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EXISTENCE OF PERIODIC SOLUTION FOR PERTURBED GENERALIZED LIÉNARD EQUATIONS

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Abstract. Under conditions of Levinson-Smith type, we prove the existence of a τ-periodic solution for the perturbed generalized Liénard equation

$$u'' + \varphi(u,u')u' + \psi(u) = \epsilon\omega(\frac{t}{\tau}, u, u')$$

with periodic forcing term. Also we deduce sufficient condition for existence of a periodic solution for the equation

$$u'' + \sum_{k=0}^{2s+1} p_k(u){u'}^k = \epsilon\omega(\frac{t}{\tau}, u, u').$$

Our method can be applied also to the equation

$$u'' + [u^2 + (u+u')^2 - 1]u' + u = \epsilon\omega(\frac{t}{\tau}, u, u').$$

The results obtained are illustrated with numerical examples.

1. Introduction

Consider Liénard equation

$$u'' + \varphi(u)u' + \psi(u) = 0$$

where $u' = \frac{du}{dt}$, $u'' = \frac{d^2u}{dt^2}$, φ and ψ are C^1. Studying the existence of periodic solution of period τ_0 has been purpose of many authors: Farkas [3] presents some typical works on this subject, where the Poincaré-Bendixson theory plays a crucial role. In general, a periodic perturbation of the Liénard equation does not possess a periodic solution as described by Moser; see for example [1].

Let us consider the perturbed Liénard equation

$$u'' + \varphi(u)u' + \psi(u) = \epsilon\omega(\frac{t}{\tau}, u, u')$$

where ω is a *controllably periodic perturbation* in the Farkas sense; i.e., it is periodic with a period τ which can be choosen appropriately. The existence of a non trivial periodic solution for (2) was studied by Chouikha [1]. Under very mild conditions it is proved that to each small enough amplitude of the perturbation there belongs a one parameter family of periods τ such that the perturbed system has a unique periodic solution with this period.

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Let us consider now the following generalized Liénard equation, which is "a more realistic assumption in modelling many real world phenomena" as stated in [3, page 105]

$$u'' + \varphi(u, u')u' + \psi(u) = 0. \tag{1.1}$$

Where φ and ψ are C^1 and satisfy some assumptions that will be specified below. The leading work of investigation for the existence of periodic solution of generalized Liénard systems was established by Levinson-Smith [4]. Let us define conditions C_{LS}.

Definition. The functions φ and ψ satisfy the condition C_{LS} if: $x\psi(x) > 0$ for $|x| > 0$,

$$\int_0^x \psi(s)ds = \Psi(x) \quad \text{and} \quad \lim_{x\to+\infty} \Psi(x) = +\infty, \quad \varphi(0,0) < 0.$$

Moreover, there exist some numbers $0 < x_0 < x_1$ and $M > 0$ such that:

$$\varphi(x, y) \geq 0 \quad \text{for } |x| \geq x_0,$$
$$\varphi(x, y) \geq -M \quad \text{for } |x| \leq x_0$$
$$x_1 > x_0, \quad \int_{x_0}^{x_1} \varphi(x, y(x))dx \geq 10Mx_0$$

for every decreasing function $y(x) > 0$.

Proposition 1.1 (Levinson-Smith [4]). *When the functions φ and ψ are of class C^1 and satisfy condition C_{LS} then the generalized Lienard equation* (1.1) *has at least one non-constant τ_0-periodic solution.*

A non trivial solution will be denoted $u_0(t)$, and its period τ_0. This proposition has many improvements (under weaker hypotheses) due to Zheng and Wax Ponzo; see [3], among other authors.

This article is organized as follows: At first, we prove the existence of a periodic solution for the perturbed generalized Liénard equation

$$u'' + \varphi(u, u')u' + \psi(u) = \epsilon\omega(\frac{t}{\tau}, u, u'), \tag{1.2}$$

Where $t, \epsilon, \tau \in \mathbb{R}$ are such that $|\tau - \tau_0| < \tau_1 < \tau_0$, $|\epsilon| < \epsilon_0$ with $\epsilon_0 \in \mathbb{R}$ sufficient small and τ_1 is a fixed real scalar. We will use the Farkas method which was effective for perturbed Liénard equation. In the third section, we will propose a criteria for the existence of periodic solution for

$$u'' + \sum_{k=0}^{2s+1} p_k(u){u'}^k = \epsilon\omega(\frac{t}{\tau}, u, u'), \tag{1.3}$$

with $s \in \mathbb{N}$ and p_k are C^1 functions, for all $k \leq 2s + 1$. In the second part of the section, using a result of De Castro [2] we will prove uniqueness of a periodic solution for the equation

$$u'' + [u^2 + (u + u')^2 - 1]u' + u = 0. \tag{1.4}$$

Sufficient condition for the existence of periodic solution to

$$u'' + [u^2 + (u + u')^2 - 1]u' + u = \epsilon\omega(\frac{t}{\tau}, u, u'). \tag{1.5}$$

will be found. At the end of the paper, some phase plane examples are given in order to illustrate the above results. In particular, we describe uniqueness of a solution for equation (1.4) and the existence of a solution of equation (1.5) for $\omega(\frac{t}{\tau}, u, u') = (\sin 2t)\ u'$.

2. Periodic solution of perturbed generalized Lienard equation

In this part of this paper we prove the existence of periodic solution of the perturbed generalized Lienard equation (1.2) such that the unperturbed one (1.1) has at least one periodic solution. The method of proof that we will employ was described in [1, 3].

Consider the equation (1.1) We assume that φ and ψ are C^1 and satisfy C_{LS}. Then by Proposition 1.1 there exists at least a non trivial periodic solution denoted $u_0(t)$.

Let the least positive period of the solution $u_0(t)$ be denoted by τ_0 and U be an open subset of $\mathbb{R}^2$ containing $(0, 0)$. These notation will be used in the rest of the paper.

Theorem 2.1. *Let φ and ψ be C^1 and satisfy C_{LS}. Suppose 1 is a simple characteristic multiplier of the variational system associated to (1.1). Then there are two real functions τ, h defined on $U \subset \mathbb{R}^2$ and constants $\tau_1 < \tau_0$ such that the periodic solution $\nu(t, \alpha, a + h(\epsilon, \alpha), \epsilon, \tau(\epsilon, \alpha))$ of the equation*

$$u'' + \varphi(u, u')u' + \psi(u) = \epsilon\omega(\frac{t}{\tau}, u, u'),$$

exists for $(\epsilon, \alpha) \in U$, $|\tau - \tau_0| < \tau_1$, $\tau(0, 0) = \tau_0$ and $h(0, 0) = 0$.

We point out that the characteristic multipliers are the eigenvalues of the characteristic matrix which is the fundamental matrix in the time τ_0.

Proof of Theorem 2.1. Following the method used in [3], we set $x_2 = u$, $x_1 = \frac{du}{dt} = u'$ and note $x = \mathrm{col}(x_1, x_2) = \mathrm{col}(u', u)$. The plane equivalent system of (1.1) is

$$x' = f(x) \Longleftrightarrow \begin{cases} x_1' = -\varphi(x_2, x_1)x_1 - \psi(x_2) \\ x_2' = x_1 \end{cases} \tag{2.1}$$

with

$$f(x) = \mathrm{col}(-\varphi(x_2, x_1)x_1 - \psi(x_2), x_1).$$

Then the system (2.1) has the periodic solution $q(t)$ with period τ_0. We define

$$q(t) = \mathrm{col}({u_0}'(t), u_0(t))$$

and therefore

$$q'(t) = \mathrm{col}(-\varphi(u_0(t), {u_0}'(t)){u_0}'(t) - \psi(u_0(t)), {u_0}'(t)).$$

The variational system associated with (2.1) is

$$y' = {f_x}'(q(t))y, \tag{2.2}$$

Without loss of generality, we take the initial conditions

$$t = 0, \quad u_0(0) = a < 0 \quad \text{and} \quad u_0'(0) = 0$$

Hence ${f_x}'(q(t))$ is the matrix

$$\begin{pmatrix} -{\varphi'}_{x_1}(u_0(t), {u_0}'(t)){u_0}'(t) - \varphi(u_0(t), {u_0}'(t)) & -{\varphi'}_{x_2}(u_0(t), {u_0}'(t)){u_0}'(t) - \psi'(u_0(t)) \\ 1 & 0 \end{pmatrix}$$

Notice that $q'(t) = \mathrm{col}(-\varphi(u_0(t), {u_0}'(t)){u_0}'(t) - \psi(u_0(t), {u_0}'(t))$ is the first solution of the variational system. Now we calculate the second one, denoted by $\widehat{y}(t) =$

$\mathrm{col}(\widehat{y}_1(t), \widehat{y}_2(t))$ and linearly independent with $q'(t) = y(t)$, in order to write the fundamental matrix. Consider

$$I(s) = \exp\Big[- \int_0^s ({\varphi'}_{x_1}(u_0(\rho), {u_0}'(\rho)){u_0}'(\rho) + \varphi(u_0(\rho), {u_0}'(\rho)))d\rho\Big]$$

and

$$\begin{aligned}\pi(t) = -\int_0^t &\big(\varphi(u_0(\rho), {u_0}'(\rho)){u_0}'(\rho) + \psi(u_0(\rho))\big)^{-2}\big({\varphi'}_{x_2}(u_0(t), {u_0}'(t)){u_0}'(t) \\ &+ \psi'(u_0(t))\big)I(\rho)d\rho\end{aligned}$$

We then obtain

$$\begin{gathered}\widehat{y}_1(t) = -[\varphi(u_0(t), {u_0}'(t)){u_0}'(t) + \psi(u_0(t)]\pi(t), \\ \widehat{y}_2(t) = {u_0}'(t)\pi(t) + \pi'(t)\frac{\varphi(u_0(t), {u_0}'(t)){u_0}'(t) + \psi(u_0(t)}{{\varphi'}_{x_2}(u_0(t), {u_0}'(t)){u_0}'(t) + \psi'(u_0(t))}\end{gathered}$$

It is known, [1, 3], that the fundamental matrix satisfying $\Phi(0) = Id_2$ is $\Phi(t)$ equals to

$$\begin{pmatrix} \frac{\varphi(u_0(t),{u_0}'(t)){u_0}'(t)+\psi(u_0(t))}{\psi(a)} & \psi(a)\pi(t)[\varphi(u_0(t), {u_0}'(t)){u_0}'(t) + \psi(u_0(t)] \\ -\frac{{u_0}'(t)}{\psi(a)} & -\psi(a)u_0'(t)\pi(t) - \psi(a)\pi'(t)\frac{\varphi(u_0(t),{u_0}'(t)){u_0}'(t)+\psi(u_0(t))}{{\varphi'}_{x_2}(u_0(t),{u_0}'(t)){u_0}'(t)+\psi'(u_0(t))} \end{pmatrix}$$

Thus,

$$\Phi(\tau_0) = \begin{pmatrix} 1 & \psi(a)^2\pi(\tau_0) \\ 0 & \rho_2 \end{pmatrix}.$$

We use the Liouville's formula

$$\det \Phi(t) = \det \Phi(0) \exp \int_0^t \mathrm{Tr}({f_x}'(q(\tau)))d\tau.$$

Since $\det(\Phi(0)) = 1$, we deduce the characteristic multipliers associated with (2.2): $\rho_1 = 1$ and $\rho_2 = I(\tau_0) = \exp\Big[- \int_0^{\tau_0} ({\varphi'}_{x_1}(u_0(\rho), {u_0}'(\rho)){u_0}'(\rho) + \varphi(u_0(\rho), {u_0}'(\rho)))d\rho\Big]$.

From [3], we have:

$$J(\tau_0) = -Id_2 + \begin{bmatrix} -\psi(a) & 0 \\ 0 & 0 \end{bmatrix} + \Phi(\tau_0)$$

Hence we obtain the jacobian matrix

$$J(\tau_0) = \begin{pmatrix} -\psi(a) & \psi(a)^2\pi(\tau_0) \\ 0 & \rho_2 - 1 \end{pmatrix},$$

Since 1 is a simple characteristic multiplier ($\rho_2 \neq 1$), $\det J(0,0,0,\tau_0) \neq 0$. We define the periodicity condition

$$z(\alpha, h, \epsilon, \tau) := \nu(\alpha + \tau, a + h, \epsilon, \tau) - (a + h) = 0. \tag{2.3}$$

By the Implicit Function Theorem there are $\epsilon_0 > 0$ and $\alpha_0 > 0$ and uniquely determined functions τ and h defined on $U = \{(\alpha, \epsilon) \in \mathbb{R}^2 : |\epsilon| < \epsilon_0, |\alpha| < \alpha_0\}$ such that: $\tau, h \in C^1$, $\tau(0,0) = T_0, h(0,0) = 0$ and $z(\alpha, h, \epsilon, \tau) \equiv 0$. Because of (2.3), the periodic solution of (1.2) has period $\tau(\epsilon, \alpha)$ near T_0 and has path near the path of the unperturbed solution. $\square$

In particular if $\rho_2 < 1$, the periodic solution is orbitally asymptotically stable i.e. stable in the Liapunov sense and it is attractive see [3, page 346]. Thus, the following inequality is a criteria of the existence of orbital asymptotical stable periodic solution of the equation (1.2).

$$\rho_2 < 1 \iff \int_0^{\tau_0} ({\varphi'}_{x_1}(u_0(\rho), {u_0}'(\rho)){u_0}'(\rho) + \varphi(u_0(\rho), {u_0}'(\rho)))d\rho > 0. \tag{2.4}$$

Using Proposition 1.1, we conclude the existence of non trivial periodic solution for perturbed generalized Liénard equation.

3. Results on the periodic solutions

Special case. Let us now consider the equation

$$u'' + \sum_{k=0}^{2s+1} p_k(u){u'}^k = 0. \tag{3.1}$$

Let p_k be C^1 function, for all$k \leq 2s+1$ for $s \in \mathbb{N}$. This is a special case of Liénard equation with $p_0(u) = \psi(u)$ and

$$\varphi(u, u') = \sum_{k=1}^{2s+1} p_k(u){u'}^{k-1}.$$

We will suppose φ and ψ verify C_{LS} conditions. Let U be an open subset of $\mathbb{R}^2$ containing $(0, 0)$. The associated perturbed equation, as denoted previously (1.3), is equation

$$u'' + \sum_{k=0}^{2s+1} p_k(u){u'}^k = \epsilon\omega(\frac{t}{\tau}, u, u').$$

Remark. The last non-zero term of the finite sum $\sum_{k=0}^{2s+1} p_k(u){u'}^k$ has an odd index. Then it is necessary to have the element $x_0 \neq 0$ in the C_{LS} conditions.

Theorem 3.1. *Let φ and ψ be C^1 and satisfy C_{LS}. If 1 is a simple characteristic multiplier of the variational system associated to (3.1) then there are two functions $\tau, h : U \to R$ and constants $\tau_1 < \tau_0$ such that the periodic solution $\nu(t, \alpha, a + h(\epsilon, \alpha), \epsilon, \tau(\epsilon, \alpha))$ of the equation*

$$u'' + \sum_{k=0}^{n} p_k(u){u'}^k = \epsilon\omega(\frac{t}{\tau}, u, u')$$

exists for $(\epsilon, \alpha) \in U$ with $|\tau - \tau_0| < \tau_1$, $\tau(0, 0) = \tau_0$ and $h(0, 0) = 0$.

Proof. We will use the same method as in the existence theorem for non-trivial periodic solution of the perturbed system. Consider the unperturbed equation to compute some useful elements. First we assume that $2s + 1 = n$, to simplify notation. Let $x_2 = u$ and $x_1 = \frac{du}{dt} = u'$. The equivalent plane system of (3.1) is

$$x' = f(x) \iff \begin{cases} x_1' = -\sum_{k=0}^{n} p_k(x_2){x_1}^k \\ x_2' = x_1 \end{cases} \tag{3.2}$$

with

$$f(x) = \operatorname{col}(-\sum_{k=0}^{n} p_k(x_2){x_1}^k, x_1).$$

Let $q(t) = \mathrm{col}(u_0'(t), u_0(t))$ the periodic solution of (3.2). The variational system associated to (3.2) is

$$y' = f_x'(q(t))y$$

with the periodic solution

$$q'(t) = \mathrm{col}(-\sum_{k=0}^{n} p_k(u_0)(t){u_0'}^k(t), u_0'(t)),$$

hence

$$f_x'(q(t)) = \begin{pmatrix} -\sum_{k=1}^{n} kp_k(u_0(t))u_0'(t)^{k-1} & -\sum_{k=0}^{n} p_k'(u_0(t))u_0'(t)^k \\ 1 & 0 \end{pmatrix}.$$

We assume the initial values:

$$t = 0, \quad u_0(0) = a < 0 \quad \text{and} \quad {u_0}'(0) = 0.$$

Then $q(0) = \mathrm{col}(0, a)$ and $q'(0) = \mathrm{col}(-\psi(a), 0)$.

In same way as the previous section we compute the fundamental matrix associated with (3.2), denoted $\Phi(t)$. Determine the second vector solution (linearly independent with $q'(t) = y(t)$). A trivial calculation described in [1, 3] gives us the second solution denoted $\widehat{y}(t)$, hence $\Phi(t) = (\frac{y(t)}{y(0)}, y(0)\widehat{y}(t))$. For that consider

$$I(s) = \exp\Big[-\int_0^s (\sum_{k=1}^{n} kp_k(u_0(\rho))u_0'(\rho)^{k-1})d\rho\Big],$$

and denote as in the previous section

$$\pi(t) = -\int_0^t (\sum_{k=0}^{n} p_k(u_0)(\rho)u_0'(\rho)^k)^{-2}(\sum_{k=0}^{n} p_k'(u(t)){u'}^k(t))I(\rho)d\rho.$$

Sine $\widehat{y}(t) = \mathrm{col}(\widehat{y}_1(t), \widehat{y}_2(t))$, where

$$\widehat{y}_1(t) = -(\sum_{k=0}^{n} p_k(u_0)(t)u_0'(t)^k)\pi(t),$$

$$\widehat{y}_2(t) = u_0'(t)\pi(t) + \pi'(t)\frac{\sum_{k=0}^{n} p_k(u_0)(t){u_0'}^k(t)}{\sum_{k=0}^{n} p_k'(u_0(t))u_0'(t)^k}.$$

Hence the fundamental matrix associated with our variational system is

$$\Phi(t) = \begin{pmatrix} \frac{\sum_{k=0}^{n} p_k(u_0)(t){u_0'}^k(t)}{\psi(a)} & \psi(a)(\sum_{k=0}^{n} p_k(u_0)(t)u_0'(t)^k)\pi(t) \\ -\frac{{u_0}'(t)}{\psi(a)} & -\psi(a)u_0'(t)\pi(t) - \psi(a)\pi'(t)\frac{\sum_{k=0}^{n} p_k(u_0)(t)u_0'(t)^k}{\sum_{k=0}^{n} p_k'(u_0(t))u_0'(t)^k} \end{pmatrix}.$$

We deduce the principal matrix (the fundamental one with $t = \tau_0$).

$$\Phi(\tau_0) = \begin{pmatrix} 1 & \psi(a)^2\pi(\tau_0) \\ 0 & \rho_2 \end{pmatrix}.$$

By the Liouville's formula, we have the characteristic multipliers $\rho_1 = 1$ and

$$\begin{aligned}\rho_2 &= \det(\Phi(\tau_0)) \\ &= \exp\Big(\int_0^{\tau_0} (Trf_x{}'(q(\tau))d\tau\Big) \\ &= \exp\Big(-\int_0^{\tau_0} \sum_{k=1}^{n} kp_k(u_0(\tau))u_0'(\tau)^{k-1})d\tau\Big)\end{aligned}$$

Then we define the equivalence (2.4):

$$\rho_2 < 1 \Longleftrightarrow \int_0^{\tau_0} \Big(\sum_{k=1}^{n} kp_k(u_0(\tau))u_0'(\tau)^{k-1}\Big)d\tau > 0 \tag{3.3}$$

and the associated Jacobian matrix is

$$J(\tau_0) = \begin{pmatrix} -\psi(a) & \psi(a)^2\pi(\tau_0) \\ 0 & \rho_2 - 1 \end{pmatrix}.$$

□

Uniqueness of the periodic solution for an unperturbed equation. Let us consider now equation (1.4):

$$u'' + [u^2 + (u + u')^2 - 1]u' + u = 0,$$

which is a special case of generalized Liénard equation with

$$\varphi(u, u') = (u^2 + (u' + u)^2 - 1) \text{ and } \psi(u) = u.$$

We will prove existence and uniqueness of non trivial periodic solution for equation (1.4). Existence will be ensured by C_{LS} conditions and for proving uniqueness we use a De Castro's result [5] (see also [2]).

Proposition 3.2 (De Castro [1]). *Suppose the following system has at least one periodic orbit*

$$\begin{gathered} y' = -\varphi(x, y)y - \psi(x) \\ x' = y. \end{gathered}$$

Then under the following two assumptions:

(a) $\psi(x) = x$;
(b) $\varphi(x, y)$ *increases, when* $|x|$ *or* $|y|$ *or the both increase*

this periodic orbit is unique.

Let us verify that (1.4) satisfies the above assumptions: Equation (1.4) is satisfied if and only if

$$\begin{gathered} u'' + \sum_{k=0}^{3} p_k(u)u'^k = 0, \\ p_0(u) = \psi(u) = u, \quad p_1(u) = 2u^2 - 1, \quad p_2(u) = 2u, \quad p_3(u) = 1. \end{gathered} \tag{3.4}$$

Also if and only if

$$\begin{gathered} u'' + \varphi(u, u')u' + \psi(u) = 0, \\ \varphi(u, u') = (u^2 + (u' + u)^2 - 1), \quad \psi(u) = u. \end{gathered} \tag{3.5}$$

Clearly, the assumptions of Proposition 3.2 are satisfied. In the following, we firstly verify conditions C_{LS}. In that case the equation

$$u'' + \varphi(u, u')u' + \psi(u) = 0$$

has at least a non trivial periodic solution. It is easy to see that $\psi(u) = u$ satisfies

$$x\psi(x) > 0 \quad \text{for } |x| > 0,$$
$$\int_0^x \psi(s)ds = \Psi(x), \quad lim_{x\to+\infty}\Psi(x) = +\infty$$

Now we have $\varphi(0, 0) = -1 < 0$. By taking $x_0 = 1$, $M = 1$, we have

$$\varphi(x, y) \geq 0 \quad \text{for } |x| \geq x_0,$$
$$\varphi(x, y) \geq -M \quad \text{for } |x| \leq x_0\,.$$

The following calculation gives us the optimal value of $x_1 > x_0$. Let

$$\begin{aligned}
H &= \int_{x_0}^{x_1} \varphi(x, y)dx \\
&= \int_1^{x_1} [x^2 + (x + y)^2 - 1]dx \\
&= \int_1^{x_1} [2x^2 + 2xy + y^2 - 1]dx \\
&= \Big[\frac{2}{3}x^3 + x^2y + x(y^2 - 1)\Big]_1^{x_1} \\
&= (x_1 - 1)(\frac{{x_1}^2 - 2x_1 + 1}{6} + 2(\frac{x_1 + 1}{2})^2 + 2y(\frac{x_1 + 1}{2}) + (y^2 - 1)) \\
&= (x_1 - 1)\Big(\frac{{x_1}^2 - 2x_1 + 1}{6} + \varphi(\frac{x_1 + 1}{2}, y)\Big)
\end{aligned}$$

Since $\frac{x_1+1}{2} \geqslant x_0 = 1$, using the inequality $\varphi(x, y) \geq 0$ for $|x| \geq x_0$, we obtain $H \geqslant \frac{(x_1-1)^3}{6}$. Hence, if $\frac{(x_1-1)^3}{6} = 10Mx_0 = 10$, then $x_1 = 1 + (60)^{\frac{1}{3}}$ which satisfies

$$x_1 > x_0, \quad \int_{x_0}^{x_1} \varphi(x, y)\, dx \geq 10Mx_0,$$

for every decreasing function $y(x) > 0$.

Existence of periodic solution for perturbed equation satisfying C_{LS}. In the following we are dealing with the existence of periodic solution for the equation (1.5). We assume the initial values:

$$t = 0, \quad u_0(0) = a < 0, \quad {u_0}'(0) = 0.$$

Theorem 3.3. *Suppose* 1 *is a simple characteristic multiplier of the variational system associated to* (1.4)*. Then there are two functions* $\tau, h : U \to R$ *and constants* $\tau_1 < \tau_0$ *such that the periodic solution* $\nu(t, \alpha, a + h(\epsilon, \alpha), \epsilon, \tau(\epsilon, \alpha))$ *of the equation*

$$u'' + {u'}^3 + 2u{u'}^2 + (2u^2 - 1)u' + u = \epsilon\omega(\frac{t}{\tau}, u, u'),$$

exists for $(\epsilon, \alpha) \in U$ *with* $|\tau - \tau_0| < \tau_1$*,* $\tau(0, 0) = \tau_0$ *and* $h(0, 0) = 0$*.*

Proof. We proceed similarly as in the proof of Theorem 3.1. We substitute the fundamental matrix, the second characteristic multiplier is ρ_2. The following holds for equation (1.4),

$$\rho_2 < 1 \Longleftrightarrow \int_0^{\tau_0} (\sum_{k=1}^{3} kp_k(u_0(\tau))u_0'(\tau)^{k-1})d\tau > 0,$$

then

$$\rho_2 < 1 \Longleftrightarrow \int_0^{\tau_0} [2{u_0}^2(\tau) + 4u_0(\tau)u_0'(\tau) + 3u_0'(\tau)^2 - 1]d\tau > 0.$$

It ensures that 1 is a simple characteristic multiplier of the variational system associated to (1.4) it implies $J(\tau_0) \neq 0$. Then a periodic solution for the perturbed equation (1.5) exists. $\square$

Using Scilab we will describe the phase plane of equation (1.4) $u'' + [u^2 + (u + u')^2 - 1]u' + u = 0$. We take $x_0 = u_0(0) = a = -0.7548829$, $y_0 = {u_0}'(0) = 0$ and the step time of integration ($step = .0001$). Recall that the periodic orbit is unique.

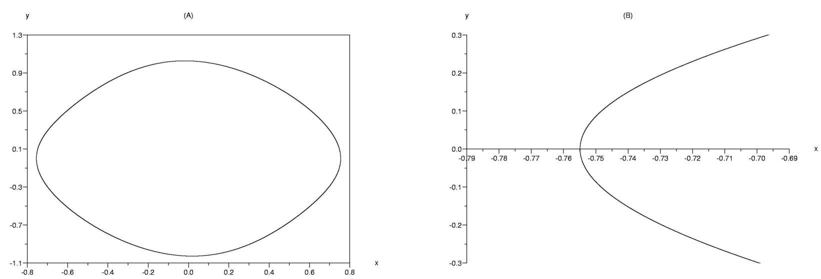

FIGURE 1. (A) The unique periodic orbit for $u''+[u^2+(u+u')^2-1]u' + u = 0$. (B) Zoom on the periodic orbit ($\times 20$)

We take $\epsilon\omega(\frac{t}{\tau}, u, u') = \epsilon sin(2t)u'$. Some illustrations of the phase portrait for the perturbed equation (1.5), those can explain existence of a bound ϵ_0, from which periodicity of the orbit will be not insured. In order to localize ϵ_0, we have taken several values of ϵ.

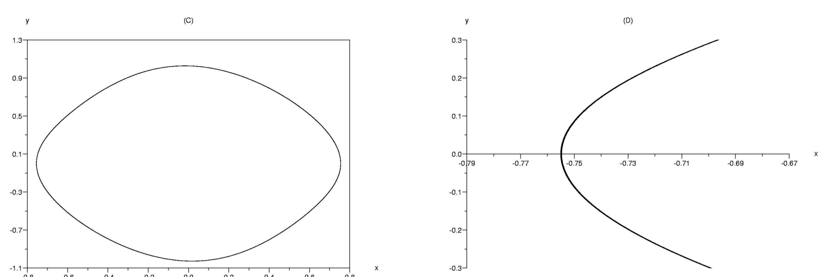

FIGURE 2. (C) The periodic orbit for $u''+[u^2+(u+u')^2-1]u'+u = \epsilon\omega(\frac{t}{\tau}, u, u')$, $\epsilon = 0.001$. (D) Zoom on the periodic orbit ($\times 20$)

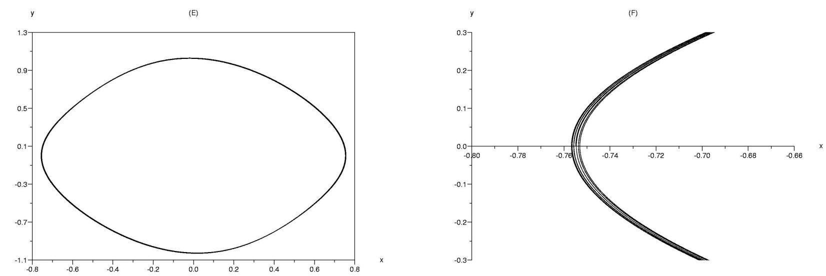

FIGURE 3. (E) Orbit for $u''+[u^2+(u+u')^2-1]u'+u = \epsilon\omega(\frac{t}{\tau}, u, u')$, $\epsilon = 0.01$. (F) Zoom on the orbit ($\times 10$) and loss of periodicity.

We see that from the range of $\epsilon = 0.01$ the orbit loses the periodicity.

TABLE 1. Period τ for some values of ϵ

ϵ	0	1/1000	1/900	1/800	1/700
τ	5.4296	5.4287	5.4286	5.4285	5.4283
ϵ	1/600	1/500	1/400	1/300	1/200
τ	5.4281	5.4278	5.4274	5.4267	5.4252

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