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OSCILLATION FOR FORCED SECOND-ORDER NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. By means of Riccati transformation techniques, we present oscillation criteria for forced second-order nonlinear dynamic equations on time scales. These results are based on the information on a sequence of subintervals of $[a, \infty)$ only, rather than on the whole half-line.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [6] in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis. A time scale $\mathbb{T}$, is an arbitrary nonempty closed subset of the reals. Many authors have expanded on various aspects of this new theory; see the survey paper by Agarwal et al. [1] and the book by Bohner and Peterson [3] which summarizes and organizes much of the time scale calculus. For the notion used below we refer to the next section that provides some basic facts on time scale extracted from [3].

There are many interesting time scales and they give rise to plenty of applications, the cases when the time scale is equal to reals or the integers represent the classical theories of differential and of difference equations. Another useful time scale is $\mathbb{P}_{a,b} = \cup_{n=0}^{\infty}[n(a+b), n(a+b)+a]$ which is widely used to study population in biological communities, electric circuit and so on.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solution of various equations on time scales, and we refer the reader to papers [2, 4, 5, 7, 8, 9] and references cited therein.

Bohner and Saker[4] considered the perturbed nonlinear dynamic equation

$$(\alpha(t)(x^{\Delta})^{\gamma})^{\Delta} + F(t, x^{\sigma}) = G(t, x^{\sigma}, x^{\Delta}), \quad t \in [a, b]. \tag{1.1}$$

Assuming that $\frac{F(t,u)}{f(u)} \geq q(t)$, $\frac{G(t,u,v)}{f(u)} \leq p(t)$, they change (1.1) into the inequality

$$(\alpha(t)(x^{\Delta})^{\gamma})^{\Delta} + (q(t) - p(t))f(x^{\sigma}) \leq 0. \tag{1.2}$$

Then using Riccati transformation techniques, they obtain sufficient conditions for the solution to be oscillatory, or to converge to zero.

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Saker [8] considered the second-order forced nonlinear dynamic equation

$$(a(t)x^{\Delta})^{\Delta} + p(t)f(x^{\sigma}) = r(t), \quad t \in [t_0, \infty),$$

assuming that $\int_{t_0}^{\infty} |r(s)|\Delta s < \infty$; that is, the forcing terms are "small" enough for all large $t \in \mathbb{T}$. Some additional assumptions have to be imposed on the unknown solutions. He obtained sufficient condition on the forcing terms directly, for solution to be oscillatory or to converge to zero.

Following this trend, to develop the qualitative theory of dynamic equations on time scales, in this paper, we consider the following second-order forced nonlinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)f(x^{\sigma}(t)) = e(t), \tag{1.3}$$

on the time scale interval $[a, \infty) = \{t \in \mathbb{T}, t \geq a\}$, where $x^{\sigma}(t) = x(\sigma(t))$, $e, p \in C_{\rm rd}(\mathbb{T}, \mathbb{R})$.

In this paper, we apply Riccati transformation technique to obtain some oscillation criteria for (1.3). Our results do not require that $p(t)$ and $e(t)$ be of definite sign and are based on the information only on a sequence of subintervals of $[a, \infty)$ rather than the whole half-line. Our results in this paper improve the results given in [4, 8].

By a solution of (1.3), we mean a nontrivial real-valued function x satisfying (1.3) for $t \geq a$. A solution x of (1.3) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Equation (1.3) is called oscillatory if all solutions are oscillatory. Our attention is restricted to those solution x of (1.3) which exist on half line $[t_x, \infty)$ with $\sup\{|x(t)| : t \geq t_0\} \neq 0$ for any $t_0 \geq t_x$.

2. Preliminaries

Let $\mathbb{T}$ be a time scale, we define the forward and backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$, and $\emptyset$ denotes the empty set. A nonmaximal element $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and right-scattered if $\sigma(t) > t$. A nonminimal element $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and left-scattered if $\rho(t) < t$. The graininess μ of the time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

A mapping $f : \mathbb{T} \to \mathbb{X}$ is said to be differentiable at $t \in \mathbb{T}$, if there exists $b \in \mathbb{X}$ such that for every $\varepsilon > 0$, there exists a neighborhood U of t satisfying $|[f(\sigma(t)) - f(s)] - b[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$, for all $s \in U$. We say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}$ provided $f^{\Delta}(t)$ exist for all $t \in \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{\rm rd}(\mathbb{T}, \mathbb{R})$.

The derivative and forward jump operator σ are related by the formula

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t). \tag{2.1}$$

Let f be a differentiable function on $[a, b]$. If $f^{\Delta} > 0$, $f^{\Delta} < 0$, $f^{\Delta} \geq 0$, $f^{\Delta} \leq 0$ for all $t \in [a, b)$; then f is increasing, decreasing, nondecreasing, nonincreasing on $[a, b]$, respectively.

We use the following product and quotient rules for derivative of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \tag{2.2}$$

$$\big(\frac{f}{g}\big)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}, \tag{2.3}$$

where $f^{\sigma} = f \circ \sigma$ and $gg^{\sigma} \neq 0$.

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(t)g(t)|_a^b - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t, \tag{2.4}$$

Chain Rule: Assume $g : \mathbb{T} \to \mathbb{R}$ is $\Delta-$differentiable on $\mathbb{T}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable and satisfies

$$(f \circ g)^{\Delta}(t) = \Big\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \Big\} g^{\Delta}(t). \tag{2.5}$$

From (2.5), we obtain (see [9]).

$$(x^{\gamma})^{\Delta}(t) = \gamma \int_0^1 [hx^{\sigma}(t) + (1-h)x(t)]^{\gamma-1} dh x^{\Delta}(t). \tag{2.6}$$

In order to prove our main results, we need the following auxiliary result.

Lemma 2.1. *If A and B are nonnegative, then*

$$A^{\lambda} - \lambda AB^{\lambda-1} + (\lambda-1)B^{\lambda} \geq 0, \quad \lambda > 1, \tag{2.7}$$

and the equality holds if and only if $A = B$.

3. Main results

Our interest is to establish oscillation criteria for (1.3) that do not assume that $p(t)$ and $e(t)$ being of definite sign. In this section, we give some new oscillation criteria. Since we are interested in oscillatory behavior, we suppose that the time scale $\mathbb{T}$ under consideration is not bounded above, i.e. it is a time scale interval of the form $[a, \infty)$. Let

$$D(a_i, b_i) = \big\{ u \in C^1_{\rm rd}[a_i, b_i] : u(t) \not\equiv 0, u(a_i) = u(b_i) = 0 \big\}, \quad i = 1, 2.$$

Theorem 3.1. *Let $f(x)/x \geq k > 0$ for $x \neq 0$. Assume that for any $T \geq a$, there exist constants $a_1, b_1, a_2, b_2 \in \mathbb{T}$ such that $T \leq a_1 < b_1$, $T \leq a_2 < b_2$, and*

$$\begin{gathered} p(t) \geq 0, \quad t \in [a_1, b_1] \cup [a_2, b_2], \\ e(t) \leq 0, \quad t \in [a_1, b_1]; \\ e(t) \geq 0, \quad t \in [a_2, b_2]. \end{gathered} \tag{3.1}$$

If there exists $u \in D(a_i, b_i)$ such that

$$\int_{a_i}^{b_i} \Big\{ kp(t)u^2(\sigma(t)) - \frac{\mu(t) + t - a_i}{4(t - a_i)} \big[\frac{u(t) + u(\sigma(t))}{u(\sigma(t))} u^{\Delta}(t)\big]^2 \Big\} \Delta t \geq 0, \tag{3.2}$$

for $i = 1, 2$, then (1.3) is oscillatory.

Proof. Suppose, to the contrary, that x is a nonoscillatory solution of (1.3), which is eventually positive. Say $x(t) > 0$, $x^\sigma(t) > 0$ for $t \geq t_0 \geq a$. Denote $w(t) = -\frac{x^\Delta(t)}{x(t)}$ for $t \geq t_0$. It follows from (1.3) that $w(t)$ satisfies the dynamic equation

$$\begin{aligned} w^\Delta(t) &= -\frac{x^{\Delta\Delta}(t)}{x^\sigma(t)} + \frac{x(t)}{x^\sigma(t)}\Big[\frac{x^\Delta(t)}{x(t)}\Big]^2 \\ &= \frac{1}{1+\mu(t)\frac{x^\Delta(t)}{x(t)}} w^2(t) + p(t)\frac{f(x^\sigma(t))}{x^\sigma(t)} - \frac{e(t)}{x^\sigma(t)}. \end{aligned} \tag{3.3}$$

By assumption, we can choose $a_1, b_1 \in \mathbb{T}$ such that $b_1 > a_1 \geq t_0$ and $p(t) \geq 0, e(t) \leq 0, t \in [a_1, b_1]$. From (1.3), we get $x^{\Delta\Delta}(t) = e(t) - p(t)f(x^\sigma(t)) \leq 0$ for $t \in [a_1, b_1]$. Therefore, we have that for $t \in [a_1, b_1]$

$$x(t) \geq x(t) - x(a_1) = \int_{a_1}^t x^\Delta(s)\Delta s \geq x^\Delta(t)(t - a_1);$$

i.e.,

$$\frac{x^\Delta(t)}{x(t)} \leq \frac{1}{t - a_1}, \quad t \in (a_1, b_1]. \tag{3.4}$$

Using the above inequality and $\frac{f(x)}{x} \geq k$, (3.3) yields

$$w^\Delta(t) \geq \frac{t - a_1}{\mu(t) + t - a_1} w^2(t) + kp(t). \tag{3.5}$$

Let $u(t) \in D(a_1, b_1)$ be as in the hypothesis. Multiply both sides of (3.5) by $u^2(\sigma(t))$ and integrate it from a_1 to b_1, we obtain

$$\int_{a_1}^{b_1} u^2(\sigma(t))w^\Delta(t)\Delta t \geq \int_{a_1}^{b_1} \big[\lambda_1(t)w^2(t)u^2(\sigma(t)) + kp(t)u^2(\sigma(t))\big]\Delta t, \tag{3.6}$$

where $\lambda_1(t) = \frac{t-a_1}{\mu(t)+t-a_1}$. Using the integration by parts, (2.4), and $u(a_1) = u(b_1) = 0$, we have

$$\begin{aligned} 0 &= w(t)u^2(t)|_{a_1}^{b_1} \\ &\geq \int_{a_1}^{b_1} [(u(t) + u(\sigma(t)))u^\Delta(t)w(t) + \lambda_1(t)w^2(t)u^2(\sigma(t)) + kp(t)u^2(\sigma(t))]\Delta t \\ &= \int_{a_1}^{b_1} \Big[(\lambda_1(t))^{1/2}u(\sigma(t))w(t) + \frac{u(t) + u(\sigma(t))}{2(\lambda_1(t))^{1/2}u(\sigma(t))}u^\Delta(t)\Big]^2\Delta t \\ &\quad + \int_{a_1}^{b_1} \Big[kp(t)u^2(\sigma(t)) - \frac{(u(t) + u(\sigma(t)))^2}{4\lambda_1(t)u^2(\sigma(t))}(u^\Delta(t))^2\Big]\Delta t \\ &> \int_{a_1}^{b_1} \Big[kp(t)u^2(\sigma(t)) - \frac{(u(t) + u(\sigma(t)))^2}{4\lambda_1(t)u^2(\sigma(t))}(u^\Delta(t))^2\Big]\Delta t, \end{aligned} \tag{3.7}$$

which contradicts (3.2).

In the case of $x(t) < 0$ for $t \geq t_0 \geq a$, we use the function $y = -x$ as a positive solution of the dynamic equation $x^{\Delta\Delta}(t) + p(t)f(x^\sigma(t)) = -e(t)$ and repeat the above procedure on the interval $[a_2, b_2]$. This completes the proof of theorem 3.1. □

Theorem 3.2. *Let $xf(x) > 0$ for $x \neq 0$ and $|f(x)| \geq |x|^r$ for $r > 1$. Assume, in addition, that for any $T \geq a$, there exist constants $a_1, b_1, a_2, b_2 \in \mathbb{T}$ such that (3.1) holds. If there exists $u(t) \in D(a_i, b_i)$ such that*

$$\begin{aligned}\int_{a_i}^{b_i} \Big\{ & r(r-1)^{\frac{1-r}{r}} p^{1/r}(t)|e(t)|^{\frac{r-1}{r}} u^2(\sigma(t)) \\ & - \frac{\mu(t)+t-a_i}{4(t-a_i)} \big[\frac{u(t)+u(\sigma(t))}{u(\sigma(t))} u^\Delta(t)\big]^2 \Big\} \Delta t \geq 0,\end{aligned} \tag{3.8}$$

for $i = 1, 2$, then (1.3) is oscillatory.

Proof. As before, we suppose $x(t) > 0$, $t \geq t_0 \geq a$, be a nonoscillatory solution of (1.3). Let $A = p^{1/r}(t)x^\sigma(t), B = \big(\frac{-e(t)}{r-1}\big)^{1/r}$. By the assumption, we can choose $a_1, b_1 \in \mathbb{T}$ such that $b_1 > a_1 \geq t_0 \geq a$ and $p(t) \geq 0$, $e(t) \leq 0$ for $t \in [a_1, b_1]$. Hence, $A > 0$, $B > 0$ for $r > 1$. From lemma 2.1, we obtain

$$\begin{aligned} p(t)x^r(\sigma(t)) - e(t) &\geq r(r-1)^{\frac{1-r}{r}} p^{1/r}(t)|e(t)|^{\frac{r-1}{r}} x(\sigma(t)) \\ &= \lambda_2 p^{1/r}(t)|e(t)|^{\frac{r-1}{r}} x(\sigma(t)), \end{aligned} \tag{3.9}$$

where $\lambda_2 = r(r-1)^{\frac{1-r}{r}}$ is a constant. By (1.3) and (3.9), we obtain

$$x^{\Delta\Delta}(t) + \lambda_2 p^{1/r}(t)|e(t)|^{\frac{r-1}{r}} x(\sigma(t)) \leq 0. \tag{3.10}$$

Let $w(t) = x^\Delta(t)/x(t)$ and use (2.1), (2.3) and (3.4), then

$$w^\Delta(t) = \frac{x^{\Delta\Delta}(t)}{x^\sigma(t)} - \frac{x(t)}{x^\sigma(t)}[\frac{x^\Delta(t)}{x(t)}]^2 \leq -\lambda_2 p^{1/r}(t)|e(t)|^{\frac{r-1}{r}} - \lambda_1(t)w^2(t), \tag{3.11}$$

where $\lambda_1(t) = \frac{t-a_1}{\mu(t)+t-a_1}$. Let $u(t) \in D(a_1, b_1)$, product both sides of (3.11) by $u^2(\sigma(t))$ and integrate it from a_1 to b_1, we get

$$\int_{a_1}^{b_1} u^2(\sigma(t))w^\Delta(t)\Delta t \leq \int_{a_1}^{b_1} \big[-\lambda_1(t)w^2(t)u^2(\sigma(t)) - \lambda_2 p^{1/r}(t)|e(t)|^{\frac{r-1}{r}} u^2(\sigma(t))\big]\Delta t.$$

Using integration by parts formula (2.4), and $u(a_1) = u(b_1) = 0$, we have

$$\begin{aligned} 0 &= w(t)u^2(t)|_{a_1}^{b_1} \\ &\leq \int_{a_1}^{b_1} \Big[(u(t)+u(\sigma(t)))u^\Delta(t)w(t) - \lambda_1(t)w^2(t)u^2(\sigma(t)) \\ &\quad - \lambda_2 p^{1/r}(t)|e(t)|^{\frac{r-1}{r}} u^2(\sigma(t))\Big]\Delta t \\ &= -\int_{a_1}^{b_1} \Big[(\lambda_1(t))^{1/2}u(\sigma(t))w(t) - \frac{u(t)+u(\sigma(t))}{2(\lambda_1(t))^{1/2}u(\sigma(t))}u^\Delta(t)\Big]^2 \Delta t \\ &\quad + \int_{a_1}^{b_1} \Big[\frac{(u(t)+u(\sigma(t)))^2}{4\lambda_1(t)u^2(\sigma(t))}(u^\Delta(t))^2 - \lambda_2 p^{1/r}(t)|e(t)|^{\frac{r-1}{r}} u^2(\sigma(t))\Big]\Delta t \\ &< \int_{a_1}^{b_1} \Big[\frac{(u(t)+u(\sigma(t)))^2}{4\lambda_1(t)u^2(\sigma(t))}(u^\Delta(t))^2 - \lambda_2 p^{1/r}(t)|e(t)|^{\frac{r-1}{r}} u^2(\sigma(t))\Big]\Delta t, \end{aligned}$$

which contradicts (3.8). $\square$

Theorem 3.3. *Let $xf(x) > 0$ for $x \neq 0$ and $|f(x)| \geq k|x|^r$. Suppose, furthermore, that for any $T \geq a$, there exist constants $a_1, b_1, a_2, b_2 \in \mathbb{T}$ such that (3.1) holds. If $\mu(t) \leq k't$ and there exists $u(t) \in D(a_i, b_i)$ such that*

$$\int_{a_i}^{b_i} \Big[kp(t)u^2(\sigma(t)) - \frac{1}{4M}\big(\frac{u(t)+u(\sigma(t))}{u(\sigma(t))}u^\Delta(t)\big)^2\Big]\Delta t \geq 0,$$

for $i = 1, 2$ and M, k, k' are some positive constants, then

(1) *every unbounded solution of (1.3) with $r > 1$ is oscillatory.*
(2) *every bounded solution of (1.3) with $0 < r < 1$ is oscillatory.*

Proof. As before, we assume $x(t) > 0$, $x(\sigma(t)) > 0$, $t \geq t_0 \geq a$, be a nonoscillatory solution of (1.3). Let $w(t) = -\frac{x^\Delta(t)}{x^r(t)}$ for $t \geq t_0$. It follows from (1.3), the condition $f(x) \geq kx^r(t)$ and (2.6) that $w(t)$ satisfies

$$\begin{aligned} w^\Delta(t) &= -\frac{x^{\Delta\Delta}(t)}{x^r(\sigma(t))} + \frac{(x^\Delta(t))^2}{x^r(t)x^r(\sigma(t))} r \int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{r-1} dh \\ &= p(t)\frac{f(x(\sigma(t)))}{x^r(\sigma(t))} - \frac{e(t)}{x^r(\sigma(t))} \\ &\quad + r\frac{(x^\Delta(t))^2}{x^r(t)x^r(\sigma(t))}\int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{r-1} dh. \end{aligned} \tag{3.12}$$

By the assumption, we can choose $a_1, b_1 \in \mathbb{T}$ such that $b_1 > a_1 \geq t_0 \geq a$ and $p(t) \geq 0, e(t) \leq 0$ for $t \in [a_1, b_1]$. Then $x^{\Delta\Delta}(t) = e(t) - p(t)f(x^\sigma(t)) \leq 0$ for $t \in [a_1, b_1]$, and (3.12) satisfies

$$w^\Delta(t) \geq kp(t) + r\frac{(x^\Delta(t))^2}{x^r(t)x^r(\sigma(t))}\int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{r-1} dh. \tag{3.13}$$

There are three cases to be considered
(i) $x^\Delta(t) \geq 0$, $t \in [a_1, b_1]$. Then we obtain

$$\int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{r-1} dh \geq \int_0^1 x^{r-1}(t) dh = x^{r-1}(t). \tag{3.14}$$

Using (3.14) and (3.4), (3.13) yields

$$\begin{aligned} w^\Delta(t) &\geq kp(t) + rx^{r-1}(t)\big[\frac{x(t)}{x(\sigma(t))}\big]^r w^2(t) \\ &\geq kp(t) + r\big[\frac{t-a_1}{\mu(t)+t-a_1}\big]^r x^{r-1}(t)w^2(t) \\ &= kp(t) + r\lambda_1^r(t)x^{r-1}(t)w^2(t), \quad t \in [a_1, b_1], \end{aligned} \tag{3.15}$$

where $\lambda_1(t) = \frac{t-a_1}{\mu(t)+t-a_1}$.
(ii) $x^\Delta(t) < 0$, $t \in [a_1, b_1]$. Then we get

$$\int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{r-1} dh \geq \int_0^1 x^{r-1}(\sigma(t)) dh = x^{r-1}(\sigma(t)). \tag{3.16}$$

Using (3.16) and (3.4), (3.13) yields

$$\begin{aligned} w^{\Delta}(t) &\geq kp(t) + rx^{r-1}(t)\frac{x(t)}{x(\sigma(t))}w^2(t) \\ &\geq kp(t) + r\frac{t-a_1}{\mu(t)+t-a_1}x^{r-1}(t)w^2(t) \\ &= kp(t) + r\lambda_1(t)x^{r-1}(t)w^2(t), \quad t \in [a_1, b_1]. \end{aligned} \tag{3.17}$$

(iii) there exist $a_1 < c_1 < b_1$ ($c_1 \in \mathbb{T}$) such that $x^{\Delta}(t) \geq 0$, $t \in [a_1, c_1]$ and $x^{\Delta}(t) < 0, t \in (c_1, b_1]$. Proceeding as in the proof of (i) and (ii), we obtain that

$$w^{\Delta}(t) \geq kp(t) + r\lambda_1^r(t)x^{r-1}(t)w^2(t), \quad t \in [a_1, c_1], \tag{3.18}$$

$$w^{\Delta}(t) \geq kp(t) + r\lambda_1(t)x^{r-1}(t)w^2(t), \quad t \in (c_1, b_1]. \tag{3.19}$$

Next, we consider the following two cases
(I) If x is an unbounded nonoscillatory solution of (1.3) with $r > 1$. Since $\mu(t) \leq k't$ for $k' > 0$ is a positive constant, then there exists a positive constant $0 < k'' < \frac{1}{k'+1}$ such that $k'' < \frac{t-a_1}{\mu(t)+t-a_1} \leq 1$ for t large enough, from (3.15), (3.17), (3.18), and (3.19), we get

$$w^{\Delta}(t) \geq kp(t) + r\lambda_1^r(t)x^{r-1}(t)w^2(t). \tag{3.20}$$

Since $x^{\Delta\Delta}(t) \leq 0, t \in [a_1, b_1]$, then there exists a constant $M_1 > 0$ such that $x(t) \geq M_1$ on $[a_1, b_1]$, such that

$$r\lambda_1^r(t)x^{r-1}(t) \geq r\lambda_1^r(t)M_1^{r-1} \geq M, \quad t \in [a_1, b_1], \tag{3.21}$$

where $M > 0$ is a constant. Using (3.20) and (3.21), and proceeding as in the proof of theorem 3.1, we obtain the desired contradiction.
(II) If x is a bounded nonoscillatory solution of (1.3) with $0 < r < 1$ on $[a_1, b_1]$. Since $0 < k'' < \frac{t-a_1}{\mu(t)+t-a_1} \leq 1$ for t large enough, from (3.15), (3.17), (3.18), and (3.19), we obtain

$$w^{\Delta}(t) \geq kp(t) + r\lambda_1(t)x^{r-1}(t)w^2(t).$$

Since $x^{\Delta\Delta}(t) \leq 0, t \in [a_1, b_1]$, then there exists a constant $M_2 > 0$ such that $x(t) \leq M_2$ on $[a_1, b_1]$, hence

$$r\lambda_1(t)x^{r-1}(t) \geq r\lambda_1(t)M_2^{r-1} \geq M', \quad t \in [a_1, b_1],$$

where $M' > 0$ is a constant. The rest of the proof is similar to that in the previous case and we obtain the desired contradiction. □

4. Example

Since the time scale $\mathbb{P}_{a,b} = \cup_{n=0}^{\infty}[n(a+b), n(a+b)+a]$ can be used to study many models of real world, for instance, population in biological communities, electric circuit and so on, we give an example in such a time scale to demonstrate how the theory may be applied to specific problems.

Consider the forced second order dynamic equation

$$x^{\Delta\Delta}(t) + msintx(\sigma(t)) = \cos t, \quad \text{for } t \in \mathbb{P}_{\pi,\pi} = \cup_{n=0}^{\infty}[2n\pi, (2n+1)\pi], \tag{4.1}$$

with the transition condition

$$x(2n\pi) = x((2n-1)\pi), \quad n \geq 1, \tag{4.2}$$

where $m > 0$ is a constant, $p(t) = m\sin t$, $e(t) = \cos t$, $f(x(\sigma(t))) = x(\sigma(t))$. For any $T \geq 0$, if we choose $a_1 = 2n\pi + \frac{\pi}{2}$, $b_1 = 2n\pi + \frac{3\pi}{4}$, $a_2 = 2n\pi + \frac{\pi}{4}$, $b_2 = 2n\pi + \frac{\pi}{2}$,

($a_i, b_i \in \mathbb{P}_{\pi,\pi}$, $i = 1,2$) such that $a_i \geq T$ for sufficiently large $n, i = 1,2$, then we have $p(t) \geq 0$ for $t \in [a_1, b_1] \bigcup [a_2, b_2]$, $e(t) \leq 0$ for $t \in [a_1, b_1]$, and $e(t) \geq 0$ for $t \in [a_2, b_2]$. Choose $u(t) = sin2tcos2t$, then $u(t) \in D(a_i, b_i), i = 1,2$. Furthermore, we have $\sigma(t) = t$, $\mu(t) = 0$ for $t \in [a_i, b_i], i = 1,2$. Noting that for $i = 1,2$,

$$\begin{aligned}
&\int_{a_i}^{b_i} \big\{kp(t)u^2(\sigma(t)) - \frac{\mu(t) + t - a_i}{4(t - a_i)}\big[\frac{u(t) + u(\sigma(t))}{u(\sigma(t))}u^{\Delta}(t)\big]^2\big\}\Delta t \\
&= \int_{a_i}^{b_i} \big[p(t)u^2(t) - (u'(t))^2\big]dt \\
&= \int_{a_i}^{b_i} m \sin t \sin^2(2t)\cos^2(2t) - 4\cos^2(4t)\big]dt.
\end{aligned}$$

On the other hand, we have

$$\int_{a_i}^{b_i} 4\cos^2(4t)dt = \frac{\pi}{2},$$

and

$$\int_{a_i}^{b_i} m \sin t \sin^2(2t)\cos^2(2t)dt = \frac{\sqrt{2}}{2}m\big[\frac{1}{8} - \frac{1}{9 \times 16} + \frac{1}{7 \times 16}\big].$$

Then, $\int_{a_i}^{b_i} m \sin t \sin^2(2t)\cos^2(2t)dt \geq \pi/2$ for sufficiently large m, hence (3.2) holds. By Theorems 3.1, we obtain that (4.1) and (4.2) is oscillatory. However, the results in Saker [8] and Bohner and Saker [4] cannot be applied the oscillation of (4.1) and (4.2).

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