behavior of the global weak solution of the hemivariational inequality (1.1)-(1.2). The background of these variational problems are in physics, especially in continuum mechanics, where nonmonotone, multi-valued constitutive laws lead to the above-cited hemivariational inequalities.

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At this point it is important to mention that such hemivariational inequalities were studied by some authors [4, 5, 8], but, in their works no decay rates were obtained as in this present paper. The plan of this paper as follows. In section 2, the assumptions and the main results are given. In section 3, the existence of a solution to the (1.1)-(1.2) is proved by using the Faedo-Galerkin method and finally in section 4, the decay of solutions is investigated.

2. Assumptions and main results

First we explain the notation used throughout this paper. Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $Q = [0, T] \times \Omega$, where T be any positive real number. Let $H = L^2(\Omega)$ with inner product and norm respectively denoted by $(\cdot, \cdot)$ and $\|\cdot\|$. Let A be a linear operator in H, with domain $D(A) = V$ dense in H and the graph norm denoted by $\|\cdot\|_V$. Let V' be the dual of V and let $\langle \cdot, \rangle$ denote the duality pairing between V' and V. We assume that the imbedding $V \subset H$ is compact. Identifying H and H', it follows that $V \subset H \subset V'$ and the imbedding $H \subset V'$ is also compact.

The following hypothesis will be used throughout this paper.

(H1) A is self-adjoint and positive, i.e., there is a constant $\mu_0 > 0$ such that

$$(Av, v) \geq \mu_0 \|v\|^2, \quad \forall v \in V. \tag{2.1}$$

Define $A^2 : V \to V'$ by

$$\langle A^2 u, v\rangle = (Au, Av), \quad u, v \in V$$

with $W = D(A^2) = \{u \in V | A^2 u \in H\}$.

(H2) $M(s)$ is a C^1 real function and there exist $\alpha, \beta > 0$ such that

$$M(s) \geq \alpha + \beta s, \quad M'(s) \geq 0. \tag{2.2}$$

Let $\bar{M}(t)$ be defined by

$$\bar{M}(t) = \int_0^t M(s) ds.$$

(H3) (1) $b \in L^\infty_{\rm loc}(\mathbb{R})$. (2) There exist $\mu_1, \mu_2 > 0$ such that $b(s)s \geq \mu_1 |s|^2$ and $|b(s)| \leq \mu_2 |s|$ for all $s \in \mathbb{R}$.

The multi-valued function $\varphi : \mathbb{R} \to 2^{\mathbb{R}}$ is obtained by filling in jumps of a function $b : \mathbb{R} \to \mathbb{R}$ by means of the functions $\underline{b}_\epsilon$, $\overline{b}_\epsilon$, $\underline{b}$, $\overline{b}$ from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$\underline{b}_\epsilon(t) = \operatorname*{ess\,inf}_{|s-t|\leq\epsilon} b(s), \quad \overline{b}_\epsilon(t) = \operatorname*{ess\,sup}_{|s-t|\leq\epsilon} b(s);$$

$$\underline{b}(t) = \lim_{\epsilon\to 0^+} \underline{b}_\epsilon(t), \quad \overline{b}(t) = \lim_{\epsilon\to 0^+} \overline{b}_\epsilon(t); \quad \varphi(t) = [\underline{b}(t), \overline{b}(t)].$$

We shall need a regularization of b defined by

$$b^m(t) = m \int_{-\infty}^{\infty} b(t-\tau)\rho(m\tau) d\tau,$$

where $\rho \in C_0^\infty((-1, 1)), \rho \geq 0$ and $\int_{-1}^1 \rho(\tau) d\tau = 1$. It is easy to show that b^m is continuous for all $m \in \mathbb{N}$ and $\underline{b}_\epsilon, \overline{b}_\epsilon, \underline{b}, \overline{b}, b^m$ satisfy the same condition (H3)(2) with a possibly different constant if b satisfies (H3)(2).

Now we are in a position to state our existence result.

Theorem 2.1. *Assume that the conditions (H1)-(H3) hold. Then given* $u_0 \in W$ *and* $u_1 \in V$*, there exist* $\Xi \in L^\infty(0,T;H)$ *and a function* $u : [0,T] \times \Omega \to \mathbb{R}$ *such that*

$$u \in L^\infty(0,T;V), u' \in L^\infty(0,T;H), u'' \in L^\infty(0,T;V')$$

and

$$u'' + A^2u + M(\|A^{1/2}u\|^2)Au + \Xi = 0 \quad \text{in } L^\infty(0,T;V'), \tag{2.3}$$

$$\Xi(t,x) \in \varphi(u'(t,x)) \quad \text{a.e. } (t,x) \in Q, \tag{2.4}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.5}$$

Next, we establish the decay result. Let us define the energy of the system (1.1)-(1.2) as

$$E(t) = \frac{1}{2}\big\{\|u'(t)\|^2 + \|Au(t)\|^2 + \bar{M}(\|A^{1/2}u(t)\|^2)\big\}. \tag{2.6}$$

Then we have

Theorem 2.2. *Assume that the conditions of Theorem 2.1 hold. Then given* $u_0 \in W$ *and* $u_1 \in V$*, there exist* $\Xi \in L^\infty(0,T;H)$ *and a function* $u : [0,\infty) \times \Omega \to \mathbb{R}$ *such that*

$$\begin{gathered} u \in L^\infty(0,\infty;V), \quad u' \in L^\infty(0,T;H), \quad u'' \in L^\infty(0,T;V'), \\ u'' + A^2u + M(\|A^{1/2}u\|^2)Au + \Xi = 0 \quad \text{in } L^\infty(0,\infty;V'), \\ \Xi(t,x) \in \varphi(u'(t,x)) \quad \text{a.e. } (t,x) \in (0,\infty) \times \Omega, \\ u(0) = u_0, \quad u'(0) = u_1 \end{gathered}$$

and u *satisfies the decay property*

$$E(t) \le Ce^{-\gamma t}, \quad \forall t \ge 0 \tag{2.7}$$

for some positive constants C *and* γ*.*

3. Proof of Theorem 2.1

The proof will be done by applying the Faedo-Galerkin method.

Step 1 : A priori estimate I. Assume, for simplicity, $V = D(A)$ is separable, then there is a sequence $(w_j)_{j\ge1}$ consisting of eigenfunctions of the operator A corresponding to positive real eigenvalues λ_j tending to ∞. Hence $Aw_j = \lambda_j w_j, j \ge 1$. Let us define $W_m = Span\{w_1, w_2, \dots, w_m\}$. Note that $(w_j)_{j\ge1}$ is a basis of H, V and W.

Consider a regularized Galerkin equation

$$(u_m''(t)+A^2u_m(t)+M(\|A^{1/2}u_m(t)\|^2)Au_m(t)+b^m(u_m'(t)), v) = 0, \quad \forall v \in W_m \tag{3.1}$$

with the initial conditions

$$u_m(0) = u_{0m} = \sum_{j=1}^m (u_0, w_j)w_j, \quad u_{0m} \to u_0 \quad \text{in } W, \tag{3.2}$$

$$u_m'(0) = u_{1m} = \sum_{j=1}^m (u_1, w_j)w_j, \quad u_{1m} \to u_1 \quad \text{in } V. \tag{3.3}$$

Substituting $u_m(t)=\sum_{j=1}^m g_{mj}(t)w_j$ in (3.1) gives a second-order ordinary differential equations and its local solution $g_{mj}(t)$ exists on $[0,t_m), 0<t_m<T$. Replacing v by $u_m'(t)$ in (3.1), we find

$$\frac{1}{2}\frac{d}{dt}\big\{\|u_m'(t)\|^2+\|Au_m(t)\|^2+\bar{M}(\|A^{1/2}u_m(t)\|^2)\big\}=-(b^m(u_m'(t)),u_m'(t)). \quad (3.4)$$

Integrating in $(0,t), t\le t_m$,

$$\begin{aligned}&\frac{1}{2}\big\{\|u_m'(t)\|^2+\|Au_m(t)\|^2+\bar{M}(\|A^{1/2}u_m(t)\|^2)\big\}\\&=\frac{1}{2}\big\{\|u_{1m}\|^2+\|Au_{0m}\|^2+\bar{M}(\|A^{1/2}u_{0m}\|^2)\big\}-\int_0^t(b^m(u_m'(s)),u_m'(s))ds.\end{aligned} \quad (3.5)$$

By (H3)(2) and Hölder inequality,

$$\begin{aligned}\int_0^t(b^m(u_m'(s)),u_m'(s))ds&\le\Big(\int_0^t\|b^m(u_m'(s))\|^2ds\Big)^{1/2}\Big(\int_0^t\|u_m'(s)\|^2ds\Big)^{1/2}\\&\le\mu_2\int_0^t\|u_m'(s)\|^2ds.\end{aligned} \quad (3.6)$$

From (3.5) and (3.6) we have

$$\begin{aligned}&\frac{1}{2}\big\{\|u_m'(t)\|^2+\|Au_m(t)\|^2+\bar{M}(\|A^{1/2}u_m(t)\|^2)\big\}\\&\le\frac{1}{2}\big\{\|u_{1m}\|^2+\|Au_{0m}\|^2+\bar{M}(\|A^{1/2}u_{0m}\|^2)\big\}+\mu_2\int_0^t\|u_m'(s)\|^2ds.\end{aligned} \quad (3.7)$$

In what follows, we use C to denote a generic positive constant independent of m. Using (3.2), (3.3), (3.7) and Gronwall's inequality we obtain

$$\|u_m'(t)\|^2+\|Au_m(t)\|^2+\bar{M}(\|A^{1/2}u_m(t)\|^2)\le C. \quad (3.8)$$

Using (2.2), we deduce that

$$\|u_m'(t)\|^2+\|Au_m(t)\|^2+\alpha\|A^{1/2}u_m(t)\|^2+\frac{\beta}{2}\|A^{1/2}u_m(t)\|^4\le C. \quad (3.9)$$

By (H3)(2) and (3.8) we also obtain

$$\begin{aligned}\|b^m(u_m'(t))\|&=\Big(\int_\Omega(b^m(u_m'(t,x)))^2dx\Big)^{1/2}\\&\le\mu_2\Big(\int_\Omega|u_m'(t,x)|^2dx\Big)^{1/2}\\&=\mu_2\|u_m'(t)\|\le C.\end{aligned} \quad (3.10)$$

So we can extend the approximated solutions $u_m(t)$ to the whole interval $[0,T]$ and we get

$$(u_m') \text{ is bounded in } L^\infty(0,T;H), \quad (3.11)$$
$$(Au_m) \text{ is bounded in } L^\infty(0,T;H), \quad (3.12)$$
$$(b^m(u_m')) \text{ is bounded in } L^\infty(0,T;H). \quad (3.13)$$

Moreover by assumption (2.1),

$$(u_m) \text{ is bounded in } L^\infty(0,T;H). \quad (3.14)$$

Step 2 : A priori estimate II. It follows from (3.1) that for all v in W_m,

$$\begin{aligned} &|\langle u_m''(t), v\rangle| \\ &\le \|Au_m(t)\|\|Av\| + |M(\|A^{1/2}u_m(t)\|^2)|\|Au_m(t)\|\|v\| + \|b^m(u_m'(t))\|\|v\| \\ &\le (\|Au_m(t)\| + |M(\|A^{1/2}u_m(t)\|^2)|\|Au_m(t)\| + \|b^m(u_m'(t))\|)\|v\|_V . \end{aligned} \tag{3.15}$$

Since M is a C^1 real function, we have from (3.9) and (3.10), by a density argument, that

$$(u_m'') \text{ is bounded in } L^\infty(0,T;V'). \tag{3.16}$$

Step 3: Passage to the limit. From the priori estimates (3.11)-(3.14) and (3.16), we have subsequences (in the sequence we denote subsequences by the same symbols as original sequences) such that

$$u_m \to u \text{ weakly star in } L^\infty(0,T;V), \tag{3.17}$$

$$Au_m \to Au \text{ weakly star in } L^\infty(0,T;H), \tag{3.18}$$

$$u_m' \to u' \text{ weakly star in } L^\infty(0,T;H), \tag{3.19}$$

$$u_m'' \to u'' \text{ weakly star in } L^\infty(0,T;V'), \tag{3.20}$$

$$b^m(u_m') \to \Xi \text{ weakly star in } L^\infty(0,T;H), \tag{3.21}$$

$$M(\|A^{1/2}u_m\|^2)Au_m \to \Psi \text{ weakly star in } L^\infty(0,T;H). \tag{3.22}$$

By the Aubin-Lions compactness lemma [1], from (3.17), (3.19) and (3.20) that

$$u_m \to u \text{ strongly in } L^2(0,T;H), \tag{3.23}$$

$$u_m' \to u' \text{ strongly in } L^2(0,T;V'). \tag{3.24}$$

Now we shall prove that $\Psi = M(\|A^{1/2}u\|^2)Au$. For $v \in L^2(0,T;H)$, we have

$$\begin{aligned} &\int_0^T (\Psi(t) - M(\|A^{1/2}u(t)\|^2)Au(t), v)dt \\ &= \int_0^T (\Psi(t) - M(\|A^{1/2}u_m(t)\|^2)Au_m(t), v)dt \\ &\quad + \int_0^T (M(\|A^{1/2}u(t)\|^2)(Au_m(t) - Au(t)), v)dt \\ &\quad + \int_0^T (M(\|A^{1/2}u_m(t)\|^2) - M(\|A^{1/2}u(t)\|^2))(Au_m(t), v)dt. \end{aligned} \tag{3.25}$$

On the other hand, the fact that M is C^1 and (3.9) give

$$\begin{aligned} &\int_0^T (M(\|A^{1/2}u_m(t)\|^2) - M(\|A^{1/2}u(t)\|^2))(Au_m(t), v)dt \\ &\le C\int_0^T \big|\|A^{1/2}u_m(t)\|^2 - \|A^{1/2}u(t)\|^2\big|\|Au_m(t)\|\|v\|dt \\ &\le C\int_0^T \big|(A(u_m(t)+u(t)), u_m(t)-u(t))\big|dt \\ &\le C\Big(\int_0^T \|u_m(t)-u(t)\|^2 dt\Big)^{1/2}. \end{aligned} \tag{3.26}$$

Considering (3.18), (3.22), (3.23) and (3.26), we deduce from (3.25) that

$$M(\|A^{1/2}u_m\|^2)Au_m \to M(\|A^{1/2}u\|^2)Au \quad \text{weakly star in } L^\infty(0,T;H). \tag{3.27}$$

Now we may take the limit $m \to \infty$ in the approximated (3.2). Therefore, we obtain

$$\langle u''(t) + A^2u(t) + M(\|A^{1/2}u\|^2)Au + \Xi, v\rangle = 0. \forall v \in V. \tag{3.28}$$

Step 4: (u,Ξ) is a solution of (2.3)-(2.5). Let $\phi \in C^1[0,T]$ with $\phi(T) = 0$. By replacing v by $\phi(t)w_j$ in (3.1) and integrating by parts the result over $(0,T)$, we have

$$\begin{aligned}
&(u_m'(0), \phi(0)w_j) + \int_0^T (u_m'(s), \phi'(s)w_j)ds \\
&= \int_0^T (Au_m(s), \phi(s)Aw_j)ds + \int_0^T (M(\|A^{1/2}u_m\|^2)Au_m(s) + b^m(u_m'(s)), \phi(s)w_j)ds.
\end{aligned} \tag{3.29}$$

Similarly, from (3.28),

$$\begin{aligned}
&(u'(0), \phi(0)w_j) + \int_0^T (u'(s), \phi'(s)w_j)ds \\
&= \int_0^T (Au(s), \phi(s)Aw_j)ds + \int_0^T (M(\|A^{1/2}u\|^2)Au(s) + \Xi(s), \phi(s)w_j)ds.
\end{aligned} \tag{3.30}$$

Comparing (3.29) and (3.30), we infer that

$$\lim_{m\to\infty} (u_m'(0) - u'(0), w_j) = 0, \quad \forall j \in \mathbb{N}.$$

This implies that $u_m'(0) \to u'(0)$ weakly in H. By the uniqueness of limit, $u'(0) = u_1$. Analogously, taking $\phi \in C^2[0,T]$ with $\phi(T) = \phi'(T) = 0$, we may obtain that $u(0) = u_0$.

Next we will show that $\Xi(t,x) \in \varphi(u'(t,x))$ a.e. $(t,x) \in Q$. For this purpose, we show the conclusion

$$u_m'(t,x) \to u'(t,x) \quad \text{a.e. } (t,x) \in Q. \tag{3.31}$$

First we mention the following lemmas.

Lemma 3.1 ([11, 11, p. 221]). *Assume that $(g_m(t))$ is an absolutely continuous sequence defined on $[a,b]$ and $|g_m'(t)| \le F(t)$ a.e. ($m = 1,2,3,\dots$), $F \in L(a,b)$. If $\lim_{m\to\infty} g_m(t) = g(t)$ and $\lim_{m\to\infty} g_m'(t) = f(t)$ a.e. $t \in [a,b]$, then $g'(t) = f(t)$ a.e. $t \in [a,b]$.*

Lemma 3.2 ([10, p. 152]). *If $(u_m) \subset C[0,T]$, $u \in C[0,T]$ and $u_m \to u$ weakly, then $\lim_{m\to\infty} u_m(t) = u(t)$, $t \in [0,T]$.*

Set $u(t,x) = \sum_{j=1}^\infty g_j(t)w_j$ and $u'(t,x) = \sum_{j=1}^\infty f_j(t)w_j$. Let $j \in \mathbb{N}$ be fixed. Since $u_m(t) \to u(t)$ in H for a.e. $t \in [0,T]$,

$$\lim_{m\to\infty} (u_m(t,x) - u(t,x), w_j) = 0, \quad \lim_{m\to\infty} (g_{mj}(t) - g_j(t)) = 0$$

a.e. $t \in [0,T]$. Since $|g_{mj}'(t)|$ is bounded a.e. $t \in [0,T]$ (see (3.11)), there exists a function $\xi_j(t)$ defined on $[0,T]$ such that

$$\lim_{m\to\infty} g_{mj}'(t) = \xi_j(t) \quad \text{a.e. } t \in [0,T].$$

By Lemma 3.1, $\xi_j(t) = g'_j(t)$ a.e. $t \in [0,T]$. Hence $\lim_{m\to\infty} g'_{mj}(t) = g'_j(t)$. Let

$$v(s,x) = \begin{cases} w_j & \text{if } s \in [0,t], \\ 0 & \text{if } s \in [t,T]. \end{cases}$$

Since $u'_m \to u'$ weakly star in $L^\infty(0,T;H)$, $\int_Q u'_m(s,x)v\,dx\,ds \to \int_Q u'(s,x)v\,dx\,ds$ as $m \to \infty$, and hence

$$g_j(t) = g_j(0) + \int_0^t f_j(s)ds.$$

This implies that $g'_j(t) = f_j(t)$ a.e. $t \in [0,T]$. Since $u'_m \to u'$ weakly in H for a.e. $t \in [0,T]$ and $u' \in C(0,T;V'), u'(t,x) = \sum_{j=1}^\infty g'_j(t)w_j \in C[0,T]$ and $u'_m(t,x) \to u'(t,x)$ weakly in $C[0,T]$ for a.e. $x \in \Omega$. Thus by Lemma 3.2, we conclude that (3.31). Thus, for given $\eta > 0$, using the theorems of Lusin and Egoroff, we can choose a subset $\omega \subset Q$ such that $\operatorname{meas}(\omega) < \eta$, $u' \in L^\infty(Q \setminus \omega)$ and $u'_m \to u'$ uniformly on $Q \setminus \omega$. Thus, for each $\epsilon > 0$, there is an $N > \frac{2}{\epsilon}$ such that

$$|u'_m(t,x) - u'(t,x)| < \frac{\epsilon}{2}, \quad \forall (t,x) \in Q \setminus \omega.$$

Then, if $|u'_m(t,x) - s| < 1/m$, we have $|u'(t,x) - s| < \epsilon$ for all $m > N$ and $(t,x) \in Q \setminus \omega$. Therefore we have

$$\underline{b}_\epsilon(u'(t,x)) \le b^m(u'_m(t,x)) \le \overline{b}_\epsilon(u'(t,x)), \quad \forall m > N, (t,x) \in Q \setminus \omega.$$

Let $\phi \in L^\infty(Q)$, $\phi \ge 0$. Then

$$\begin{aligned} \int_{Q\setminus\omega} \underline{b}_\epsilon(u'(t,x))\phi(t,x)\,dx\,dt &\le \int_{Q\setminus\omega} b^m(u'_m(t,x))\phi(t,x)\,dx\,dt \\ &\le \int_{Q\setminus\omega} \overline{b}_\epsilon(u'(t,x))\phi(t,x)\,dx\,dt. \end{aligned} \tag{3.32}$$

Letting $m \to \infty$ in (3.32) and using (3.21), we obtain

$$\begin{aligned} \int_{Q\setminus\omega} \underline{b}_\epsilon(u'(t,x))\phi(t,x)\,dx\,dt &\le \int_{Q\setminus\omega} \Xi(t,x)\phi(t,x)\,dx\,dt \\ &\le \int_{Q\setminus\omega} \overline{b}_\epsilon(u'(t,x))\phi(t,x)\,dx\,dt. \end{aligned} \tag{3.33}$$

Letting $\epsilon \to 0^+$ in (3.33), we infer that

$$\Xi(t,x) \in \varphi(u'(t,x)) \quad \text{a.e. in } Q \setminus \omega,$$

and letting $\eta \to 0^+$, we obtain

$$\Xi(t,x) \in \varphi(u'(t,x)) \quad \text{a.e. in} Q.$$

Therefore, the proof of Theorem 2.1 is complete.

Remark 3.1. Even if we replace the condition (H3)(2) by the weaker linear growth condition:

$$|b(s)| \le \mu_2(1 + |s|), \quad \forall s \in \mathbb{R},$$

we obtain the same results as in Theorem 2.1.

Remark 3.2. If in Theorem 2.1 we impose the condition that b is nondecreasing, then we obtain the stronger results. In other words, the solution u of (2.3)-(2.5) satisfies

$$u \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H).$$

Since the proof of this result is similar to that of [7, Theorem 1.1], we omit it here.

4. Energy decay of solutions

In this section we shall prove Theorem 2.2 by applying the following lemma by Nakao [6].

Lemma 4.1. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a bounded nonnegative function for which there exist constant $\delta > 0$ such that*

$$\sup_{t \le s \le t+1} \phi(s) \le \delta(\phi(t) - \phi(t+1)), \quad \forall t \ge 0.$$

Then there exist positive constants C and γ such that

$$\phi(t) \le Ce^{-\gamma t}, \quad \forall t \ge 0.$$

Proof of Theorem 2.2. The existence part of solution of Theorem 2.2 is a consequence of the proof of Theorem 2.1. To prove the decay property, we first obtain uniform estimates for the approximated energy,

$$E_m(t) = \frac{1}{2}\big(\|u_m'(t)\|^2 + \|Au_m(t)\|^2 + \bar{M}(\|A^{1/2}u_m\|^2)\big) \tag{4.1}$$

and then pass to the limit. Note that $E_m(t)$ is non-negative and uniformly bounded. Let us fix an arbitrary $t > 0$. From the approximated problem (3.1) with $v = u_m'(t)$, by (H3)(2) we have

$$\frac{d}{dt}E_m(t) = -(b^m(u_m'(t)), u_m'(t)) \le -\mu_1\|u_m'(t)\|^2. \tag{4.2}$$

This implies that $E_m(t)$ is a non-increasing function. Setting $F_m^2(t) = E_m(t) - E_m(t+1)$ and integrating (4.2) over $(t, t+1)$ we have

$$F_m^2(t) \ge \mu_1 \int_t^{t+1} \|u_m'(s)\|^2 ds. \tag{4.3}$$

By applying the mean value theorem, there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_m'(t_i)\| \le \frac{2}{\sqrt{\mu_1}}F_m(t), \quad i = 1, 2. \tag{4.4}$$

Now, replacing v by $u_m(t)$ in the approximated problem we have

$$\begin{aligned} &(A^2u_m(t), u_m(t)) + (M(\|A^{1/2}u_m(t)\|^2)Au_m(t), u_m(t)) \\ &= -(u_m''(t), u_m(t)) - (b^m(u_m'(t)), u_m(t)). \end{aligned} \tag{4.5}$$

Integrating (4.5) over (t_1, t_2) and using (H3)(2), we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} \|Au_m(s)\|^2 + M(\|A^{1/2}u_m\|^2)\|A^{1/2}u_m(s)\|^2 ds \\ &= -(u_m'(t_2), u_m(t_2)) + (u_m'(t_1), u_m(t_1)) \\ &\quad + \int_{t_1}^{t_2} \|u_m'(s)\|^2 ds - \int_{t_1}^{t_2}\int_\Omega b^m(u_m'(s,x))u_m(s,x)\,dx\,ds \\ &\le \|u_m'(t_2)\|\|u_m(t_2)\| + \|u_m'(t_1)\|\|u_m(t_1)\| \\ &\quad + \int_{t_1}^{t_2} \|u_m'(s)\|^2 ds + \mu_2 \int_{t_1}^{t_2} \|u_m'(s)\|\|u_m(s)\| ds. \end{aligned} \tag{4.6}$$

Using Hölder's inequality and (2.1), from (4.3), (4.4) and (4.6), we have

$$\begin{aligned} &\int_{t_1}^{t_2} E_m(s) ds \\ &\le \frac{3}{2\mu_1}F_m^2(t) + \frac{2}{\mu_0\sqrt{\mu_1}}F_m(t)\|Au_m(t_2)\| \\ &\quad + \frac{2}{\mu_0\sqrt{\mu_1}}F_m(t)\|Au_m(t_1)\| + \frac{\mu_2}{\mu_0}\big(\int_{t_1}^{t_2} \|u_m'(s)\|^2 ds\big)^{1/2} \sup_{t_1\le s\le t_2}\|Au_m(s)\| \end{aligned} \tag{4.7}$$

and then we have

$$\int_{t_1}^{t_2} E_m(s) ds \le C_1 F_m^2(t) + C_2 F_m(t) E_m(t)^{1/2}, \tag{4.8}$$

where C_1, C_2 are a generic positive constant independent of m. Noting that $E_m(t+1) \le 2\int_{t_1}^{t_2} E_m(s)ds$ and $E_m(t+1) = E_m(t) - F_m^2(t)$, from (4.8) we have

$$\begin{aligned} E_m(t) &\le 2\int_{t_1}^{t_2} E_m(s)ds + F_m^2(t) \\ &\le (2C_1+1)F_m^2(t) + 2C_2F_m(t)E_m(t)^{1/2}. \end{aligned} \tag{4.9}$$

Young's inequality implies

$$E_m(t) \le C_3 F_m^2(t) \tag{4.10}$$

for some positive constant C_3. Since E_m is non-increasing, from (4.10), we have

$$\sup_{t\le s\le t+1} E_m(s) \le C_3(E_m(t) - E_m(t+1)), \quad \forall t \ge 0.$$

Applying Lemma 4.1, there exist positive constants C and γ such that

$$E_m(t) \le Ce^{-\gamma t}, \forall t \ge 0.$$

Passing to the limit $m \to \infty$, we get (2.7). This completes the proof of Theorem 2.2. □

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