Electronic Journal of Differential Equations, Vol. 2006(2006), No. 144, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A QUASISTATIC UNILATERAL CONTACT PROBLEM WITH SLIP-DEPENDENT COEFFICIENT OF FRICTION FOR NONLINEAR ELASTIC MATERIALS

AREZKI TOUZALINE

Abstract. Existence of a weak solution under a smallness assumption of the coefficient of friction for the problem of quasistatic frictional contact between a nonlinear elastic body and a rigid foundation is established. Contact is modelled with the Signorini condition. Friction is described by a slip dependent friction coefficient and a nonlocal and regularized contact pressure. The proofs employ a time-discretization method, compactness and lower semicontinuity arguments.

1. Introduction

Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Because of the importance of this process a considerable effort has been made in its modelling and numerical simulations. An early attempt to study frictional contact problems within the framework of variational inequalities was made in [8]. The mathematical, mechanical and numerical state of the art can be found in [12]. In this paper we investigate a mathematical model for the process of unilateral frictional contact of a nonlinear elastic body with a rigid foundation. We assume that slowly varying time-dependent volume forces and surface tractions act on it, and as a result its mechanical state evolves quasistatically. The contact is modelled with the Signorini condition and the friction is described by a slip-dependent friction and a nonlocal and regularized contact pressure. The model of slip-dependent is considered in geophysics and solid mechanics corresponding to a smooth dependence of the friction coefficient on the slip u_τ, i.e. $\mu = \mu(|u_\tau|)$. The quasistatic contact problem with slip-dependent coefficient of friction for linear elastic materials was studied in [5] by using a new result obtained in [11]. In [9], the contact problem with slip-dependent coefficient of friction was studied in dynamic elasticity. By using the Galerkin method and regularization techniques, the authors of [9] proved the existence of a solution in the two-dimensional case (in-plane and anti-plane problems), hence for the case one-dimensional shearing problem, the solution that has been found in two dimensions is unique. The quasistatic problem with unilateral

2000 *Mathematics Subject Classification.* 35J85, 49J40, 47J20, 74M15.
Key words and phrases. Existence; quasistatic; nonlinear elastic; slip-dependent friction; incremental; variational inequality.

Submitted May 5, 2006. Published November 16, 2006.

contact which used a normal compliance law has been studied in [1] by considering incremental problems and in [10] by another method using a time regularization. In [15] the quasistatic unilateral contact problem involving a nonlocal friction law for nonlinear elastic materials was solved by the time-discretization method. By using a fixed point method, Signorini's problem with friction for nonlinear elastic materials has been solved in [6]. The same method was used in [14] to study the quasistatic contact problem with normal compliance and friction for nonlinear viscoelastic materials. Here, we try to complete the study of the elastic contact problem presented in [5]. Based on a time-discretization method, we prove the existence of a solution for a variational formulation of the quasistatic frictional problem, where this problem is given in terms of two variational inequalities as in [4, 15]. Thus this method is similar to the one that has been used in [4, 13] in order to study quasistatic contact problems for linear elastic materials. Given a time step, we construct a sequence of quasivariational inequalities for which we pove the existence of the solution. Then, we interpolate the discrete solution in time and, using compactness and lower semicontinuity, we derive the existence of a solution of the quasistatic contact problem if the coefficient of friction is sufficiently small.

2. Variational formulation

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be the reference domain occupied by the nonlinear elastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. Γ is decomposed into three parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets. Let $T > 0$ and let $[0, T]$ be the time interval of interest. We assume that the body is fixed on $\Gamma_1 \times (0, T)$ where the displacement field vanishes and that $\operatorname{meas}\Gamma_1 > 0$. The body is acted upon by a volume force of density φ_1 on $\Omega \times (0, T)$ and a surface traction of density φ_2 on $\Gamma_2 \times (0, T)$. On $\Gamma_3 \times (0, T)$ the body is in unilateral contact with friction with a rigid foundation.

Under these conditions the classical formulation of the mechanical problem is the following.

Problem (P1). Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$\operatorname{div}\sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma = F(\varepsilon(u)) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$\sigma_\nu(u) \le 0, u_\nu \le 0, \sigma_\nu(u)u_\nu = 0 \quad \text{on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$\left.\begin{array}{l} |\sigma_\tau| \le \mu(|u_\tau|)|R\sigma_\nu(u)| \\ |\sigma_\tau| < \mu(|u_\tau|)|R\sigma_\nu(u)| \Rightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = \mu(|u_\tau|)|R\sigma_\nu(u)| \Rightarrow \exists\lambda \ge 0 : \sigma_\tau = -\lambda\dot{u}_\tau \end{array}\right\} \quad \text{on } \Gamma_3 \times (0, T), \tag{2.6}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{2.7}$$

Here (2.1) represents the equilibrium equation; (2.2) represents the nonlinear elastic constitutive law in which F is a given function and $\varepsilon(u)$ denotes the small strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions on Γ_1 and Γ_2 respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress tensor; (2.5) represent the unilateral contact boundary conditions. Conditions (2.6) represent the associate friction law in which σ_τ denotes the tangential stress, $\dot{u}_\tau$ denotes the tangential velocity on the boundary, μ is the coefficient of friction and R is a regularization operator. Finally, (2.7) represents the initial condition. In (2.6) and below, a dot above a variable represents its derivative which respect to time. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ and it is endowed with its natural inner product. Moreover, in the sequel, the index that follows a comma indicates a partial derivative, e.g., $u_{i,j} = \partial u_i/\partial x_j$.

Here ε and div are the *deformation* and *divergence* operators defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{div}\sigma = (\sigma_{ij,j}),$$

respectively, where we denote by u and σ the displacement and stress fields in the body.

To proceed with the variational formulation, we consider the following spaces (repeated convention indexes is used):

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d,$$
$$Q = \{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega)_s^{d\times d},$$
$$Q_1 = \{\sigma \in Q; \operatorname{div}\sigma \in H\}.$$

The spaces H, Q and Q_1 are real Hilbert spaces endowed with the inner products

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx, \quad \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx,$$
$$\langle \sigma, \tau\rangle_{Q_1} = \langle \sigma, \tau\rangle_Q + \langle \operatorname{div}\sigma, \operatorname{div}\tau\rangle_H.$$

Keeping in mind the boundary condition (2.3), we introduce the closed subspace of H_1 defined by

$$V = \{v \in H_1; v = 0 \quad \text{on } \Gamma_1\}.$$

and K be the set of admissible displacements

$$K = \{v \in V; v_\nu \le 0 \quad \text{on } \Gamma_3\}.$$

Since $\operatorname{meas}\Gamma_1 > 0$, we have Korn's inequality [8],

$$\|\varepsilon(v)\|_Q \ge c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.8}$$

where the constant c_Ω depends only on Ω and Γ_1. We equip V with the inner product

$$\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and let $\|.\|_V$ be the associated norm. It follows from Korn's inequality (2.8) that the norms $\|\cdot\|_{H_1}$and $\|\cdot\|_V$ are equivalent on V. Therefore $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_3 and Γ_1 such that

$$\|v\|_{L^2(\Gamma_3)^d} \le d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.9}$$

For every $v \in H_1$, we denote by v_ν and v_τ the normal and tangential components of v on Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Similarly, σ_ν and σ_τ denote the normal and the tangential traces of a function $\sigma \in Q_1$. When σ is a regular function, then $\sigma_\nu = (\sigma\nu).\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$, and the following Green's formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_Q + \langle \operatorname{div}\sigma, v\rangle_H = \int_\Gamma \sigma\nu.v da \quad \forall v \in H_1. \tag{2.10}$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

For $p \in [1,\infty]$, we use the standard notation of $L^p(0,T;V)$ spaces. We also use the Sobolev space $W^{1,\infty}(0,T;V)$ with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)},$$

where a dot now represents the weak derivative with respect to the time variable.

In the study of contact problem (P1) we assume that the nonlinear elasticity operator $F : \Omega \times S_d \to S_d$ that satisfies:

$$\begin{aligned}
&\text{(a) There exists } L_1 > 0 \text{ such that} \\
&\qquad |F(x,\varepsilon_1) - F(x,\varepsilon_2)| \le L_1|\varepsilon_1 - \varepsilon_2|, \\
&\quad \text{for all } \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Omega; \\
&\text{(b) there exists } L_2 > 0 \text{ such that} \\
&\qquad (F(x,\varepsilon_1) - F(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \ge L_2|\varepsilon_1 - \varepsilon_2|^2, \\
&\quad \text{for all } \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Omega; \\
&\text{(c) } x \to F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \text{ for all } \varepsilon \in S_d; \\
&\text{(d) } F(x,0) = 0 \text{ for almost all } x \text{ in } \Omega.
\end{aligned} \tag{2.11}$$

Remark 2.1. From the hypotheses on F we have $F(x,\tau(x)) \in Q$, for all $\tau \in Q$ and thus we can consider F as an operator defined from Q to Q.

The coefficient of friction satisfies

$$\begin{aligned}
&\text{(a) } \mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+; \\
&\text{(b) there exists } L_\mu > 0 \text{ such that} \\
&\qquad |\mu(.,u) - \mu(.,v)| \le L_\mu|u - v| \\
&\quad \text{for all } u, v \in \mathbb{R}_+, \text{ a.e. on } \Gamma_3 \\
&\text{(c) There exists } \mu^* > 0 \text{ such that } \mu(x,u) \le \mu^* \text{ for all } u \in \mathbb{R}_+, \text{ a.e. } x \in \Gamma_3; \\
&\text{(d) the function } x \to \mu(x,u) \text{ is Lebesgue measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}_+.
\end{aligned} \tag{2.12}$$

We suppose that the body forces and surface tractions satisfy

$$\varphi_1 \in W^{1,\infty}(0,T;H), \quad \varphi_2 \in W^{1,\infty}(0,T;L^2(\Gamma_2)^d). \tag{2.13}$$

Using Riesz'representation theorem we define the element $f(t)$ by

$$\langle f(t), v\rangle_V = \int_\Omega \varphi_1(t).v dx + \int_{\Gamma_2} \varphi_2(t).v da \quad \forall v \in V,\ t \in [0,T].$$

The hypotheses on φ_1 and φ_2 imply that

$$f \in W^{1,\infty}(0,T;V).$$

Let us define the subset $\tilde{V}$ of H_1 by

$$\tilde{V} = \{v \in H_1; \operatorname{div}\sigma(v) \in H\}.$$

Similarly define

$$H(\Gamma_3) = \{w\big|_{\Gamma_3} : w \in H^{1/2}(\Gamma), w = 0 \text{ on } \Gamma_1\}$$

equipped with the norm of $H^{1/2}(\Gamma)$ and $\langle ., . \rangle$ shall denote the duality pairing between $H(\Gamma_3)$ and its dual $H'(\Gamma_3)$. We define the normal component of the stress vector $\sigma\nu$ on Γ_3 at time t as follows. Let $u \in \tilde{V}$ such that $\operatorname{div}\sigma(u) = -\varphi_1(t)$ in Ω and $\sigma(u)\nu = \varphi_2(t)$ on Γ_2. Then $\sigma_\nu(u(t)) \in H'(\Gamma_3)$ is given by

$$\begin{aligned} &\forall w \in H(\Gamma_3) : \\ &\langle \sigma_\nu(u(t)), w \rangle = \langle F(\varepsilon(u(t))), \varepsilon(v) \rangle_Q - \langle f(t), v \rangle_V, \\ &\forall v \in V; v_\nu = w, v_\tau = 0 \quad \text{on } \Gamma_3. \end{aligned} \tag{2.14}$$

Next we define the functional j: $\tilde{V} \times V \to \mathbb{R}$ by

$$j(u, v) = \int_{\Gamma_3} \mu(|u_\tau(a)|)|R\sigma_\nu(u)||v_\tau(a)|da \quad \forall (u, v) \in \tilde{V} \times V,$$

and da is the surface measure on Γ_3. We assume that $R : H'(\Gamma_3) \to L^\infty(\Gamma_3)$ is a linear and continuous mapping.

Finally we assume that the initial data u_0 satisfy

$$\begin{aligned} &u_0 \in K \cap \tilde{V}, \\ &\langle F(\varepsilon(u_0)), \varepsilon(v - u_0) \rangle_Q + j(u_0, v - u_0) \geq \langle f(0), v - u_0 \rangle_V \quad \forall v \in K. \end{aligned} \tag{2.15}$$

Using Green's formula (2.10) it is straightforward to see that if u is a sufficiently regular function which satisfy (2.1)-(2.6) then for almost all $t \in [0, T]$:

$$\begin{gathered} u(t) \in K, \\ \langle F(\varepsilon(u(t)), \varepsilon(v - \dot{u}(t)) \rangle_Q + j(u(t), v) - j(u(t), \dot{u}(t)) \\ \geq \langle f(t), v - \dot{u}(t) \rangle_V + \langle \sigma_\nu(u(t)), v_\nu - \dot{u}_\nu(t) \rangle \quad \forall v \in V, \\ \langle \sigma_\nu(u(t)), z_\nu - u_\nu(t) \rangle \quad \forall z \in K. \end{gathered}$$

Therefore, using (2.7) and the previous inequalities yields to the following variational formulation of problem (P1).

Problem (P2). Find a displacement field $u \in W^{1,\infty}(0, T; V)$ such that $u(0) = u_0$ in Ω and for almost all $t \in [0, T]$, $u(t) \in K \cap \tilde{V}$ and

$$\begin{aligned} &\langle F(\varepsilon(u(t))), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q + j(u(t), v) - j(u(t), \dot{u}(t)) \\ &\geq \langle f(t), v - \dot{u}(t) \rangle_V + \langle \sigma_\nu(u(t)), v_\nu - \dot{u}_\nu(t) \rangle \quad \forall v \in V, \end{aligned} \tag{2.16}$$

$$\langle \sigma_\nu(u(t)), z_\nu - u_\nu(t) \rangle \quad \forall z \in K. \tag{2.17}$$

The main result of this paper is the following.

Theorem 2.2. *Let $T > 0$ and assume that* (2.11), (2.12), (2.13) *and* (2.15) *hold. Then problem (P2) has at least one solution u for a sufficiently small friction coefficient.*

3. Incremental formulation

This evolution problem can be integrated in time by an implicit scheme as in [4, 15]. We need a partition of the time interval $[0,T]$, $0=t_0<t_1<\dots<t_n=T$, where $t_i=i\Delta t$, $0\le i\le n$, with step size $\Delta t=T/n$. We denote by u^{t_i} the approximation of u at the time t_i and by the symbol Δu^{t_i} the backward difference $u^{t_{i+1}}-u^{t_i}$. For a continuous function $w(t)$ we use the notation $w^{t_i}=w(t_i)$. Then we obtain a sequence of incremental problems $(P_n^{t_i})$ defined for $u^0=u_0$ by:

Problem $(P_n^{t_i})$. Find $u^{t_{i+1}}\in K\cap\tilde V$ such that

$$\begin{gathered}\langle F(\varepsilon(u^{t_{i+1}})),\varepsilon(w)-\varepsilon(u^{t_{i+1}})\rangle_Q+j(u^{t_{i+1}},w-u^{t_i})-j(u^{t_{i+1}},\Delta u^{t_i})\\ \ge\langle f^{t_{i+1}},w-u^{t_{i+1}}\rangle_V+\langle\sigma_\nu(u^{t_{i+1}}),w_\nu-u_\nu^{t_{i+1}}\rangle\quad\forall w\in V,\\ \langle\sigma_\nu(u^{t_{i+1}}),z_\nu-u_\nu^{t_{i+1}}\rangle\ge0\quad\forall z\in K.\end{gathered}$$

Lemma 3.1. *Problem $(P_n^{t_i})$ is equivalent to the following problem.*

Problem $(Q_n^{t_i})$. Find $u^{t_{i+1}}\in K\cap\tilde V$ such that

$$\begin{aligned}&\langle F(\varepsilon(u^{t_{i+1}})),\varepsilon(w)-\varepsilon(u^{t_{i+1}})\rangle_Q+j(u^{t_{i+1}},w-u^{t_i})-j(u^{t_{i+1}},\Delta u^{t_i})\\ &\ge\langle f^{t_{i+1}},w-u^{t_{i+1}}\rangle_V\quad\forall w\in K\end{aligned}\tag{3.1}$$

For the proof of the lemma above, we refer the reader to [4].

Lemma 3.2. *There exists $\mu_0>0$ such that for $\mu^*<\mu_0$, problem $(Q_n^{t_i})$ has a unique solution.*

To show this lemma, we introduce an intermediate problem. First, we define the convex set

$$C_+^*=\{g\in L^2(\Gamma_3);g\ge0\text{ a.e. on }\Gamma_3\}$$

and the function

$$\varphi(w)=\int_{\Gamma_3}g|w_\tau|da.$$

We introduce the intermediate problem $(Q_{ng}^{t_i})$ for $g\in C_+^*$ by replacing in (3.1) $\mu(|u_\tau^{t_{i+1}}|)|R\sigma_\nu(u^{t_{i+1}})|$ by g as follows.

Problem $(Q_{ng}^{t_i})$. Find $u_g\in K$ such that for all $w\in K$,

$$\langle F(\varepsilon(u_g)),\varepsilon(w)-\varepsilon(u_g)\rangle_Q+\varphi(w-u^{t_i})-\varphi(u_g-u^{t_i})\ge\langle f^{t_{i+1}},w-u_g\rangle_V\,.\tag{3.2}$$

Then we have the following lemma.

Lemma 3.3. *For any $g\in C_+^*$ problem $(Q_{ng}^{t_i})$ has a unique solution u_g. Moreover, there exists a constant $c_1>0$ such that*

$$\|u_g\|_V\le c_1\|f^{t_{i+1}}\|_V.\tag{3.3}$$

The proof of the above lemma can be found in [15]. Now we prove the following lemma.

Lemma 3.4. *Let $\Psi:C_+^*\to C_+^*$ be the mapping defined by*

$$\Psi(g)=\mu(|u_{g\tau}|)|R\sigma_\nu(u_g)|.$$

There exists $L_1^*>0$ such that if $\mu^*+L_\mu<L_1^*$, then Ψ has a fixed point g^* and u_{g^*} is a solution to problem $(Q_n^{t_i})$.

Proof. Since for $g \in L^2(\Gamma_3)$, $\sigma_\nu(u_g)$ is defined on Γ_3 and belongs to the dual space $H'(\Gamma_3)$, we have

$$\begin{aligned}\|\Psi(g_1)-\Psi(g_2)\|_{L^2(\Gamma_3)} &= \|\mu(|u_{g_{1\tau}}|)|R\sigma_\nu(u_{g_1})| - \mu(|u_{g_{2\tau}}|)|R\sigma_\nu(u_{g_2})|\|_{L^2(\Gamma_3)} \\ &\le \||\mu(|u_{g_{1\tau}}|)-\mu(|u_{g_{2\tau}}|)||R\sigma_\nu(u_{g_1})|\|_{L^2(\Gamma_3)} \\ &\quad + \|\mu(|u_{g_{2\tau}}|)(|R\sigma_\nu(u_{g_1})|-|R\sigma_\nu(u_{g_2})|)\|_{L^2(\Gamma_3)}.\end{aligned}$$

Using the relation (2.14), the continuity of R and (3.3), it follows that there exists a constant $C > 0$ such that

$$\|R\sigma_\nu(u_{g_1})\|_{L^\infty(\Gamma_3)} \le C\|f\|_{C([0,T];V)}.$$

Using (2.9), (2.12)(c), (2.14) and the continuity of R, yield that there exists a constant $C_1 > 0$ such that

$$\|\Psi(g_1)-\Psi(g_2)\|_{L^2(\Gamma_3)} \le C_1(\mu^* + L_\mu)\|u_{g_1} - u_{g_2}\|_V.$$

On the other hand set $v = u_{g_1}$ in $(Q^{t_i}_{ng_2})$ and $v = u_{g_2}$ in $(Q^{t_i}_{ng_1})$ and adding them, we obtain by using (2.9) and (2.11)(b), that there exists a constant $C_2 > 0$ such that

$$\|u_{g_1} - u_{g_2}\|_V \le C_2\|g_1 - g_2\|_{L^2(\Gamma_3)}.$$

Hence we deduce

$$\|\Psi(g_1)-\Psi(g_2)\|_{L^2(\Gamma_3)} \le C_1C_2(\mu^* + L_\mu)\|g_1 - g_2\|_{L^2(\Gamma_3)},$$

and when $L_1^* = \frac{1}{C_1C_2}$, we have for $\mu^* + L_\mu < L_1^*$, that the mapping Ψ is a contraction. Thus it has a fixed point g^* and u_{g*} is the solution of problem $(Q^{t_i}_n)$. We remark that $g^* \in L^\infty(\Gamma_3)$ as $\Psi(g^*) \in L^\infty(\Gamma_3)$ and $u_{g*} \in K \cap \tilde{V}$ yields that $u^{t_{i+1}} \in K \cap \tilde{V}$. □

Lemma 3.5. *We have the following estimates: There exists a constant $L_2^* > 0$ such that for $\mu^* + L_\mu < L_2^*$, there exist $d_i > 0$, $i = 1, 2$, such that*

$$\|u^{t_{i+1}}\|_V \le d_1\|f^{t_{i+1}}\|_V, \tag{3.4}$$

$$\|\Delta u^{t_i}\|_V \le d_2\|\Delta f^{t_i}\|_V. \tag{3.5}$$

Proof. By setting $w = 0$ in the inequality (3.1) we deduce the inequality

$$\langle F(\varepsilon(u^{t_{i+1}})), \varepsilon(u^{t_{i+1}})\rangle_Q \le j(u^{t_{i+1}}, u^{t_{i+1}}) + \langle f^{t_{i+1}}, u^{t_{i+1}}\rangle_V.$$

Using the properties of j we have

$$j(u^{t_{i+1}}, u^{t_{i+1}}) \le \mu^*\|R\sigma_\nu(u^{t_{i+1}})\|_{L^\infty(\Gamma_3)} d_\Omega (\operatorname{meas}\Gamma_3)^{1/2}\|u^{t_{i+1}}\|_V.$$

Then using the continuity of R and (2.14), there exists a constant $c > 0$ such that

$$\|R\sigma_\nu(u^{t_{i+1}})\|_{L^\infty(\Gamma_3)} \le c(\|u^{t_{i+1}}\|_V + \|f^{t_{i+1}}\|_V).$$

Using (2.11)(b) and (2.9), there exists a constant $c_1 > 0$ such that

$$L_2\|u^{t_{i+1}}\|_V^2 \le d_\Omega \mu^* c(\operatorname{meas}\Gamma_3)^{1/2}\|u^{t_{i+1}}\|_V^2 + c_1\|f^{t_{i+1}}\|_V\|u^{t_{i+1}}\|_V,$$

from which we deduce if we take

$$\mu_1 = \frac{L_2}{2d_\Omega c(\operatorname{meas}\Gamma_3)^{1/2}},$$

that for $\mu^* + L_\mu < \mu_1$, there exists $d_1 > 0$ such that (3.4) hold. To show the inequality (3.5) we consider the translated inequality of (3.1) at the time t_i, that is

$$\begin{aligned}&\langle F(\varepsilon(u^{t_i})), \varepsilon(w) - \varepsilon(u^{t_i})\rangle_Q + j(u^{t_i}, w - u^{t_{i-1}}) - j(u^{t_i}, u^{t_i} - u^{t_{i-1}})\\ &\geq \langle f^{t_i}, w - u^{t_i}\rangle_V \quad \forall \quad w \in K.\end{aligned} \tag{3.6}$$

By setting $w = u^{t_i}$ in (3.1) and $w = u^{t_{i+1}}$ in (3.6) and adding them up, we obtain the inequality

$$\begin{aligned}&-\langle F(\varepsilon(u^{t_{i+1}})) - F(\varepsilon(u^{t_i})), \varepsilon(\Delta u^{t_i})\rangle_Q - j(u^{t_{i+1}}, \Delta u^{t_i})\\ &+ j(u^{t_i}, u^{t_{i+1}} - u^{t_{i-1}}) - j(u^{t_i}, u^{t_i} - u^{t_{i-1}})\\ &\geq \langle -\Delta f^{t_i}, \Delta u^{t_i}\rangle_V.\end{aligned}$$

Then using the inequality

$$||u_\tau^{t_{i+1}} - u_\tau^{t_{i-1}}| - |u_\tau^{t_i} - u_\tau^{t_{i-1}}|| \leq |u_\tau^{t_{i+1}} - u_\tau^{t_i}|,$$

we have

$$j(u^{t_i}, u^{t_{i+1}} - u^{t_{i-1}}) - j(u^{t_i}, u^{t_i} - u^{t_{i-1}}) \leq \ j(u^{t_i}, \Delta u^{t_i}).$$

Therefore,

$$\begin{aligned}&\langle F(\varepsilon(u^{t_{i+1}})) - F(\varepsilon(u^{t_i})), \varepsilon(\Delta u^{t_i})\rangle_Q - j(u^{t_i}, \Delta u^{t_i}) + j(u^{t_{i+1}}, \Delta u^{t_i})\\ &\leq \langle \Delta f^{t_i}, \Delta u^{t_i}\rangle_V.\end{aligned} \tag{3.7}$$

Using the hypothesis (2.11) (b) on μ, inequality (2.9) and the properties of j, there exist two positive constants c_2 and c_3 such that

$$| - j(u^{t_i}, \Delta u^{t_i}) + j(u^{t_{i+1}}, \Delta u^{t_i})| \leq c_2(\mu^* + L_\mu)\|\Delta u^{t_i}\|_V^2 + c_3\|\Delta f^{t_i}\|_V\|\Delta u^{t_i}\|_V.$$

Then using the hypothesis (2.10)(b) on F, we obtain from the previous inequality that

$$L_2\|\Delta u^{t_i}\|_V^2 \leq c_2(\mu^* + L_\mu)\|\Delta u^{t_i}\|_V^2 + c_3\|\Delta f^{t_i}\|_V\|\Delta u^{t_i}\|_V.$$

Then if we take $\mu_2 = \frac{L_2}{2c_2}$, for $\mu^* + L_\mu < \mu_2$, there exists $d_2 > 0$ such that

$$\|\Delta u^{t_i}\|_V \leq d_2\|\Delta f^{t_i}\|_V.$$

and the lemma is proved with $L_2^* = \min(\mu_1, \mu_2)$. □

4. Existence

In this section we prove our main result, Theorem 2.2, which guarantees the existence of a weak solution for problem (P2) obtained as a limit of the interpolate function in time of the discrete solution. For thus, we shall define the following sequence of functions u^n in $[0, T] \to V$ by

$$u^n(t) = u^{t_i} + \frac{(t - t_i)}{\Delta t}\Delta u^{t_i} \quad \text{on } [t_i, t_{i+1}], i = 0, ..., n - 1.$$

As in [15] we have the following lemma.

Lemma 4.1. *There exists $u \in W^{1,\infty}(0, T; V)$ and a subsequence of the sequence (u^n), still denoted (u^n), such that*

$$u^n \to u \quad \textit{weak} * \textit{ in } W^{1,\infty}(0, T; V).$$

Proof. As in [15], from (3.4) we deduce that the sequence (u^n) is bounded in $C([0,T];V)$ and there exists $c_3 > 0$ such that

$$\max_{0\le t\le T} \|u^n(t)\|_V \le c_3 \|f\|_{C([0,T];V)}.$$

From (3.5) we deduce that the sequence $(\dot u^n)$ is bounded in $L^\infty(0,T;V)$ and that there exists $c_4 > 0$ such that

$$\|\dot u^n\|_{L^\infty(0,T;V)} = \max_{0\le i\le n-1} \|\frac{\Delta u^{t_i}}{\Delta t}\|_V \le c_4 \|\dot f\|_{L^\infty(0,T;V)}.$$

Consequently the sequence (u^n) is bounded in $W^{1,\infty}(0,T;V)$. Therefore, there exists a function u in $W^{1,\infty}(0,T;V)$ and a subsequence, still denoted by (u^n), such that

$$u^n \to u \quad \text{weak} * \text{ in } W^{1,\infty}(0,T;V) \text{ as } n\to\infty \text{ satisfying}$$
$$\|u\|_{W^{1,\infty}(0,T;V)} \le c_5 \|f\|_{W^{1,\infty}(0,T.V)},$$

with $c_5 = \max(c_3, c_4)$. □

Let us introduce the following piecewise constant functions $\widetilde{u}^n : [0,T] \to V$, $\widetilde{f}^n : [0,T] \to V$ defined as follows

$$\widetilde{u}^n(t) = u^{t_{i+1}}, \widetilde{f}^n(t) = f(t_{i+1}), \quad \forall t \in (t_i, t_{i+1}], i = 0,\dots,n-1.$$

We have the following result.

Lemma 4.2. *Passing to a subsequence again denoted* $(\tilde u^n)$ *we have*

(i) $\widetilde{u}^n \to u$ *weak* $*$ *in* $L^\infty(0,T;V)$,
(ii) $\widetilde{u}^n(t) \to u(t)$ *weakly in* V *a.e.* t *in* $[0,T]$,
(iii) $u(t) \in K \cap \tilde V$ *a.e.* $t \in [0,T]$.

Proof. From (3.1) we deduce that the sequence $(\widetilde{u}^n)$ is bounded in $L^\infty(0,T;V)$. Then, there exists a subsequence still denoted $(\widetilde{u}^n)$ which converges weakly $*$ in $L^\infty(0,T;V)$. On the other hand as in [11] we deduce for every $t \in (0,T)$ the inequality

$$\|u^n(t) - \widetilde{u}^n(t)\|_V \le \frac{T}{n} \|\dot u^n(t)\|_V, \tag{4.1}$$

from which we deduce

$$\|u^n(t) - \widetilde{u}^n(t)\|_{L^\infty(0,T;V)} \le c_4 \frac{T}{n} \|\dot f\|_{L^\infty(0,T;V)}.$$

This inequality proves that

$$\widetilde{u}^n \to u \quad \text{weak} * \text{ in } L^\infty(0,T;V),$$

whence (i) follows. To prove (ii), since $W^{1,\infty}(0,T;V) \hookrightarrow C([0,T];V)$, we have $u^n(t) \to u(t)$ weakly in V, for all $t \in [0,T]$, and from (4.1) we have immediately (ii). We turn now to the proof of (iii). To this end we remark that we have $\tilde u^n(t) \in K$ a.e. $t \in [0,T]$, so we deduce that $u(t) \in K$ a.e. $t \in [0,T]$. Then it suffices only to show that $u(t) \in \tilde V$ a.e. $t \in [0,T]$. Indeed, from the inequality (3.1) we deduce the inequality

$$\langle F(\varepsilon(\tilde u^n(t))), \varepsilon(w) - \varepsilon(\tilde u^n(t))\rangle_Q + j(\tilde u^n(t), w - \tilde u^n(t))$$
$$\ge \langle \tilde f^n(t), w - \tilde u^n(t)\rangle_V, \quad \forall w \in K, \text{ a.e. } t \in (0,T).$$

From this inequality we deduce that for a fixed $t \in (0,T)$, $\operatorname{div}\sigma(\tilde{u}^n(t))$ is bounded in H and so we can extract a subsequence again denoted $\operatorname{div}\sigma(\tilde{u}^n(t))$ such that it converges weakly in H. Since $\operatorname{div}\sigma(\tilde{u}^n(t)) \to \operatorname{div}\sigma(u(t))$ in the sense of distributions we conclude that $\operatorname{div}\sigma(u(t)) \in H$ a.e. $t \in [0,T]$. Then $u(t) \in \tilde{V}$ a.e. $t \in [0,T]$, which concludes that $u(t) \in K \cap \tilde{V}$ a.e. $t \in [0,T]$. □

Remark 4.3. Since $f \in W^{1,\infty}(0,T;V)$, it follows that

$$\widetilde{f}^n \to f \quad \text{strongly in } L^2(0,T;V). \tag{4.2}$$

Now we have all the ingredients to prove the following proposition.

Proposition 4.4. *The sequence $(\tilde{u}^n)$ converges strongly to u in $L^2(0,T;V)$ and u is a solution to problem (P2) if the coefficient of friction is sufficiently small.*

Proof. To show the strong convergence of the sequence $(\tilde{u}^n)$ in $L^2(0,T;V)$ we consider the following inequality deduced from inequality (3.1):

$$\langle F(\varepsilon(u^{t_{i+1}})), \varepsilon(v) - \varepsilon(u^{t_{i+1}})\rangle_Q + j(u^{t_{i+1}}, v - u^{t_{i+1}}) \geq \langle f^{t_{i+1}}, v - u^{t_{i+1}}\rangle_V \quad \forall v \in K.$$

Whence we get the inequality

$$\langle F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v) - \varepsilon(\widetilde{u}^n(t))\rangle_Q + j(\widetilde{u}^n(t), v - \widetilde{u}^n(t)) \geq \langle \widetilde{f}^n(t), v - \widetilde{u}^n(t)\rangle_V \tag{4.3}$$

for all $v \in K$, a. e. $t \in [0,T]$. Also we shall consider the inequality

$$\begin{aligned} &\langle F(\varepsilon(\widetilde{u}^{n+m}(t))), \varepsilon(v) - \varepsilon(\widetilde{u}^{n+m}(t))\rangle_Q + j(\widetilde{u}^{n+m}(t), v - \widetilde{u}^{n+m}(t)) \\ &\geq \langle \widetilde{f}^{n+m}(t), v - \widetilde{u}^{n+m}(t)\rangle_V \quad \forall v \in K, \text{a.e. } t \in [0,T]. \end{aligned} \tag{4.4}$$

In the next, setting $v = \widetilde{u}^n(t)$ in (4.4) and $v = \widetilde{u}^{n+m}(t)$ in (4.3) and adding them, we obtain by using the hypothesis (2.12)(b) on μ the inequality

$$\begin{aligned} &\langle F(\varepsilon(\widetilde{u}^{n+m}(t))) - F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(\widetilde{u}^n(t)) - \varepsilon(\widetilde{u}^{n+m}(t))\rangle_Q \\ &+ 2\mu^* \int_{\Gamma_3} |\tilde{u}^{n+m}_\tau(t) - \tilde{u}^n_\tau(t)| da \\ &\geq -\langle \widetilde{f}^{n+m}(t) - \widetilde{f}^n(t), \widetilde{u}^{n+m}(t) - \widetilde{u}^n(t)\rangle_V. \end{aligned}$$

Therefore, there exists a constant $C_3 > 0$ such that

$$\begin{aligned} &\|\widetilde{u}^{n+m}(t) - \widetilde{u}^n(t)\|_V^2 \\ &\leq C_3(2\mu^* \|\tilde{u}^{n+m}_\tau(t) - \tilde{u}^n_\tau(t)\|_{L^2(\Gamma_3)^d} + \|\widetilde{f}^{n+m}(t) - \widetilde{f}^n(t)\|_V^2). \end{aligned}$$

To complete the proof we refer the reader to [15, Proposition 4.5] and conclude that

$$\widetilde{u}^n \to u \quad \text{strongly in } L^2(0,T;V). \tag{4.5}$$

Now to prove that u is a solution of problem (P2), in the first inequality of problem $(P_n^{t_i})$, for $v \in V$ set $w = u^{t_i} + v\Delta t$ and divide by Δt; we obtain the inequality:

$$\begin{aligned} &\langle F(\varepsilon(u^{t_{i+1}})), \varepsilon(v) - \varepsilon(\frac{\Delta u^{t_i}}{\Delta t})\rangle_Q + j(u^{t_{i+1}}, v) - j(u^{t_{i+1}}, \frac{\Delta u^{t_i}}{\Delta t}) \\ &\geq \langle f(t_{i+1}), v - \frac{\Delta u^{t_i}}{\Delta t}\rangle_V + \langle \sigma_\nu(u^{t_{i+1}}), v_\nu - \frac{\Delta u^{t_i}_\nu}{\Delta t}\rangle. \end{aligned}$$

Whence for any $v \in L^2(0,T;V)$, we have

$$\begin{aligned} &\langle F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q + j(\widetilde{u}^n(t), v(t)) - j(\widetilde{u}^n(t), \dot{u}^n(t)) \\ &\geq \langle \widetilde{f}^n(t), v(t) - \dot{u}^n(t)\rangle_V + \langle \sigma_\nu(\tilde{u}^n(t)), v_\nu(t) - \dot{u}^n_\nu(t)\rangle, \quad \text{a.e. } t \in [0,T]. \end{aligned}$$

Integrating both sides of the previous inequality on $(0,T)$, we obtain

$$\begin{aligned}
&\int_0^T \langle F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q dt \\
&+ \int_0^T j(\widetilde{u}^n(t), v(t))dt - \int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t))dt \\
&\geq \int_0^T \langle \widetilde{f}^n(t), v(t) - \dot{u}^n(t)\rangle_V dt + \int_0^T \langle \sigma_\nu(\tilde{u}^n(t)), v_\nu(t) - \dot{u}^n_\nu(t)\rangle dt.
\end{aligned} \tag{4.6}$$

□

Lemma 4.5. *We have the following properties:*

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \langle F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q dt \\
&= \int_0^T \langle F(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}(t))\rangle_Q dt \quad \forall v \in L^2(0,T;V),
\end{aligned} \tag{4.7}$$

$$\liminf_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t))dt \geq \int_0^T j(u(t), \dot{u}(t))dt, \tag{4.8}$$

$$\lim_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), v(t))dt = \int_0^T j(u(t), v(t))dt \quad \forall v \in L^2(0,T;V), \tag{4.9}$$

$$\lim_{n\to\infty} \int_0^T \langle \widetilde{f}^n(t), v(t) - \dot{u}^n(t)\rangle_V dt = \int_0^T \langle f(t), v(t) - \dot{u}(t)\rangle_V dt \tag{4.10}$$

for all $v \in L^2(0,T;V)$.

Proof. For the proof of (4.7), we refer the reader to [15]. To prove (4.8) we write

$$\begin{aligned}
j(\widetilde{u}^n(t), \dot{u}^n(t)) = &\int_{\Gamma_3} (\mu(|\tilde{u}^n_\tau|) - \mu(|u_\tau|))|R\sigma_\nu(\widetilde{u}^n)||\dot{u}^n_\tau| da \\
&+ \int_{\Gamma_3} \mu(|u_\tau|)(|R\sigma_\nu(\widetilde{u}^n)| - |R\sigma_\nu(u)|)|\dot{u}^n_\tau| da + j(u(t), \dot{u}^n(t)).
\end{aligned}$$

Using hypothesis (2.12)(b) on μ, we obtain

$$\Big|\int_{\Gamma_3} (\mu(|\tilde{u}^n_\tau|) - \mu(|u_\tau|))|R\sigma_\nu(\widetilde{u}^n)||\dot{u}^n_\tau| da\Big| \leq L_\mu \int_{\Gamma_3} |\tilde{u}^n_\tau - u_\tau||R\sigma_\nu(\widetilde{u}^n)||\dot{u}^n_\tau| da,$$

which implies

$$\begin{aligned}
&\Big|\int_{\Gamma_3} (\mu(|\tilde{u}^n_\tau|) - \mu(|u_\tau|))|R\sigma_\nu(\widetilde{u}^n)||\dot{u}^n_\tau| da\Big| \\
&\leq L_\mu \|\tilde{u}^n_\tau - u_\tau\|_{L^2(\Gamma_3)^d} \|R\sigma_\nu(\widetilde{u}^n)\|_{L^\infty(\Gamma_3)} \|\dot{u}^n_\tau\|_{L^2(\Gamma_3)^d}.
\end{aligned}$$

Now, the continuity of R and the relation (2.14) imply that there exists a constant $C_4 > 0$ such that

$$\|R(\sigma_\nu(\widetilde{u}^n))\|_{L^\infty(\Gamma_3)} \leq C_4 \|f\|_{W^{1,\infty}(0,T;V)}.$$

Therefore, using $\|\dot{u}^n\|_{L^\infty(0,T;V)} \leq c_5 \|f\|_{W^{1,\infty}(0,T.V)}$, we find from (2.9) that

$$\begin{aligned}
&\Big|\int_0^T \int_{\Gamma_3} (\mu(|\tilde{u}^n_\tau|) - \mu(|u_\tau|))|R\sigma_\nu(\widetilde{u}^n)||\dot{u}^n_\tau| da\, dt\Big| \\
&\leq C_5 \|f\|^2_{W^{1,\infty}(0,T.V)} \|\widetilde{u}^n - u\|_{L^2(0,T;V)},
\end{aligned}$$

where $C_5 > 0$. As previously the continuity of R and the relation (2.14) yield that there exists a constant $C_6 > 0$ such that

$$\|R(\sigma_\nu(\tilde{u}^n(t)) - \sigma_\nu(u(t)))\|_{L^\infty(\Gamma_3)} \le C_6\big(\|\tilde{u}^n(t) - u(t)\|_V + \|\tilde{f}^n(t) - \tilde{f}(t)\|_V\big),$$

a.e. $t \in (0,T)$. So, we deduce that there exists a constant $C_7 > 0$ such that

$$\begin{aligned}
&\Big|\int_0^T \int_{\Gamma_3} \mu(|u_\tau|)(|R\sigma_\nu(\widetilde{u}^n)| - |R\sigma_\nu(u)|)|\dot{u}^n_\tau|\,da\,dt\Big| \\
&\le C_7\|f\|_{W^{1,\infty}(0,T.V)}\big(\|\widetilde{u}^n - u\|_{L^2(0,T;V)} + \|\tilde{f}^n - \tilde{f}\|_{L^2(0,T;V)}\big).
\end{aligned}$$

Hence using (4.2) and (4.5), we get

$$\lim_{n\to+\infty}\int_0^T \int_{\Gamma_3} \mu(|u_\tau|)(|R\sigma_\nu(\widetilde{u}^n)| - |R\sigma_\nu(u)|)|\dot{u}^n_\tau|\,da\,dt = 0,$$

$$\lim_{n\to+\infty}\int_0^T \int_{\Gamma_3} (\mu(|\tilde{u}^n_\tau|) - \mu(|u_\tau|))|R\sigma_\nu(\widetilde{u}^n)||\dot{u}^n_\tau|\,da\,dt = 0.$$

Finally as by Mazur's lemma we have

$$\liminf_{n\to+\infty}\int_0^T j(u(t),\dot{u}^n(t))dt \ge \int_0^T j(u(t),\dot{u}(t))dt,$$

then we obtain

$$\liminf_{n\to+\infty}\int_0^T j(\tilde{u}^n(t),\dot{u}^n(t))dt \ge \int_0^T j(u(t),\dot{u}(t))dt.$$

To prove (4.9) and (4.10) it suffices to use (4.5) and (4.2), and (4.2) respectively. $\square$

Now passing to the limit in inequality (4.6), we obtain the inequality:

$$\begin{aligned}
&\int_0^T (\langle F(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}(t))\rangle_Q + j(u(t),v(t)) - j(u(t),\dot{u}(t)))dt \\
&\ge \int_0^T \langle f(t), v(t) - \dot{u}(t)\rangle_V dt + \int_0^T \langle \sigma_\nu(u(t)), v_\nu - \dot{u}_\nu(t)\rangle dt.
\end{aligned} \tag{4.11}$$

If we set in (4.11) $v \in L^2(0,T;V)$ defined by:

$$v(s) = \begin{cases} w & \text{for } s \in (t, t+\lambda) \\ \dot{u}(s) & \text{elsewhere,} \end{cases}$$

we obtain the inequality

$$\begin{aligned}
&\frac{1}{\lambda}\int_t^{t+\lambda} (\langle F(\varepsilon(u(s))), \varepsilon(w) - \varepsilon(\dot{u}(s))\rangle_Q + j(u(s),w) - j(u(s),\dot{u}(s)))ds \\
&\ge \frac{1}{\lambda}\int_t^{t+\lambda} \langle f(s), w - \dot{u}(s)\rangle_V ds + \frac{1}{\lambda}\int_t^{t+\lambda} \langle \sigma_\nu(u(s)), w_\nu - \dot{u}_\nu(s)\rangle ds.
\end{aligned}$$

Passing to the limit, one obtains that u satisfies the inequality (2.16) and consequently u is a solution of problem (P2). To complete the proof, integrate both sides of (4.3); that is,

$$\begin{aligned}
&\int_0^T \langle F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\widetilde{u}^n(t))\rangle_Q dt + \int_0^T j(\widetilde{u}^n(t), v(t) - \widetilde{u}^n(t))dt \\
&\ge \int_0^T \langle \widetilde{f}^n(t), v(t) - \widetilde{u}^n(t)\rangle_V dt
\end{aligned} \tag{4.12}$$

for all $v \in L^2(0,T;V)$ such that $v(t) \in K$ a.e. $t \in [0,T]$. Passing to the limit in the above inequality, with (4.2) and (4.5), we obtain the inequality

$$\begin{aligned}
&\int_0^T \langle F(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(u(t))\rangle_Q dt + \int_0^T j(u(t), v(t) - u(t))dt \\
&\geq \int_0^T \langle f(t), v(t) - u(t)\rangle_V \, dt \quad \forall v \in L^2(0,T;V); v(t) \in K, \quad \text{a.e. } t \in [0,T].
\end{aligned}$$

Proceeding in a similar way, we deduce that u satisfies the inequality

$$\langle F(\varepsilon(u(t))), \varepsilon(w) - \varepsilon(u(t))\rangle_Q + j(u(t), w - u(t)) \geq \langle f(t), w - u(t)\rangle_V$$

for all $w \in K$ a.e. $t \in [0,T]$. Using Green's formula in the above inequality, as in[4], we obtain that u satisfies the inequality (2.17) and consequently u is a solution of problem (P2).

Remark 4.6. We can state another variational formulation of the problem (P1) defined as follows

Problem (P3). Find a displacement field $u \in W^{1,\infty}(0,T;V)$ such that $u(0) = u_0$ in Ω and for almost all $t \in [0,T]$, $u(t) \in K \cap \tilde{V}$ and

$$\begin{gathered}
\langle F(\varepsilon(u(t))), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_Q + j(u(t), v) - j(u(t), \dot{u}(t)) \\
\geq \langle f(t), v - \dot{u}(t)\rangle_V + \langle \theta\sigma_\nu(u(t)), v_\nu - \dot{u}_\nu(t)\rangle_\Gamma \geq 0 \quad \forall v \in V, \\
\langle \theta\sigma_\nu(u(t)), z_\nu - u_\nu(t)\rangle_\Gamma \geq 0 \quad \forall z \in K.
\end{gathered}$$

Here, $R : H^{-\frac{1}{2}}(\Gamma) \to L^\infty(\Gamma_3)$ is a linear and continuous mapping and $\langle .,. \rangle_\Gamma$ denotes the duality pairing on $H^{-\frac{1}{2}}(\Gamma) \times H^{1/2}(\Gamma)$. The cut-of function $\theta \in C_0^\infty(\mathbb{R}^d)$ has the property that $\theta = 1$ on $\overline{\Gamma}_3$ and $\theta = 0$ on $\overline{S}_2$ with S_2 an open subset such that for all $t \in [0,T]$ $\operatorname{supp} \varphi_2(t) \subset S_2 \subset \overline{S}_2 \subset \Gamma_2$.

Conclusion. In this paper we have shown the existence of a solution of the quasistatic unilateral contact problem of slip-dependent coefficient of friction for non-linear elastic materials under a smallness assumption of the friction coefficient. The important question of uniqueness of the solution, as far as we know still remains open.

References

[1] L.-E. Andersson; *A quasistatic frictional problem with normal compliance*, Nonlinear Anal. Th. Appl.16, 4, 347-369, 1991.
[2] H. Brezis; *Equations et inéquations non linéaires dans les espaces vectoriels en dualité.* Annales Inst. Fourier, 18, 115-175, 1968.
[3] C. Ciulcu, D. Motreanu, M. Sofonea; *Analysis of an elastic contact problem with slip dependent coefficient of friction*; Mathematical inequalities & Applications, 4, 3, 465-479, 2001.
[4] M. Cocou, E. Pratt, M. Raous; *Formulation and approximation of quasistatic frictional contact*, Int. J. Engng Sc., 34, 7, 783-798, 1996.
[5] C. Corneshi, T.-V. Horau-Mantel, M. Sofonea; *A Quasistatic contact problem with slip dependent coefficient of friction for elatic materials*, J. Appl. Anal. 8(1), 63-82, 2002.
[6] S. Drabla, M. Sofonea; *Analysis of a Signorini problem with friction*, IMA Journal of Applied Mathematics, 63, 2, 113-130, 1999.
[7] G. Duvaut; *Equilibre d'un solide élastique avec contact unilatéral et frottement de Coulomb.* Cr Acad. Sci. Paris, Ser A, 290, 263-265, 1980.
[8] G. Duvaut, J.-L. Lions; *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.
[9] I.-R. Ionescu, Q.-L. Nguyen, S. Wolf; *Slip-dependent friction in dynamic elasticity*; Nonlinear Analysis, 53, 375-390, 2003.

[10] A. Klarbring, A. Mikelic, M. Shillor; *A global existence result for the quasistatic problem with normal compliance.* Internat. Ser. Numer. Math. 101. Birkhäuser Verlag. basel, 85-111, 1991.
[11] D. Motreanu, M. Sofonea; *Quasivariational inequalities and applications in frictional contact problems with normal compliance*, Adv. Math. Sci. Appl. 10(1), 103-118, 2000.
[12] M. Raous , M. Jean, J. J. Moreau (Eds.); *Contact Mechanics,* Plenum Press, New York, 1995.
[13] R. Rocca; *Existence of a solution for a quasistatic problem of unilateral contact with local friction.* CR. Acad. Sci Paris. Ser1, t. 328, 1253-1258, 1999.
[14] M. Rochi, M. Schillor, M. Sofonea; *Quasistatic viscoplastic contact with normal compliance and friction*, J. Elasticity, 51, 105-126, 1998.
[15] A. Touzaline. A. Mignot; *Existence of solutions for quasistatic problems of unilateral contact with nonlocal friction for nonlinear elastic materials*, Electronic Journal of Differential Equations, Vol. 2005, no. 99, 1-13, 2005.

Corrigendum posted February 8, 2007

The author would like to correct the following misprints:

Page 5, Line 24: The last line of the displayed equation should be

$$\langle \sigma_\nu(u(t)), z_\nu - u_\nu(t)\rangle \geq 0 \quad \forall z \in K.$$

Page 5: Equation (2.17) should be

$$\langle \sigma_\nu(u(t)), z_\nu - u_\nu(t)\rangle \quad \forall z \in K. \tag{2.17}$$

Page 9, Line 27: The argument (t) should be deleted; so that the inequality becomes

$$\|u^n - \widetilde{u}^n\|_{L^\infty(0,T;V)} \leq c_4 \frac{T}{n} \|\dot{f}\|_{L^\infty(0,T;V)}.$$

Page 13: The symbol "≥ 0" should be delteted in both inequalitites: This is,

$$\begin{gathered} \langle F(\varepsilon(u(t))), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_Q + j(u(t), v) - j(u(t), \dot{u}(t)) \\ \geq \langle f(t), v - \dot{u}(t)\rangle_V + \langle \theta\sigma_\nu(u(t)), v_\nu - \dot{u}_\nu(t)\rangle_\Gamma \quad \forall v \in V, \\ \langle \theta\sigma_\nu(u(t)), z_\nu - u_\nu(t)\rangle_\Gamma \quad \forall z \in K. \end{gathered}$$

End of corrigendum.

Arezki Touzaline
Faculté des Mathématiques, University of Sciences and Technology Houari Boumediene, USTHB, BP 32 El Alia, Bab Ezzouar, 16111, Algérie
E-mail address: atouzaline@yahoo.fr