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OPTIMAL REGULARIZATION METHOD FOR ILL-POSED CAUCHY PROBLEMS

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Abstract. The goal of this paper is to give an optimal regularization method for an ill-posed Cauchy problem associated with an unbounded linear operator in a Hilbert space. Key point to our proof is the use of Yosida approximation and nonlocal conditions to construct a family of regularizing operators for the considered problem. We show the convergence of this approach, and we estimate the convergence rate under a priori regularity assumptions on the problem data.

1. Introduction and motivation

Throughout this paper H will denote a Hilbert space, endowed with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, $\mathcal{L}(H)$ denotes the Banach algebra of bounded linear operators on H.

Consider the backward Cauchy problem

$$u'(t) + Au(t) = 0, \quad 0 < t < T, \quad u(T) = \varphi, \tag{1.1}$$

where A is a positive ($A \geq \gamma > 0$) self-adjoint ($A = A^*$), unbounded linear operator on H, and $\varphi \in H$.

> The problem is to determine $u(t)$ for $0 \leq t < T$ from the knowledge of the final value $u(T) = \varphi$.

Such problems are not well-posed in the Hadamard sense [18], that is, even if a unique solution exists on $[0, T]$ it need not depend continuously on the final value φ.

Physically, problems of this nature arise in different contexts. Beyond their interest in connection with standard diffusion problems [15] (then A is usually the Laplace operator $-\Delta$), they also appear, for instance, in some deconvolution problem, such as deblurring processes [7] (A is often a fractional power of $-\Delta$), material sciences [35], hydrology [19, 38] and also in many other practical applications of mathematical physics and engineering sciences.

In the mathematical literature various methods have been proposed for solving backward Cauchy problems. We can notably mention the method of quasi-solution (Q.S.-method) of Tikhonov [39], the method of quasi-reversibility (Q.R.-method)

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of Lattès and Lions [25], the method of logarithmic convexity [1, 8, 23, 26, 31], the it iterative procedures of Kozlov and Maz'ya [5, 24], the quasi boundary value method (Q.B.V.-method) [9, 13, 22, 37] and the C-regularized semigroups technique [3, 10, 12, 27, 28, 34].

In the method of quasi-reversibility, the main idea consists in replacing A in (1.1) by $A_\alpha = g_\alpha(A)$. In the original method [25] Lattès and Lions have proposed $g_\alpha(A) = A - \alpha A^2$, to obtain a well-posed problem in the backward direction. Then, using the information from the solution of the perturbed problem and solving the original problem, we get another well-posed problem and this solution sometimes can be taken to be the approximate solution of the ill-posed problem (1.1).

Difficulties may arise when using the method of quasi-reversibility discussed above. The essential difficulty is that the order of the operator is replaced by an operator of second order, which produces serious difficulties on the numerical implementation, in addition, the error ($e(\alpha)$) introduced by small change in the final value φ is of the order $e^{\frac{T}{4\alpha}}$.

In the Gajewski and Zaccharias quasi-reversibility method [17] (see also [6, 14, 20, 30, 36], $g_\alpha(A) = A(I+\alpha A)^{-1}$. The advantage of this perturbation lies in the fact that this perturbation is bounded ($A_\alpha \in \mathcal{L}(H)$), which gives a well-posedness in the forward and backward direction for the perturbed problem, the second advantage is that, this perturbation produces a best and significant approximate solution by comparison with the method proposed by Lattès and Lions. But the amplification factor of the error resulting from the approximated problem, remains always of the order $e^{\frac{T}{\alpha}}$.

In the method developed by G.W. Clark and S.F. Oppenheimer [9] (see also [13, 22, 37], they approximate problem (1.1) by

$$\begin{gathered} v_t(t) + Av(t) = 0, \quad 0 < t < T, \\ \beta v(0) + v(T) = \varphi, \end{gathered}$$

where $\beta > 0$. This method is called quasi-boundary value method (Q.B.V.-method). We note here that this method gives a better approximation than many other quasi-reversibility type methods and the error ($e(\beta)$) introduced by small change in the final value φ is of the order $\frac{1}{\beta}$.

In this paper, We combine the nice smoothing effect of Yosida approximation with advantages of quasi-boundary value method, to build an optimal approximation to problem (1.1).

2. Preliminaries and basic results

In this section we present the notation and the functional setting which will be used in this paper and prepare some material which will be used in our analysis.

If $B \in \mathcal{L}(H)$ we denote by $\mathcal{N}(B)$ the kernel of B and by $\mathcal{R}(B)$ the range of B. We denote by $\{E_\lambda,\ \lambda \geq \gamma > 0\}$ the spectral resolution of the identity associated to A.

We denote by $S(t) = e^{-tA} = \int_\gamma^\infty e^{-t\lambda}\, dE_\lambda \in \mathcal{L}(H)$, $t \geq 0$, the C_0-semigroup generated by $-A$. Some basic properties of are listed in the following theorem.

Theorem 2.1 ([33, Chap. 2, theorem 6.13]). *For this family of operators we have:*

1. $\|S(t)\| \leq 1$, *for all* $t \geq 0$;
2. *the function* $t \longmapsto S(t)$, $t > 0$, *is analytic;*

(3) *for every real* $r \geq 0$ *and* $t > 0$*, the operator* $S(t) \in \mathcal{L}(H, \mathcal{D}(A^r))$*;*
(4) *for every integer* $k \geq 0$ *and* $t > 0$*,* $\|S^{(k)}(t)\| = \|A^k S(t)\| \leq c(k)t^{-k}$*;*
(5) *for every* $x \in \mathcal{D}(A^r)$*,* $r \geq 0$ *we have* $S(t)A^r x = A^r S(t)x$*.*

Definition 2.2. We put

$$J_\alpha = (I + \alpha A)^{-1}, \quad A_\alpha = A(I + \alpha A)^{-1} = \frac{1}{\alpha}(I - J_\alpha), \quad \alpha > 0,$$

and call A_α the Yosida approximation of A.

Some basic properties of A_α are listed in the following theorem.

Theorem 2.3 ([4, chap. VII, p. 101-118])**.** *We have*
(1) A_α *is positive and self-adjoint;*
(2) $J_\alpha A h = A J_\alpha h$*, for all* $h \in \mathcal{D}(A)$*;*
(3) $J_\alpha,\ A_\alpha \in \mathcal{L}(H)$*,* $\|J_\alpha\| \leq 1$*, for all* $\alpha > 0$*;*
(4) $\|A_\alpha h\| \leq \|Ah\|$*, for all* $\alpha > 0$*, for all* $h \in \mathcal{D}(A)$*;*
(5) *for all* $h \in H$*,* $\lim_{\alpha \to 0} J_\alpha h = h$*;*
(6) *for all* $h \in \mathcal{D}(A)$*,* $\lim_{\alpha \to 0} A_\alpha h = Ah$*;*
(7) *for all* $h \in H$*, for all* $t \geq 0$*,* $\lim_{\alpha \to 0} S_\alpha(t)h = \lim_{\alpha \to 0} e^{-tA_\alpha} h = S(t)h = e^{-tA}h$*.*

Theorem 2.4. *For* $t > 0$*,* $S(t)$ *is self-adjoint and one to one operator with dense range* ($S(t) = S(t)^*$*,* $\mathcal{N}(S(t)) = \{0\}$ *and* $\overline{\mathcal{R}(S(t))} = H$*).*

Proof. A is self-adjoint and since $S(t)^* = (e^{-tA})^* = e^{-tA^*} = e^{-tA}$, then we have $S(t)^* = S(t)$.

Let $h \in \mathcal{N}(S(t_0))$, $t_0 > 0$, then $S(t_0)h = 0$, which implies that $S(t)S(t_0)h = S(t + t_0)h = 0$, $t \geq 0$. Using analyticity, one obtains that $S(t)h = 0$, $t \geq 0$. Strong continuity at 0 now gives $h = 0$. This shows that $\mathcal{N}(S(t_0)) = \{0\}$. Thanks to

$$\overline{\mathcal{R}(S(t_0))} = \mathcal{N}(S(t_0))^\perp = \{0\}^\perp = H,$$

we conclude that $\mathcal{R}(S(t_0))$ is dense in H. □

For more details, we refer the reader to a general version of theorem 2.4 in the case of analytic semigroups in Banach spaces (Lemma 2.2, [11]).

Remark 2.5 (Smoothing effect and irreversibility)**.** Thanks to Theorem 2.1 and Theorem 2.4, we observe that the solution of the direct Cauchy problem

$$u'(t) + Au(t) = 0, \ \ 0 < t \leq T, \quad u(0) = u_0,$$

has the following smoothing effect: admitting the initial value $u(0)$ to belong only to H, then for all $t > 0$,

$$\mathcal{R}(S(t)) \subset \mathcal{C}^\infty(A) \overset{def}{=} \cap_{n=1}^{\infty} \mathcal{D}(A^n)$$

(space more regular than H) (see [16]). It follows that for the final value problem (1.1) to have a solution, we should have $u(T) \in \mathcal{C}(A) \subseteq \mathcal{R}(S(T))$, where $\mathcal{C}(A)$ is an admissible class for which the (1.1) is solvable. This shows that the (1.1) is irreversible in the sense:

$$S(T - t) : H \to \mathcal{R}(S(T - t)) \subset \mathcal{C}^\infty(A) \subsetneq H, \quad 0 \leq t < T,$$

and $\mathcal{R}(S(T - t)) \neq \overline{\mathcal{R}(S(T - t))}$, in other words $S(T - t)^{-1} = S(t - T) \notin \mathcal{L}(H)$.

For notational convenience and simplicity, we denote

$$\mathcal{C}_\theta(A) = \{h \in H:\ \|h\|^2_{\mathcal{C}_\theta} = \|e^{\theta TA}h\|^2 = \int_\gamma^{+\infty} e^{2T\theta\lambda}\, d\|E_\lambda h\|^2 < +\infty\}, \quad \theta \geq 0.$$

By the definition of $\mathcal{C}_\theta(A)$ we have the following topological inclusions:

$$\begin{gathered}
\mathcal{C}_{\theta_2}(A) \subseteq \mathcal{C}_{\theta_1}(A), \quad \theta_2 \geq \theta_1, \\
\mathcal{C}_\theta(A) \subset \mathcal{D}(A^r) \subset H, \quad \theta > 0,\ r > 0, \\
\|A^r h\|^2 = \int_\gamma^{+\infty} \big(\frac{\lambda^r}{e^{\theta T\lambda}}\big)^2 e^{2\theta T\lambda}\, d\|E_\lambda h\|^2 \leq c(\theta, r, T)\|h\|^2_{C_\theta}
\end{gathered}$$

where $c(\theta, r, T) = (\frac{\theta T}{r})^{2\theta T} e^{-2r}$.

For $\lambda \geq \gamma$, we introduce the functions:

$$H_\sigma(\lambda) = F_\sigma(\lambda) + G_\sigma(\lambda),$$

where

$$\begin{gathered}
F_\sigma(\lambda) = \frac{\beta}{\beta + e^{-\frac{T\lambda}{1+\alpha\lambda}}}, \quad G_\sigma(\lambda) = \frac{e^{-\frac{T\lambda}{1+\alpha\lambda}} - e^{-T\lambda}}{\beta + e^{-\frac{T\lambda}{1+\alpha\lambda}}}, \\
F_{\sigma,\theta}(\lambda) = F_\sigma(\lambda)e^{-T\theta\lambda}, \quad G_{\sigma,\theta}(\lambda) = G_\sigma(\lambda)e^{-T\theta\lambda}, \quad \theta > 0, \\
K_\beta(\lambda) = \frac{\beta}{\beta + e^{-T\lambda}}, \quad M_\theta(\lambda) = \lambda^2 e^{-\theta T\lambda}, \quad \theta > 0, \\
F_{\sigma_1,\sigma_2}(\lambda) = \frac{|\beta_1 - \beta_2|}{\big(\beta_1 + e^{-\frac{T\lambda}{1+\alpha_1\lambda}}\big)\big(\beta_2 + e^{-\frac{T\lambda}{1+\alpha_2\lambda}}\big)}, \\
G_{\sigma_1,\sigma_2}(\lambda) = \frac{|e^{-\frac{T\lambda}{1+\alpha_1\lambda}} - e^{-\frac{T\lambda}{1+\alpha_2\lambda}}|}{\big(\beta_1 + e^{-\frac{T\lambda}{1+\alpha_1\lambda}}\big)\big(\beta_2 + e^{-\frac{T\lambda}{1+\alpha_2\lambda}}\big)}.
\end{gathered}$$

By simple differential calculus and elementary estimates, we show that

$$0 < F_\sigma(\lambda) \leq 1, \quad F_\sigma(\lambda) \leq \frac{\beta}{\beta + e^{\frac{-T}{\alpha}}}, \quad F_\sigma(\lambda) \leq \beta e^{T\lambda}. \tag{2.1}$$

$$0 < G_\sigma(\lambda) \leq 1,$$

$$G_\sigma(\lambda) = \frac{e^{-\frac{T\lambda}{1+\alpha\lambda}}\big(1 - e^{\frac{-\alpha T\lambda^2}{1+\alpha\lambda}}\big)}{\beta + e^{-\frac{T\lambda}{1+\alpha\lambda}}} \leq \big(1 - e^{\frac{-\alpha T\lambda^2}{1+\alpha\lambda}}\big) \leq \frac{\alpha T\lambda^2}{1+\alpha\lambda} \leq \alpha T\lambda^2. \tag{2.2}$$

$$M_{\theta,\infty}(\lambda) = \sup_{\lambda\geq\gamma} M_\theta(\lambda) = \big(\frac{2}{\theta Te}\big)^2 \leq \big(\frac{1}{\theta T}\big)^2. \tag{2.3}$$

$$F_{\sigma,\theta,\infty} = \sup_{\lambda\geq\gamma} F_{\sigma,\theta}(\lambda) \leq K_{\beta,\infty} = \sup_{\lambda\geq\gamma} K_\beta(\lambda) \leq \begin{cases} \beta, & \text{if } \theta \geq 1, \\ c_1(\theta)\beta^\theta, & \text{if } 0 < \theta < 1, \end{cases} \tag{2.4}$$

where $c_1(\theta) = (1-\theta)^{1-\theta}\theta^\theta \leq 1$.

$$G_{\sigma,\theta,\infty} = \sup_{\lambda\geq\gamma} G_{\sigma,\theta}(\lambda) \leq c_2(\theta, T)\frac{\alpha}{1+\beta} \leq c_2(\theta, T)\alpha, \tag{2.5}$$

where $c_2(\theta,T)=\frac{1}{T\theta^2}$.

$$F_{\sigma_1,\sigma_2}(\lambda)\le e^{T\lambda},\quad F_{\sigma_1,\sigma_2}(\lambda)\le|\beta_1-\beta_2|e^{2T\lambda}. \tag{2.6}$$

$$G_{\sigma_1,\sigma_2}(\lambda)\le e^{T\lambda},\quad F_{\sigma_1,\sigma_2}(\lambda)\le|\alpha_1-\alpha_2|T\lambda^2e^{T\lambda}. \tag{2.7}$$

$$F_\sigma(\lambda)\lambda^{-1}=\frac{\beta}{\beta\lambda+\lambda e^{\frac{-T\lambda}{1+\alpha\lambda}}}\le\frac{\beta}{\beta\lambda+\gamma e^{-T\lambda}}\le\frac{T}{1+\ln(\frac{\gamma T}{\beta})},\quad(\beta\le\gamma T). \tag{2.8}$$

Without loss of the generality, we suppose that $\lambda\ge\gamma\ge1$. By virtue of $(1-e^{-\tau}\le\sqrt{\tau},\ \tau\ge1)$, the function $G_\sigma(\lambda)$ can be estimated as follows:

$$G_\sigma(\lambda)=\frac{e^{-\frac{T\lambda}{1+\alpha\lambda}}-e^{-T\lambda}}{\beta+e^{-\frac{T\lambda}{1+\alpha\lambda}}}=\frac{e^{-\frac{T\lambda}{1+\alpha\lambda}}(1-e^{\frac{-T\alpha\lambda^2}{1+\alpha\lambda}})}{\beta+e^{-\frac{T\lambda}{1+\alpha\lambda}}}\le1-e^{\frac{-T\alpha\lambda^2}{1+\alpha\lambda}}\le\sqrt{T\alpha}\lambda. \tag{2.9}$$

Remark 2.6. Let u be a solution to the problem

$$u_t+Au=0,\quad 0<t<T,\quad u(T)=\varphi. \tag{2.10}$$

We set $U(t)=e^{-\nu t}u(t)$, $\nu\ge1$, then U is a solution of the problem

$$U_t+A_\nu U=0,\ \ 0<t<T,\quad U(T)=e^{-\nu T}\varphi=\psi, \tag{2.11}$$

with $A_\nu=A+\nu I\ge(\nu+\gamma)I\ge\nu I$. Hence, regularizing (2.10) is equivalent to regularize (2.11).

Remark 2.7. The operational calculus for a self-adjoint operator and estimates (2.1)–(2.9) play the key role in our analysis and calculations.

3. The approximate problem

Description of the method. Step 1 Let v_σ be the solution of the perturbed problem

$$\begin{gathered} v_\sigma'(t)+A_\alpha v_\sigma(t)=0,\quad 0<t<T,\\ \beta v_\sigma(0)+v_\sigma(T)=\varphi \end{gathered} \tag{3.1}$$

where the operator A is replaced by $A_\alpha=A(I+\alpha A)^{-1}$ and the final condition $v(T)=\varphi$ is replaced by the nonlocal condition $\beta v(0)+v(T)=\varphi$, where $\alpha>0$, $\beta>0$ and $\sigma=(\alpha,\beta)$.

Step 2 We use the initial value

$$\varphi_\sigma=v_\sigma(0)=\big(\beta+S_\alpha(T)\big)^{-1}\varphi,$$

in the problem

$$u_\sigma'(t)+Au_\sigma(t)=0,\quad 0<t\le T,\quad u_\sigma(0)=\varphi_\sigma. \tag{3.2}$$

Step 3 We show that

$$\begin{gathered}\|u_\sigma(T)-\varphi\|\to0,\quad\text{as }|\sigma|\to0,\\ \|u_\sigma(0)-u(0)\|\to0,\quad\text{as }|\sigma|\to0,\\ \sup_{0\le t\le T}\|u_\sigma(t)-u(t)\|\to0,\quad\text{as }|\sigma|\to0.\end{gathered}$$

4. Analysis of the method and main results

Now we are ready to state and prove the main results of this paper.

Definition 4.1 ([36])**.** A solution of (1.1) on the interval $[0,T]$ is a function $u \in \mathcal{C}([0,T];H) \cap \mathcal{C}^1((0,T);H)$ such that for all $t \in (0,T)$, $u(t) \in \mathcal{D}(A)$ and (1.1) holds.

It is useful to know exactly the admissible set for which (1.1) has a solution. The following lemma gives an answer to this question.

Lemma 4.2 ([9, Lemma 1])**.** *Problem* (1.1) *has a solution if and only if* $\varphi \in \mathcal{C}_1(A)$, *and its unique solution is represented by*

$$u(t) = e^{(T-t)A}\varphi. \tag{4.1}$$

Using semi-groups theory and the properties of $S_\alpha(t)$, we have the following theorem.

Theorem 4.3. *For all* $\varphi \in H$, *the function*

$$v_\sigma(t) = S_\alpha(t)\big(\beta + S_\alpha(T)\big)^{-1}\varphi$$

is the unique solution of (3.1) *and it depends continuously on* φ.

Proof. We consider the problem

$$y'_\sigma(t) + A_\alpha y_\sigma(t) = 0, \quad 0 < t \le T, \quad y_\sigma(0) = \big(\beta + S_\alpha(T)\big)^{-1}\varphi. \tag{4.2}$$

This problem is well-posed, and its solution is

$$y_\sigma(t) = S_\alpha(t)\big(\beta + S_\alpha(T)\big)^{-1}\varphi. \tag{4.3}$$

Observing that

$$\beta y_\sigma(0) + y_\sigma(T) = \big(\beta + S_\alpha(T)\big)\big(\beta + S_\alpha(T)\big)^{-1}\varphi = \varphi. \tag{4.4}$$

Thanks to (4.4) and the uniqueness of solution to direct problem (4.2), we deduce that the problem (3.1) admits the unique solution v_σ given by (4.3). To show the continuous dependence of v_σ on φ, we compute

$$\|v_\sigma(t)\| = \|S_\alpha(t)\big(\beta + S_\alpha(T)\big)^{-1}\varphi\| \le \|\big(\beta + S_\alpha(T)\big)^{-1}\varphi\| \le (\beta + e^{\frac{-T}{\alpha}})^{-1}\|\varphi\|.$$

∎

Theorem 4.4. *The problem* (3.2) *is well-posed, and its solution is*

$$u_\sigma(t) = S(t)\varphi_\alpha = S(t)\big(\beta + S_\alpha(T)\big)^{-1}\varphi. \tag{4.5}$$

An easy computation shows that

$$\|u_\sigma(t)\| \le \big(\frac{1}{\beta + e^{-\frac{T}{\alpha}}}\big)^{\frac{T-t}{T}}\|\varphi\|. \tag{4.6}$$

Theorem 4.5. *For all* $\varphi \in H$, $\|u_\sigma(T) - \varphi\| \to 0$, *as* $|\sigma| \to 0$.

Proof. We compute

$$\|u_\sigma(T) - \varphi\|^2 = \int_\gamma^{+\infty} H_\alpha(\lambda)^2\, d\|E_\lambda\varphi\|^2 \le 2(I_{1,\sigma} + I_{2,\sigma}), \tag{4.7}$$

where

$$I_{1,\sigma} = \int_\gamma^{+\infty} F_\alpha(\lambda)^2\, d\|E_\lambda\varphi\|^2,$$
$$I_{2,\sigma} = \int_\gamma^{+\infty} G_\alpha(\lambda)^2\, d\|E_\lambda\varphi\|^2.$$

Fix $\varepsilon > 0$. Choose N so that $\int_N^{+\infty} d\|E_\lambda\varphi\|^2 < \frac{\varepsilon}{8}$. Thus

$$I_{1,\sigma} \le \int_\gamma^{N} F_\sigma(\lambda)^2\, d\|E_\lambda\varphi\|^2 + \int_N^{+\infty} F_\sigma(\lambda)^2\, d\|E_\lambda\varphi\|^2,$$
$$I_{2,\sigma} \le \int_\gamma^{N} G_\sigma(\lambda)^2\, d\|E_\lambda\varphi\|^2 + \int_N^{+\infty} G_\sigma(\lambda)^2\, d\|E_\lambda\varphi\|^2.$$

Using inequalities (2.1) and (2.2), we derive

$$I_{1,\sigma} \le \frac{\varepsilon}{8} + \beta^2 e^{2TN}\|\varphi\|^2,$$
$$I_{2,\sigma} \le \frac{\varepsilon}{8} + \alpha^2 T^2 N^4\|\varphi\|^2.$$

So by taking σ such that

$$|\sigma|^2 = \beta^2 + \alpha^2 \le \frac{1}{\|\varphi\|^2}\Big(\frac{1}{T^2N^4} + \frac{1}{e^{2TN}}\Big)\frac{\varepsilon}{4},$$

we complete the proof. $\square$

Note that we do not have a convergence rate here.

Theorem 4.6. *If $\varphi \in \mathcal{C}_\theta(A)$, $0 < \theta < 1$, then we have*

$$\|u_\sigma(T) - \varphi\|^2 \le 2\left(c_1^2(\theta)\beta^{2\theta} + c_2^2(\theta,T)\alpha^2\right)\|\varphi\|^2_{\mathcal{C}_\theta}. \tag{4.8}$$

Moreover, if $\theta \ge 1$, then we have

$$\|u_\sigma(T) - \varphi\|^2 \le 2\left(\beta^2 + c_2^2(\theta,T)\alpha^2\right)\|\varphi\|^2_{\mathcal{C}_\theta}, \tag{4.9}$$

where $c_1(\theta) = (1-\theta)^{1-\theta}\theta^\theta \le 1$ and $c_2(\theta,T) = T^{-1}\theta^{-2}$.

Proof. By doing computation, we have

$$\begin{aligned}
\|u_\sigma(T) - \varphi\|^2 &= \int_\gamma^{+\infty} H_\sigma^2(\lambda)e^{-2\theta T\lambda}e^{2\theta T\lambda}\, d\|E_\lambda\varphi\|^2 \\
&\le 2\int_\gamma^{+\infty} F_{\sigma,\theta}^2(\lambda)e^{2\theta T\lambda}\, d\|E_\lambda\varphi\|^2 + 2\int_\gamma^{+\infty} G_{\sigma,\theta}^2(\lambda)e^{2\theta T\lambda}\, d\|E_\lambda\varphi\|^2 \\
&\le 2\left(F^2_{\sigma,\theta,\infty} + G^2_{\sigma,\theta,\infty}\right)\|\varphi\|^2_{\mathcal{C}_\theta}
\end{aligned}$$

and by virtue of inequalities (2.4), (2.5) we obtain the desired estimates. $\square$

We define

$$\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to H, \quad \sigma = (\alpha,\beta) \mapsto \mathcal{F}(\sigma) = \begin{cases} u_\sigma(0) = \varphi_\sigma, & \sigma \ne (0,0), \\ u(0) = \varphi_0, & \sigma = (0,0). \end{cases}$$

Theorem 4.7. *For all $\varphi \in H$, (1.1) has a solution u if and only if the function $\mathcal{F}$ is continuous at $(0,0)$. Furthermore, we have that $u_\sigma(t)$ converges to $u(t)$ as $|\sigma|$ tends to zero uniformly in t.*

Proof. Assume that $\lim_{|\sigma|\to 0}\varphi_\sigma = \varphi_0$ and $\|\varphi_0\| < +\infty$. Let $w(t) = S(t)\varphi_0$. We compute

$$\|w(t) - u_\sigma(t)\| = \|S(t)\varphi_0 - S(t)\varphi_\sigma\| = \|S(t)(\varphi_0 - \varphi_\sigma)\| \le \|\varphi_0 - \varphi_\sigma\|.$$

Which implies

$$\sup_{0\le t\le T}\|w(t) - u_\sigma(t)\| \le \|\varphi_0 - \varphi_\sigma\| \to 0, \quad \text{as } |\sigma| \to 0.$$

Since $\lim_{|\sigma|\to 0} u_\sigma(T) = \varphi$ and $\lim_{|\sigma|\to 0} u_\sigma(T) = w(T)$, we infer that $w(T) = \varphi$. Thus, $w(t) = S(t)\varphi_0$ solves (1.1) and satisfies the condition $w(T) = \varphi$.

Now, let us assume that $u(t)$ is the solution to (1.1). From lemma 4.2 we have $u(0) = S(-T)\varphi \in H$, i.e.,

$$\|u(0)\|^2 = \|\varphi\|^2_{\mathcal{C}_1} = \int_\gamma^{+\infty} e^{2T\lambda}\, d\|E_\lambda\varphi\|^2 < \infty.$$

Let $N > 0$ and $\varepsilon > 0$ such that $\int_N^{+\infty} e^{2T\lambda}\, d\|E_\lambda\varphi\|^2 < \frac{\varepsilon}{8}$. Let $\sigma_i = (\alpha_i, \beta_i),\ i = 1, 2$. Then

$$\begin{aligned}\|u_{\sigma_1}(0) - u_{\sigma_2}(0)\|^2 &= \int_\gamma^{+\infty} \Big((\beta_1 + e^{\frac{-T\lambda}{1+\alpha_1\lambda}})^{-1} - (\beta_2 + e^{\frac{-T\lambda}{1+\alpha_2\lambda}})^{-1}\Big)^2\, d\|E_\lambda\varphi\|^2 \\ &\le 2\int_\gamma^{+\infty} F^2_{\sigma_1,\sigma_2}(\lambda)\, d\|E_\lambda\varphi\|^2 + 2\int_\gamma^{+\infty} G^2_{\sigma_1,\sigma_2}(\lambda)\, d\|E_\lambda\varphi\|^2.\end{aligned} \tag{4.10}$$

By using (2.6) and (2.7), the right hand sand of (4.10) can be estimated as follows

$$\begin{aligned}\int_\gamma^{+\infty} F^2_{\sigma_1,\sigma_2}(\lambda)\, d\|E_\lambda\varphi\|^2 &\le \int_\gamma^{N} F^2_{\sigma_1,\sigma_2}(\lambda)\, d\|E_\lambda\varphi\|^2 + \int_N^{+\infty} F^2_{\sigma_1,\sigma_2}(\lambda)\, d\|E_\lambda\varphi\|^2 \\ &\le (\beta_2 - \beta_1)^2 e^{2TN}\|\varphi\|^2_{\mathcal{C}_1} + \frac{\varepsilon}{8},\end{aligned}$$

$$\begin{aligned}\int_\gamma^{+\infty} G^2_{\sigma_1,\sigma_2}(\lambda)\, d\|E_\lambda\varphi\|^2 &\le \int_\gamma^{N} G^2_{\sigma_1,\sigma_2}(\lambda)\, d\|E_\lambda\varphi\|^2 + \int_N^{+\infty} G^2_{\sigma_1,\sigma_2}(\lambda)\, d\|E_\lambda\varphi\|^2 \\ &\le (\alpha_2 - \alpha_1)^2 T^2 N^4 \|\varphi\|^2_{\mathcal{C}_1} + \frac{\varepsilon}{8}.\end{aligned}$$

Now if we choose $\sigma = (\alpha, \beta)$ so that

$$|\sigma|^2 = \alpha^2 + \beta^2 \le \frac{1}{\|\varphi\|^2_{\mathcal{C}_1}}\Big(\frac{1}{T^2N^4} + \frac{1}{e^{2TN}}\Big)\frac{\varepsilon}{4}$$

and $\sigma_0 = (0, 0)$, then we have $\|u_\sigma(0) - u_{\sigma_0}(0)\|^2 = \|\varphi_\sigma - \varphi_0\|^2 \le \varepsilon$. This shows that the function $\mathcal{F}$ is continuous at $(0, 0)$. □

Remark 4.8. If we suppose that $\varphi \in \mathcal{C}_1(A)$, then by the equality

$$\|u_\sigma(0) - u(0)\|^2 = \|u_\sigma(T) - \varphi\|^2_{\mathcal{C}_1}$$

and theorem 4.5, we have

$$\|u_\sigma(0) - u(0)\|^2 \to 0, \ \text{ as } \ |\sigma| \to 0.$$

Theorem 4.9. *If $\varphi \in \mathcal{C}_{1+\theta}(A)$, $0<\theta<1$, we have*
$$\|u_\sigma(0)-u(0)\|^2 \le 2\left(c_1^2(\theta)\beta^{2\theta}+c_2^2(\theta,T)\alpha^2\right)\|\varphi\|^2_{\mathcal{C}_{1+\theta}}. \tag{4.11}$$
Moreover, if $\theta \ge 1$, we have
$$\|u_\sigma(0)-u(0)\|^2 \le 2\left(\beta^2+c_2^2(\theta,T)\alpha^2\right)\|\varphi\|^2_{\mathcal{C}_{1+\theta}}. \tag{4.12}$$

Proof. By similar calculations to those used in Theorem 4.6 and 4.7, we have
$$\begin{aligned}
&\|u_\sigma(0)-u(0)\|^2\\
&=\int_\gamma^{+\infty} H_\sigma(\lambda)^2 e^{-2\theta T\lambda}e^{2(1+\theta)T\lambda}\,d\|E_\lambda\varphi\|^2\\
&\le 2\int_\gamma^{+\infty} F_{\sigma,\theta}(\lambda)^2 e^{2(1+\theta)T\lambda}\,d\|E_\lambda\varphi\|^2+2\int_\gamma^{+\infty} G_{\sigma,\theta}(\lambda)^2 e^{2(1+\theta)T\lambda}\,d\|E_\lambda\varphi\|^2\\
&\le 2\left(F^2_{\sigma,\theta,\infty}+G^2_{\sigma,\theta,\infty}\right)\|\varphi\|^2_{\mathcal{C}_{1+\theta}}
\end{aligned}$$
and by (2.4), (2.5) we obtain the desired estimates. □

From Theorem 4.7 and 4.9, we have the following result.

Corollary 4.10. *If $\varphi \in \mathcal{C}_{1+\theta}(A)$, $0<\theta<1$, then an upper bound of the rate of convergence of the method is given by*
$$\sup_{0\le t\le T}\|u_\sigma(t)-u(t)\|^2 \le \|u_\sigma(0)-u(0)\|^2 \le 2\left(c_1^2(\theta)\beta^{2\theta}+c_2^2(\theta,T)\alpha^2\right)\|\varphi\|^2_{\mathcal{C}_{1+\theta}}.$$
Moreover, if $\theta\ge 1$, then we have
$$\sup_{0\le t\le T}\|u_\sigma(t)-u(t)\|^2 \le \|u_\sigma(0)-u(0)\|^2 \le 2\left(\beta^2+c_2^2(\theta,T)\alpha^2\right)\|\varphi\|^2_{\mathcal{C}_{1+\theta}}.$$

Remark 4.11. If $\varphi\in\mathcal{D}(A^2)$, then with the help of (2.1) and (2.2), $\|u_\sigma(T)-\varphi\|^2$ can be estimated as follows:
$$\begin{aligned}
\|u_\sigma(T)-\varphi\|^2 &\le 2\int_\gamma^{+\infty} F_\sigma^2(\lambda)\,d\|E_\lambda\varphi\|^2+2\int_\gamma^{+\infty} G_\sigma^2(\lambda)\,d\|E_\lambda\varphi\|^2\\
&\le 2\Big(\frac{\beta}{\beta+e^{\frac{-T}{\alpha}}}\Big)^2\int_\gamma^{+\infty} d\|E_\lambda\varphi\|^2+2T^2\alpha^2\int_\gamma^{+\infty}\lambda^4\,d\|E_\lambda\varphi\|^2.
\end{aligned}$$
Choosing $\alpha=\frac{T}{(1-r)\ln(\frac{1}{\beta})}$, $0<r<1$, we obtain
$$\|u_\sigma(T)-\varphi\|^2 \le 2\Big(\beta^{2r}\|\varphi\|^2+\frac{T^4}{(1-r)^2\ln^2(\frac{1}{\beta})}\|A^2\varphi\|^2\Big).$$

Theorem 4.12. *Assuming that $\varphi\in\mathcal{D}(A)$ and letting $\alpha=\frac{T}{(1-r)\ln(\frac{1}{\beta})}$, $0<r<1$, the expression $\|u_\sigma(T)-\varphi\|^2$ can be estimated as follows*
$$\|u_\sigma(T)-\varphi\|^2 \le \beta^{2r}\|\varphi\|^2+\frac{4T}{(1-r)\ln(\frac{1}{\beta})}\|A\varphi\|^2.$$

Proof. We have
$$u'_\sigma(t)+Au_\sigma(t)=0, \tag{4.13}$$
$$v'_\sigma(t)+Av_\sigma(t)=(A-A_\alpha)v_\sigma(t)=\alpha J_\alpha A^2 v_\sigma(t), \tag{4.14}$$
$$u_\sigma(0)-v_\sigma(0)=0, \tag{4.15}$$
$$\beta v_\sigma(0)+v_\sigma(T)=\varphi. \tag{4.16}$$

If we put $x_\sigma(t) = v_\sigma(t) - u_\sigma(t)$, then $x_\sigma(t)$ satisfies the equation

$$x'_\sigma(t) + Ax_\sigma(t) = \alpha J_\alpha A^2 u(t). \tag{4.17}$$

Applying the operator $\mathcal{M}(t) = e^{(t-T)A}$ to (4.13), (4.14) and (4.17), we obtain

$$\frac{d}{dt}\left(\mathcal{M}(t)u_\sigma(t)\right) = 0, \tag{4.18}$$

$$\frac{d}{dt}\left(\mathcal{M}(t)v_\sigma(t)\right) = \alpha J_\alpha A^2 \mathcal{M}(t)v_\sigma(t), \tag{4.19}$$

$$\frac{d}{dt}\left(\mathcal{M}(t)x_\sigma(t)\right) = \alpha J_\alpha A^2 \mathcal{M}(t)v_\sigma(t). \tag{4.20}$$

Multiplying (4.20) by $\mathcal{M}(t)x_\sigma$ and integrating the obtained result over $(0,\tau)$, we get

$$\begin{aligned}
\int_0^\tau \frac{d}{dt}\|\mathcal{M}(t)x_\sigma(t)\|^2 &= \|\mathcal{M}(\tau)x_\sigma(\tau)\|^2 - \|\mathcal{M}(0)x_\sigma(0)\|^2 = \|\mathcal{M}(\tau)x_\sigma(\tau)\|^2 \\
&= 2\int_0^\tau Re(\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t), \mathcal{M}(t)x_\sigma)\,dt \\
&\le 2T\int_0^T \|\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t)\|^2\,dt + \frac{1}{2T}\int_0^T \|\mathcal{M}(t)x_\sigma(t)\|^2 \\
&\le 2T\int_0^T \|\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t)\|^2\,dt + \frac{1}{2}\sup_{0\le t\le T}\|\mathcal{M}(t)x_\sigma(t)\|^2.
\end{aligned}$$

This implies

$$\frac{1}{2}\sup_{0\le t\le T}\|\mathcal{M}(t)x_\sigma(t)\|^2 \le 2T\int_0^T \|\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t)\|^2\,dt.$$

In particulary for $t = T$ we have

$$\|\mathcal{M}(T)x_\sigma(T)\|^2 = \|u_\sigma(T) - v_\sigma(T)\|^2 \le 4T\int_0^T \|\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t)\|^2\,dt. \tag{4.21}$$

From (4.21) we can write

$$\begin{aligned}
\|u_\sigma(T) - v_\sigma(T)\|^2 &= \|(u_\sigma(T) - \varphi) + (\varphi - v_\sigma(T))\|^2 \\
&= \|(u_\sigma(T) - \varphi) + \beta v_\sigma(0)\|^2 \\
&= \|u_\sigma(T) - \varphi\|^2 + \|\beta v_\sigma(0)\|^2 + 2Re(u_\sigma(T) - \varphi, \beta v_\sigma(0)) \\
&\le 4T\int_0^T \|\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t)\|^2\,dt.
\end{aligned}$$

This last inequality implies

$$\|u_\sigma(T) - \varphi\|^2 \le 8T\int_0^T \|\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t)\|^2\,dt + 2\|\beta v_\sigma(0)\|^2. \tag{4.22}$$

To estimate the integral in the right-hand side, we take the inner product of (4.19) with $\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t)$ and integrate over $(0,T)$:

$$\begin{aligned}
&\int_0^T \|\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t)\|^2\,dt\\
&=\frac{1}{2}\int_0^T \mathrm{Re}(\frac{d}{dt}\left(\mathcal{M}(t)v_\sigma(t)\right), \alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t))\,dt\\
&=\frac{1}{2}\int_0^T \frac{d}{dt}\|\alpha J_\alpha A\mathcal{M}(t)v_\sigma(t)\|^2\,dt+\frac{1}{2}\int_0^T \frac{d}{dt}\|\alpha^2 J_\alpha A^{3/2}\mathcal{M}(t)v_\sigma(t)\|^2\,dt\\
&\le \frac{1}{2}\Big(\|\alpha J_\alpha A v_\sigma(T)\|^2+\|\alpha^2 J_\alpha A^{3/2}v_\sigma(T)\|^2\Big).
\end{aligned}$$

By virtue of

$$\|v_\sigma(T)\|=\|S_\alpha(T)\big(\beta+S_\alpha(T)\big)^{-1}\varphi\|\le\|\varphi\|$$

and

$$\|A\varphi\|^2=\|(I+\alpha A)J_\alpha A\varphi\|^2=\|J_\alpha A\varphi\|^2+2\alpha\|J_\alpha A^{3/2}\varphi\|^2+\alpha^2\|J_\alpha A^2\varphi\|^2,$$

we derive

$$\int_0^T \|\alpha J_\alpha A^2\mathcal{M}(t)v_\sigma(t)\|^2\le\frac{1}{2}\Big(\|\alpha J_\alpha A\varphi\|^2+\|\alpha^2 J_\alpha A^{3/2}\varphi\|^2\Big)\le\frac{1}{2}\alpha\|A\varphi\|^2.$$

Combining this inequality and (4.22), we obtain

$$\begin{aligned}
\|u_\sigma(T)-\varphi\|^2&\le 4T\alpha\|A\varphi\|^2+\|\beta v_\sigma(0)\|^2\\
&\le 4T\alpha\|A\varphi\|^2+\big(\frac{\beta}{\beta+e^{\frac{-T}{\alpha}}}\big)^2\|\varphi\|^2.
\end{aligned}\tag{4.23}$$

If we choose $\alpha=\frac{T}{(1-r)\ln(\frac{1}{\beta})}$, $0<r<1$, then (4.23) becomes

$$\|u_\sigma(T)-\varphi\|^2\le\beta^{2r}\|\varphi\|^2+\frac{4T}{(1-r)\ln(\frac{1}{\beta})}\|A\varphi\|^2.$$

□

Theorem 4.13. *Assuming that $\varphi\in\mathcal{D}(A)$ and $\gamma\ge 1$, then $\|u_\sigma(T)-\varphi\|^2$ can be estimated as follows*

$$\|u_\sigma(T)-\varphi\|^2\le 2\Big(\big(\frac{T}{1+\ln(\frac{\gamma T}{\beta})}\big)^2+T\alpha\Big)\|A\varphi\|^2.$$

Proof. We have

$$\|u_\sigma(T)-\varphi\|^2=\int_\gamma^{+\infty} H_\sigma(\lambda)^2\,d\|E_\lambda\varphi\|^2\le 2(I_{1,\sigma}+I_{2,\sigma}),$$

where

$$I_{1,\sigma}=\int_\gamma^{+\infty} F_\sigma(\lambda)^2\lambda^{-2}\lambda^2\,d\|E_\lambda\varphi\|^2,$$
$$I_{2,\sigma}=\int_\gamma^{+\infty} G_\sigma(\lambda)^2\,d\|E_\lambda\varphi\|^2.$$

Using (2.8) and (2.9), we obtain

$$I_{1,\sigma} \le \Big(\sup_{\lambda\ge\gamma} F_\sigma(\lambda)\lambda^{-1}\Big)^2 \|A\varphi\|^2 \le \Big(\frac{T}{1+\ln(\frac{\gamma T}{\beta})}\Big)^2 \|A\varphi\|^2, \tag{4.24}$$

$$I_{2,\sigma} \le T\alpha \|A\varphi\|^2. \tag{4.25}$$

Combining (4.24) and (4.25) we obtain the desired estimate. □

Theorem 4.14. *Assume that $\|Au(0)\|^2 = \int_\gamma^{+\infty} \lambda^2 e^{2T\lambda}\, d\|E_\lambda\varphi\|^2 < \infty$, i.e., $u(0) \in \mathcal{D}(A)$, and that $\gamma \ge 1$. Then $\|u_\sigma(0) - u(0)\|^2$ can be estimated as follows*

$$\|u_\sigma(0) - u(0)\|^2 \le 2\Big(\big(\frac{T}{1+\ln(\frac{\gamma T}{\beta})}\big)^2 + T\alpha\Big)\|Au(0)\|^2.$$

Proof. By a computation,

$$\|u_\sigma(0) - u(0)\|^2 = \int_\gamma^{+\infty} H_\sigma(\lambda)^2 e^{2T\lambda}\, d\|E_\lambda\varphi\|^2 \le 2(I_{1,\sigma} + I_{2,\sigma}),$$

where

$$I_{1,\sigma} = \int_\gamma^{+\infty} F_\sigma(\lambda)^2 \lambda^{-2}\lambda^2 e^{2T\lambda}\, d\|E_\lambda\varphi\|^2,$$

$$I_{2,\sigma} = \int_\gamma^{+\infty} G_\sigma(\lambda)^2 e^{2T\lambda}\, d\|E_\lambda\varphi\|^2.$$

Using (2.8) and (2.9) we obtain

$$I_{1,\sigma} \le \Big(\sup_{\lambda\ge\gamma} F_\sigma(\lambda)\Big)^2 \|Au(0)\|^2 \le \Big(\frac{T}{1+\ln(\frac{\gamma T}{\beta})}\Big)^2 \|Au(0)\|^2,$$

$$I_{2,\sigma} \le T\alpha \|Au(0)\|^2.$$

Combining the two inequalities above, we obtain the desired estimate. □

We conclude this paper by constructing a family of regularizing operators to (1.1).

Definition 4.15. A family $\{R_\sigma(t),\ \sigma > 0,\ t \in [0,T]\} \subset \mathcal{L}(H)$ is called a family of regularizing operators for the problem (1.1) if for each solution $u(t)$, $0 \le t \le T$ of (1.1) with final element φ, and for any $\eta > 0$, there exists $\sigma(\eta) > 0$, such that

$$\sigma(\eta) \to 0, \quad \eta \to 0, \tag{4.26}$$

$$\|R_{\sigma(\eta)}(t)\varphi_\eta - u(t)\| \to 0, \quad \eta \to 0, \tag{4.27}$$

for each $t \in [0,T]$ provided that φ_η satisfies $\|\varphi_\eta - \varphi\| \le \eta$.

Define $R_\sigma(t) = S(t)\big(\beta + S_\alpha(T)\big)^{-1}$, $t \ge 0$, $\sigma > 0$; it is clear that $R_\sigma(t) \in \mathcal{L}(H)$. In the following we will show that $R_\sigma(t)$ is a family of regularizing operators for (1.1).

Theorem 4.16. *Under the assumption $\varphi \in \mathcal{C}_1(A)$, the condition (4.27) holds.*

Proof. We have

$$\Delta_\sigma(t) = \|R_\sigma(t)\varphi_\eta - u(t)\| \le \|R_\sigma(t)(\varphi_\eta - \varphi)\| + \|R_\sigma(t)\varphi - u(t)\| = \Delta_1(t) + \Delta_2(t),$$

where

$$\Delta_1(t) = \|R_\sigma(t)(\varphi_\eta - \varphi)\| \le \Big(\frac{1}{\beta + e^{\frac{-T}{\alpha}}}\Big)\eta,$$
$$\Delta_2(t) = \|R_\sigma(t)f - u(t)\|.$$

We observe that

$$\Delta_1(t) \le \frac{\eta}{\beta}, \quad \Delta_1(t) \le \eta e^{\frac{T}{\alpha}}.$$

Choose $\beta = \sqrt{\eta}$ and $\alpha = \frac{2T}{\ln(\frac{1}{\eta})}$, then $\sigma(\eta) = (\alpha(\eta), \beta(\eta)) \to (0,0)$, $\eta \to 0$, and

$$\Delta_1(t) \le \sqrt{\eta} \to 0, \quad \text{as } \eta \to 0. \tag{4.28}$$

Now, by Theorem 4.7 we have

$$\Delta_2(t) = \|u_{\sigma(\eta)}(t) - u(t)\| \to 0, \quad \text{as } \eta \to 0, \tag{4.29}$$

uniformly in t. Combining (4.28) and (4.29) we obtain

$$\sup_{0\le t\le T} \|R_\sigma(t)\varphi_\eta - \varphi\| \to 0, \text{ as } \eta \to 0.$$

This shows that $R_\sigma(t)$ is a family of regularizing operators for (1.1). □

Concluding remarks. 1. Note that the error factor $e(\sigma)$ introduced by small changes in the final value φ is of order $\frac{1}{\beta + e^{\frac{-T}{\alpha}}}$.

2. When $\alpha = \frac{T}{(1-r)\ln(\frac{1}{\beta})}$, $0 < r < 1$, then

$$e(\sigma) = e(\beta) = \frac{1}{\beta + \beta^{1-r}} \le (\frac{1}{\beta})^{1-r}.$$

3. In [9] (resp. [17, 25]) the error factor $e(\beta)$ (resp. $e(\alpha)$ is of order $\frac{1}{\beta}$ (resp. $e^{\frac{T}{\alpha}}$). Observe that

$$\frac{1}{\beta + e^{\frac{-T}{\alpha}}} \le \frac{1}{\beta}, \quad \frac{1}{\beta + e^{\frac{-T}{\alpha}}} \le e^{\frac{T}{\alpha}}.$$

This shows that our approach has a nice regularizing effect and gives a better approximation with comparison to the methods developed in [9, 17, 25].

In this study we have achieved a better results than those established in [9, 17, 25]. The error resulting from approximation and the rate of convergence of the method are optimal.

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