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ON A BREZIS-NIRENBERG TYPE PROBLEM

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Abstract. In this note we discuss the existence and symmetry breaking of least energy solutions for certain weighted elliptic equations in the unit ball in $\mathbb{R}^N$, with zero Dirichlet boundary conditions. We prove a multiplicity result, which answers one of the questions we left open in [6] regarding a Brezis-Nirenberg type problem.

1. Introduction

Over the last four decades, a large amount of work has been done on existence and qualitative properties of solutions for semi-linear elliptic problems. A significant proportion of these studies deal with positive solutions for problems that lack compactness. The loss of compactness may be due to the existence of limiting problems which are invariant under translations or (in the case of critical nonlinearities) under dilations. The presence of zeros or singularities in the coefficients, in many instances plays a role in the form of the limiting problem. As a typical sample of existence results on the whole of $\mathbb{R}^N$ one may consult [1, 27, 25], and the references therein.

On bounded domains in $\mathbb{R}^N$ for $N \geq 3$, it was shown by Pohozaev as early as 1965 (see [23]) that nonlinear eigenvalue problems of the form

$$\begin{gathered} -\Delta u = u^{p-1} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

have no positive solution if Ω is star-shaped and $p \geq 2^*$ with $2^* = \frac{2N}{N-2}$ the critical Sobolev exponent.

An intriguing phenomenon was observed by Brezis and Nirenberg in [4] relative to the problem

$$\begin{gathered} -\Delta u = u^{2^*-1} + \lambda u, \\ u > 0 \quad \text{in } B, \\ u = 0 \quad \text{on } \partial B. \end{gathered} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^N$ with $N \geq 3$. The authors have proved the following theorem.

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Theorem 1.1. *Let $\lambda_1 = \lambda_1(N)$ be the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition on B. Then problem (1.1) has solution if and only if*

(a) $N \geq 4$ *and* $\lambda \in (0, \lambda_1)$*; or*
(b) $N = 3$ *and* $\lambda \in (\lambda_1/4, \lambda_1)$.

This started a flurry of work on problems in which the same phenomenon was observed. That is, for some dimensions N the branch of solutions which bifurcates from the trivial solution exists for all λ between λ_1 and zero, while for other "critical" dimensions this branch is bounded away from $\lambda = 0$ (see [3, 5, 9, 10, 11, 12, 14, 15, 18, 24], and the references therein).

The present work is motivated by one of the questions we raised in [6]. Let B be the unit ball in $\mathbb{R}^N$. Consider the problem

$$\begin{gathered} -\operatorname{div}(|x|^{-2a}\nabla u) = |x|^{-bp}u^{p-1} + \lambda|x|^{-2(a+1)+c}u, \\ u > 0 \quad \text{in } B, \quad u \in \mathcal{D}_a(B). \end{gathered} \tag{1.2}$$

Throughout we shall consider

$$\begin{gathered} 2 \leq N, \quad a < \frac{N-2}{2}, \quad a < b < a+1, \\ p = \frac{2N}{N-2(a+1-b)}, \quad 0 < c. \end{gathered} \tag{1.3}$$

Here $\mathcal{D}_a(B)$ denotes the Hilbert space obtained as the completion of smooth functions with compact support in B under the norm induced by the inner product

$$\langle u, v\rangle = \int_B |x|^{-2a}\nabla u \cdot \nabla v \, dx.$$

The exponent p given in (1.3) above, is a critical exponent for the limiting problem to (1.2) in the fact that the equation

$$-\operatorname{div}(|x|^{-2a}\nabla u) = |x|^{-bp}u^{p-1}$$

in $\mathbb{R}^N$, is invariant under the (non-compact) group of dilations centered at the origin.

In [6], we filled in a gap between results of Nicolaescu [22] and Chou and Geng [9] by giving the exact range of existence of radial ($u = u(|x|)$) solutions of (1.2). This result encompasses Theorem 1.1 for the particular values of the parameters $a = b = 0$ and $c = 2$. To our knowledge it is one of the few situations where an exact formula can be given for the break-down value of λ which separates the existence and nonexistence regions. For a precise statement summarizing the results in [22, 9, 6] see Theorem 4.1 in the concluding remarks.

In [6] we asked the following two questions.

Question 1.2. Are there cases when (1.2) has a solution, but admits no spherically symmetric (radial) solutions?

Question 1.3. Are there cases when (1.2) has both, radial and nonradial solutions?

While we still do not have an answer to the first question, the purpose of this paper is to answer the second question in the affirmative. Our main result is the following theorem.

Theorem 1.4. *Let $Z(\nu)$ denote the first positive zero of the Bessel function of the first kind J_ν, and let $\gamma = \frac{N-2-2a}{2}$. Assume $0 < b-a < 1$, $0 < c < 2\gamma$, and $\gamma^2 > \frac{N-1}{p-2}$. Then for any*

$$0 < \lambda < \Big(\frac{c}{2} Z\Big(\frac{2}{c}\sqrt{\gamma^2 - \frac{N-1}{p-2}}\Big)\Big)^2$$

the least energy solution for the problem (1.2) *exists and it is nonradial.*

As the existence of radial solutions has been discussed in [6] (see also Theorem 4.1 in the concluding remarks), we obtain the following corollary.

Corollary 1.5. *In the range of parameters stated in Theorem 1.4, the problem* (1.2) *has a (higher energy) radial and a (ground state) nonradial solution.*

In the case $N \geq 3$, $a = b = 0$ and $c = 2$, one has $p = \frac{2N}{N-2} = 2^*$, the critical exponent for the Sobolev imbedding $\mathcal{D}_0(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$, and the problem under study (1.2) becomes exactly (1.1). Due to a well known result of Gidas, Ni and Nirenberg (see [16]) every solution of (1.1) is radial. Therefore in this case, the study of the ODE obtained by symmetry reduction answers our questions completely in form of Theorem 1.1. Chou and Geng in [9], extend the moving plane method of [16], and they obtain that for $3 \leq N$, $0 \leq a < \frac{N-2}{2}$, every solution of (1.2) is radial. Hence, again the ODE results settle the questions.

We remark that in the range $a < 0$, the method of moving planes breaks down. It is in this range where Theorem 1.4 applies. Indeed, since $2^* > p > 2$ one has from $\gamma^2 > \frac{N-1}{p-2}$ that $\gamma^2 > \frac{N-1}{2^*-2}$. This leads to $\sqrt{N-2}(\sqrt{N-2} - \sqrt{N-1}) > 2a$ which forces a to be negative.

The fact that the problem may admit both radial and non-radial solutions for certain values of the parameters is somewhat expected in view of the symmetry breaking phenomenon of the least energy solutions observed in [7]. Combined with the existence result for radial solutions in Theorem 4.1, the symmetry breaking of the ground state for problem (1.2) guarantees the existence of at least two solutions. Hence our theorem in the present article should be viewed both, as a symmetry breaking result and a multiplicity theorem.

The main novelty in Theorem 1.4 is to give an explicit range of parameters in which the ground state (least energy solution) is nonradial. The existence of the ground state is based on Lemma 1 in [2]. In order to apply this Lemma we need a precise estimate on the decay at infinity of solutions of a related problem in $\mathbb{R}^N$ which we establish in Lemma 3.3. For this we employ Harnack inequality as it is presented in the Section 11 in the Lecture Notes [20]. For the symmetry breaking part, we employ the technique of [26] which originated in the work [19].

The paper is organized as follows. In Section 2 we introduce an equivalent problem, which we find more convenient to work with, and we gather some preliminary results. Section 3 contains the proof of Theorem 1.4. The paper ends with a short section of concluding remarks.

2. Preliminaries

For $u \in \mathcal{D}_a(B) \setminus \{0\}$, consider the energy

$$E(u) = E_{a,b,c,\lambda}(u) = \frac{\displaystyle\int_B |x|^{-2a}|\nabla u|^2 - \lambda |x|^{-2(a+1)+c} u^2 \, dx}{\Big(\displaystyle\int_B |x|^{-bp}|u|^p \, dx\Big)^{2/p}},$$

and denote

$$J = J(a,b,c,\lambda) = \inf_{u \in \mathcal{D}_a(B)\setminus\{0\}} E(u).$$

Note that solutions of (1.2) are critical points of E. Conversely, if the infimum J of E is positive and it is achieved by some function u in $\mathcal{D}_a(B) \setminus \{0\}$, then after an eventual multiplication by a constant, u is solution of (1.2).

As in [7], consider the cylinder $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{N-1} \subset \mathbb{R}^{N+1}$. Define the conformal diffeomorphism

$$\vartheta : \mathbb{R}^N \setminus \{0\} \to \mathcal{C} \quad \text{given by } \vartheta(x) = \Big(-\ln|x|, \frac{x}{|x|}\Big) = (t,\theta)$$

Note that ϑ takes $B\setminus\{0\}$ diffeomorphically into the half cylinder $\Omega = (0,\infty)\times\mathbb{S}^{N-1}$. The transformation

$$\Upsilon_a : H_a(\Omega) \to \mathcal{D}_a(B) \quad \Upsilon_a v = u \quad \text{with } u(x) = |x|^{-\gamma} v(\vartheta(x))$$

is a Hilbert space isomorphism. Here $H_a(\Omega)$ is obtained by completion of smooth functions with compact support in Ω under the norm

$$\|v\|^2 = \int_\Omega |\nabla v|^2 + \gamma^2 v^2 \, d\mu$$

We remark that as a function space, $H_a(\Omega)$ is independent of a. We keep the index a however, to indicate which inner product is used.

If u is solution of (1.2) then $v = \Upsilon_a^{-1} u$ satisfies the equation

$$\begin{gathered} -\Delta v + \gamma^2 v = v^{p-1} + \lambda e^{-ct} v, \\ v > 0 \quad \text{in } \Omega, \quad v \in H_a(\Omega) \end{gathered} \tag{2.1}$$

One can check that for any $v \in H_a(\Omega)$ and $\Upsilon(v) = u \in \mathcal{D}_a(B)$, we have

$$E(u) = F(v) := \frac{\displaystyle\int_\Omega |\nabla v|^2 + (\gamma^2 - \lambda e^{-ct}) v^2 \, d\mu}{\Big(\displaystyle\int_\Omega |v|^p \, d\mu\Big)^{2/p}};$$

therefore

$$J = \inf_{u \in \mathcal{D}_a(B)\setminus\{0\}} E(u) = \inf_{v \in H_a(\Omega)\setminus\{0\}} F(v).$$

It is known (see [7]) that on the whole cylinder $\mathcal{C}$, the functional

$$G(u) = \frac{\int_{\mathcal{C}} |\nabla u|^2 + \gamma^2 u^2 \, d\mu}{\big(\int_{\mathcal{C}} |u|^p \, d\mu\big)^{2/p}}$$

has a positive infimum S achieved by a positive function $U \in H^1(\mathcal{C})$ which satisfies

$$-\Delta U + \gamma^2 U = U^{p-1}, \tag{2.2}$$

and so,

$$S = S(N,a,b) = \inf_{u \in H^1(\mathcal{C})} G(u) = G(U) = \Big(\int_{\mathcal{C}} U^p \, d\mu \Big)^{(p-2)/p}$$

In [7] we also discussed the symmetry of U. While for some values of the parameters, S is achieved by the radial solutions

$$U(t) = \Big(\frac{\gamma^2 p}{2} \Big)^{\frac{1}{p-2}} \Big(\cosh \big(\frac{p-2}{2} \gamma t \big) \Big)^{-\frac{2}{p-2}},$$

for other values we proved that the ground state is not radial anymore (see [13] for an improvement of the range obtained in [7]). We have however that eventually after a translation we can assume that $U(t,\theta) = U(-t,\theta)$ for any $t \in \mathbb{R}$ and $\theta \in \mathbb{S}^{N-1}$.

We shall prove Theorem 1.4 by showing the corresponding result for the equivalent problem (2.1). That is, we prove the following result.

Theorem 2.1. *Assume* $0 < b-a < 1$, $0 < c < 2\gamma$, *and* $\gamma^2 > \frac{N-1}{p-2}$. *Then for any*

$$0 < \lambda < \Big(\frac{c}{2} Z \Big(\frac{2}{c} \sqrt{\gamma^2 - \frac{N-1}{p-2}} \Big) \Big)^2$$

problem (2.1) *has both, a radial and a nonradial solution.*

3. Existence and Multiplicity

We employ Lemma 1 in [2] to show that $J = \inf F$ is achieved. Then we prove the symmetry breaking part, from which Theorem 2.1 follows. Lemma 1 in [2] translates to our situation step by step to give the following lemma.

Lemma 3.1. *If* $J < S$ *then* J *is achieved.*

We shall also need the following decay result.

Lemma 3.2. *For any positive solution* U *of* (2.2) *which is in* $H^1(\mathcal{C})$, *there exists a constant* $C > 0$ *such that* $\frac{1}{C} e^{-\gamma|t|} \le U(t,\theta) \le C e^{-\gamma|t|}$ *for all* $t \in \mathbb{R}$ *and* $\theta \in \mathbb{S}^{N-1}$.

Proof. Let U be a positive solution of (2.2), which without loss of generality we can assume even in t, i.e. $U(-t,\theta) = U(t,\theta)$. For any $t \in (-\infty,\infty)$, let

$$f(t) = \int_{\mathbb{S}^{N-1}} U(t,\theta) \, d\theta, \quad \text{and} \quad h(t) = \int_{\mathbb{S}^{N-1}} U^{p-1}(t,\theta) \, d\theta.$$

Making the necessary modifications to Moser's proof of Harnack inequality [21], one can prove that there is a positive constant C_0 (depending on U) such that

$$\frac{1}{C_0} < \frac{U(t,\theta)}{f(t)} < C_0 \quad \text{for all } (t,\theta) \in \mathcal{C}. \tag{3.1}$$

This can be done by adapting the proof of Theorem 8.20 in [17] to the cylinder, or more directly one can follow the last section in [20]. We justify (3.1) by the fact that U satisfies

$$\Delta U \ge -qU,$$

with $q = U^{p-2} \in L^\infty(\mathcal{C})$, and

$$\Delta U \le \gamma^2 U.$$

One can then apply Lemma 11.1 and Lemma 11.2 in [20] to conclude Harnack's inequality in our setting.

Integrating (2.2) on $\mathbb{S}^{N-1}$ with respect to the variable θ, we get that the function f is a positive solution in $H^1(\mathbb{R})$ for the ODE

$$-f_{tt} + \gamma^2 f = h$$

which can be rewritten as

$$-\frac{d}{dt}\Big(e^{-2\gamma t}\frac{d}{dt}(e^{\gamma t}f(t))\Big) = e^{-\gamma t}h(t).$$

It follows that

$$\frac{d}{dt}(e^{\gamma t}f(t)) = e^{2\gamma t}\int_t^\infty e^{-\gamma s}h(s)\,ds.$$

Hence

$$e^{\gamma t}f(t) = \int_{-\infty}^t e^{2\gamma r}\int_r^\infty e^{-\gamma s}h(s)\,ds\,dr.$$

Since $e^{\gamma t}f(t)$ is increasing, there exists $C_1 = \frac{1}{f(0)} > 0$ such that $\frac{1}{C_1}e^{-\gamma t} \le f(t)$ for all $t \ge 0$. We now prove that

$$C_2 = \int_{-\infty}^\infty e^{2\gamma r}\int_r^\infty e^{-\gamma s}h(s)\,ds\,dr < \infty; \tag{3.2}$$

therefore

$$e^{\gamma t}f(t) \le C_2, \quad \text{hence} \quad f(t) \le C_2 e^{-\gamma t}.$$

Multiply (2.2) by $U^{p-2}(t,\theta)$ to get

$$-\frac{\Delta U^{p-1}}{p-1} + (p-2)U^{p-3}|\nabla U|^2 + \gamma^2 U^{p-1} = U^{2p-3}.$$

Hence

$$-\Delta U^{p-1} + (p-1)(\gamma^2 - U^{p-2})U^{p-1} \le 0. \tag{3.3}$$

From (3.1) we have that for any $\varepsilon > 0$, there exists t_0 sufficiently large, such that if $t \ge t_0$ then $U^{p-2}(t,\theta) < \varepsilon$. Let

$$0 < \varepsilon < \gamma^2\frac{p-2}{p-1} \quad \text{and so} \quad \gamma < \alpha = \sqrt{(p-1)(\gamma^2-\varepsilon)}.$$

Integrate on $\mathbb{S}^{N-1}$ in (3.3) to obtain, for $t \ge t_0$, $-h_{tt}(t) + \alpha^2 h(t) < 0$; i.e.,

$$-\frac{d}{dt}\big(e^{-2\alpha t}\frac{d}{dt}(e^{\alpha t}h(t))\big) < 0,$$

hence $e^{-2\alpha t}\frac{d}{dt}(e^{\alpha t}h(t))$ is increasing for $t \ge t_0$. If $\frac{d}{dt}(e^{\alpha t}h(t)) = A > 0$ for some $t = t_1 \ge t_0$, we obtain

$$\frac{d}{dt}(e^{\alpha t}h(t)) > Ae^{2\alpha t}, \quad \text{i.e.} \quad h(t) > \frac{A}{2\alpha}e^{\alpha t} + Be^{-\alpha t},$$

for all $t > t_1$. But this contradicts the fact that h is bounded. Therefore, $\frac{d}{dt}(e^{\alpha t}h(t)) \le 0$, and so $e^{\alpha t}h(t)$ is non-increasing. Letting $A = e^{\alpha t_0}h(t_0)$, we obtain $h(t) \le Ae^{-\alpha t}$ for all $t \ge t_0$. Since $\gamma < \alpha$ we have that

$$\int_{-\infty}^\infty e^{\gamma s}h(s)\,ds = B < \infty, \quad \text{i.e.} \quad \int_{-\infty}^\infty e^{-\gamma s}h(s)\int_{-\infty}^s e^{2\gamma r}\,dr\,ds = \frac{B}{2\gamma} = C_2 < \infty.$$

Reversing the order of integration, we obtain (3.2). The lemma now follows from (3.1) by taking $C = \max\{C_0C_1, C_0C_2\}$. □

Lemma 3.3. *For any $0 < b - a < 1$, $0 < c < 2\gamma$, and $\lambda > 0$, we have $J < S$.*

Proof. We construct test functions v such that $J \le F(v) < S$. Let $\varphi \in C_0^\infty(\Omega)$ be a cut-off function, with $0 \le \varphi(t,\theta) \le 1$, $\varphi(t,\theta) \equiv 1$ for $t \ge 1$, and $|\nabla\varphi(t,\theta)| \le 2$ for all $(t,\theta) \in \Omega$. For $\tau > 0$, let $U_\tau(t,\theta) = U(t-\tau,\theta)$ and take $v_\tau = \varphi U_\tau$. Then

$$\begin{aligned} F(v_\tau) &= \frac{\int_\Omega U_\tau^2|\nabla\varphi|^2 + \nabla U_\tau \cdot \nabla(U_\tau\varphi^2) + \gamma^2 U_\tau^2\varphi^2 - \lambda e^{-ct}U_\tau^2\varphi^2}{\Big(\int_\Omega U_\tau^p\varphi^p\Big)^{2/p}} \\ &= \frac{\int_\Omega U_\tau^2(|\nabla\varphi|^2 - \lambda e^{-ct}\varphi^2) + U_\tau\varphi^2(-\Delta U_\tau + \gamma^2 U_\tau)}{\Big(\int_\Omega U_\tau^p\varphi^p\Big)^{2/p}} \\ &= \frac{\int_\Omega U_\tau^2(|\nabla\varphi|^2 - \lambda e^{-ct}\varphi^2) + U_\tau^p\varphi^2}{\Big(\int_\Omega U_\tau^p\varphi^p\Big)^{2/p}}\,. \end{aligned}$$

We use the following estimates

$$\begin{aligned} &\int_\Omega U_\tau^2(|\nabla\varphi|^2 - \lambda e^{-ct}\varphi^2) + U_\tau^p\varphi^2 \\ &\le \|U_\tau\|_p^p - \int_{\mathcal{C}\setminus\Omega} U_\tau^p + 4\int_{[0,1]\times\mathbb{S}^{N-1}} U_\tau^2 - \int_{[1,\infty)\times\mathbb{S}^{N-1}} \lambda e^{-ct}U_\tau^2 \\ &\le \|U_\tau\|_p^p - C_1\exp(-p\gamma\tau) + C_2\exp(-2\gamma\tau) - C_3\exp(-c\tau) \end{aligned}$$

and

$$\int_\Omega U_\tau^p\varphi^p \ge \|U_\tau\|_p^p - \int_{(-\infty,1]\times\mathbb{S}^{N-1}} U_\tau^p \ge \|U_\tau\|_p^p - C_4\exp(-p\gamma\tau),$$

with C_1, C_2, C_3, and C_4 positive constants independent of τ. Therefore, if τ is sufficiently large, and since $c < 2\gamma$ we have

$$J \le F(v_\tau) < \frac{\|U_\tau\|_p^p}{(\|U_\tau\|_p^p)^{2/p}} = S\,.$$

$\square$

Under conditions (1.3) and $c < 2\gamma$, from Lemma 3.1 and Lemma 3.3 we have that for $\lambda > 0$, J is achieved. If we also assume $\lambda < \lambda_1(N,a,c) = (\frac{c}{2}Z(\frac{2\gamma}{c}))^2$, then J is positive and after an eventual multiplication by a constant, the function that achieves J solves (2.1).

The following lemma will prove that for certain values of γ and λ, the least energy solutions obtained above are nonradial. The argument is similar to that of [26], or Theorem 3, part c) in [19].

Lemma 3.4. *If the least energy solution is $V = V(t)$ (independent of θ), then*

$$0 \ge \int_0^\infty V_t^2 + \Big(-\frac{N-1}{p-2} + \gamma^2 - \lambda e^{-ct}\Big)V^2$$

Proof. For some $h \in H_a(\Omega)$, let $f(s) = F(V + sh)$. Since V is a local minimum of F, it follows that $f''(0) \geq 0$. This implies

$$\begin{aligned} &2\Big(\int_\Omega |\nabla h|^2 + (\gamma^2 - \lambda e^{-ct})h^2\Big)\Big(\int_\Omega V^p\Big)^{2/p} \\ &\geq 2\Big(\int_\Omega |\nabla V|^2 + (\gamma^2 - \lambda e^{-ct})V^2\Big) \\ &\Big[(p-1)\Big(\int_\Omega V^p\Big)^{\frac{2}{p}-1}\Big(\int_\Omega V^{p-2}h^2\Big) - (p-2)\Big(\int_\Omega V^p\Big)^{\frac{2}{p}-2}\Big(\int_\Omega V^{p-1}h\Big)^2\Big] \end{aligned} \tag{3.4}$$

By taking $h(t,\theta) = V(t)Y(\theta)$, with $Y(\theta)$ a first harmonic, from

$$-\Delta_{\mathbb{S}^{N-1}} Y = (N-1)Y, \quad \text{and} \quad \int_{\mathbb{S}^{N-1}} Y(\theta)\, d\theta = 0$$

we get that (3.4) simplifies to

$$\int_0^\infty V_t^2 + (N - 1 + \gamma^2 - \lambda e^{-ct})V^2 \geq (p-1)\int_0^\infty V_t^2 + (\gamma^2 - \lambda e^{-ct})V^2$$

from which the conclusion of the lemma follows. □

For $\gamma^2 > \frac{N-1}{p-2}$ and $0 < \lambda < \tilde{\lambda_1}(N,a,b,c) = (\frac{c}{2}Z(\frac{2}{c}\sqrt{\gamma^2 - \frac{N-1}{p-2}}))^2$ we have that

$$\int_0^\infty w_t^2 + (-\frac{N-1}{p-2} + \gamma^2 - \lambda e^{-ct})w^2 > 0,$$

for any radial $w = w(t) \in H_a(\Omega)$. Therefore in this range, and assuming $0 < c < 2\gamma$, we have that J is achieved by a function which according to Lemma 3.4 cannot be radial. We therefore conclude the proof of Theorem 2.1.

4. Concluding Remarks

For the reader's convenience, and to put things in perspective, we present the situation of radial solutions of (1.2). Recall that $Z(\nu)$ denotes the first positive zero of the Bessel function of the first kind J_ν, and

$$\gamma = \gamma(N,a) = \frac{N-2-2a}{2}.$$

The results in [22, 9, 6], imply the following theorem.

Theorem 4.1. *Let* $\lambda_1 = \lambda_1(N,a,c) = (\frac{c}{2}Z(\frac{2\gamma}{c}))^2$ *and*

$$\mu_1 = \mu_1(N,a,c) = (\frac{c}{2}Z(-\frac{2\gamma}{c}))^2.$$

Then problem (1.2) *has a radial solution if and only if*

(a) $0 < c \leq 2\gamma$ *and* $\lambda \in (0, \lambda_1)$*; or*
(b) $c > 2\gamma$ *and* $\lambda \in (\mu_1, \lambda_1)$*.*

Note that when looking only for radial solutions of (1.2), one can assume a slightly larger range for b, and the variational setting works the same way (see [6]) for

$$-\gamma = a - \frac{N-2}{2} < b < a + 1.$$

In closing we make several remarks. We believe that the existence of nonradial solutions is due to symmetry breaking, where the least energy solutions bifurcate from the radial solutions of (2.1) as $b - a$ is fixed and γ increases. It would be interesting to find the exact values $\lambda = \lambda(N, a, b, c)$ where bifurcation occurs. It is not difficult to see that in this range of the parameters one can construct multi-bump solutions (similar to [8]), and hence to conclude that the number of essentially distinct solutions of (2.1) tends to infinity as γ increases to infinity. For questions of a similar nature on a related problem in $\mathbb{S}^3$ (but qualitatively different solutions) see [5].

It remains an interesting problem to investigate what happens in the "critical" case $b = a$ (therefore $p = 2^*$) for $N \geq 3$, and the situation when $c \geq 2\gamma$. We emphasize the fact that the nonexistence results for $c \geq 2\gamma$ stated in Theorem 4.1 refer only to radial solutions of (1.2). One would like to either have an existence theorem, or to be able to extend the nonexistence results to nonradial functions.

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