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DIRICHLET PROBLEM FOR A SECOND ORDER SINGULAR DIFFERENTIAL EQUATION

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Abstract. This article concerns the existence of positive solutions to the Dirichlet problem for a second order singular differential equation. To prove existence, we use the classical method of elliptic regularization.

1. Introduction

We study the existence of positive solutions for the second order singular differential equation

$$u'' + \lambda \frac{u'}{t-1} - \gamma \frac{|u'|^2}{u} + f(t) = 0, \quad 0 < t < 1, \tag{1.1}$$

with the Dirichlet boundary conditions

$$u(1) = u(0) = 0, \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $\gamma > 0$, $f(t) \in C^1[0,1]$ and $f(t) > 0$ on $[0,1]$.

It is well known that boundary value problems for singular ordinary differential equations arise in the field of gas dynamics, flow mechanics, theory of boundary layer, and so on. In recent years, singular second order ordinary differential equations with dependence on the first order derivative have been studied extensively, see for example [1, 5, 7, 8, 9, 10, 11, 12] and references therein where some general existence results were obtained. We point out that the case considered here is not in their considerations since it does not satisfy some sufficient conditions of those papers. Our considerations were motivated by the model, which arises in the studies of a degenerate parabolic equation (see for instance [2, 3, 4]), considered by Bertsch and Ughi [4] in which they studied (1.1) with $\lambda = 0$ and $f \equiv 1$ and the boundary boundary conditions: $u(1) = u'(0) = 0$. By ordinary differential equation theories, they obtained a decreasing positive solution. However, it is easy to see from the boundary conditions (1.2) that any positive solution to the Dirichlet problem for (1.1) must not be decreasing. Recently, in [13] the authors studied the Dirichlet problem for (1.1) with $\lambda = 0$, and proved that if $\gamma > 0$, then the problem has a positive solution u; moreover, if $\gamma > \frac{1}{2}$, then u satisfies also $u'(1) = u'(0) = 0$.

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Note that the equation considered here is more general since it is also singular at $t=1$ for $\lambda \neq 0$. Thus the existence result obtained here is not a simple extension of [4, 13].

We say $u \in C^2(0,1) \cap C[0,1]$ is a solution to the Dirichlet problem (1.1), (1.2) if it is positive in $(0,1)$ and satisfies (1.1) and (1.2).

The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let $\lambda > -1, \gamma > \frac{1}{2}(1+\lambda)$, $f(t) \in C^1[0,1]$ and $f>0$ on $[0,1]$. Then the Dirichlet problem (1.1), (1.2) has a solution u. Moreover, u satisfies $u'(1)=0$. If in addition we assume that λ is non-negative, then u satisfies also $u'(0)=0$.*

2. Proof of Theorem 1.1

We will use the classical method of elliptic regularization to prove Theorem 1.1. For this, we consider the following regularized problem:

$$
\begin{gathered}
u'' + \lambda \frac{u'}{t-1-\varepsilon^{1/2}} - \gamma \frac{|u'|^2 \operatorname{sgn}_{\varepsilon}(u)}{I_\varepsilon(u)} + f(t) = 0, \quad 0<t<1, \\
u(1)=u(0)=\varepsilon,
\end{gathered}
$$

where $\varepsilon \in (0,1)$, $I_\varepsilon(s)$ and $\operatorname{sgn}_\varepsilon(s)$ are defined as follows:

$$
I_\varepsilon(s) = \begin{cases} s, & s \geqslant \varepsilon, \\ \frac{s^2+\varepsilon^2}{2\varepsilon}, & |s| < \varepsilon, \\ -s, & s \leqslant -\varepsilon, \end{cases} \qquad \operatorname{sgn}_\varepsilon(s) = \begin{cases} 1, & s \geqslant \varepsilon, \\ \frac{2s}{\varepsilon} - \frac{s^2}{\varepsilon^2}, & 0 \leqslant s < \varepsilon, \\ \frac{2s}{\varepsilon} + \frac{s^2}{\varepsilon^2}, & -\varepsilon \leqslant s < 0, \\ -1, & s < -\varepsilon. \end{cases}
$$

Clearly, $I_\varepsilon(s), \operatorname{sgn}_\varepsilon(s) \in C^1(\mathbb{R})$, and $I_\varepsilon(s) \geqslant \varepsilon/2, 1 \geqslant |\operatorname{sgn}_\varepsilon(s)|, \operatorname{sgn}_\varepsilon(s)\operatorname{sgn}(s) \geqslant 0$ in $\mathbb{R}$.

It follows from [6, Theorem 4.1, Chapter 7] that for any fixed $\varepsilon \in (0,1)$, the above regularized problem admits a classical solution $u_\varepsilon \in C^2(0,1) \cap C^1[0,1]$. By the maximal principle, it is easy to see that $u_\varepsilon(t) \geqslant \varepsilon$ on $[0,1]$. Thus u_ε satisfies

$$
\begin{gathered}
u_\varepsilon'' + \lambda \frac{u_\varepsilon'}{t-1-\varepsilon^{1/2}} - \gamma \frac{|u_\varepsilon'|^2}{u_\varepsilon} + f(t) = 0, \quad 0<t<1, \\
u_\varepsilon(0) = u_\varepsilon(1) = \varepsilon.
\end{gathered} \tag{2.1}
$$

Note that this system is equivalent to

$$
\begin{gathered}
\Big[(1+\varepsilon^{1/2}-t)^\lambda u_\varepsilon'\Big]' - \gamma \frac{(1+\varepsilon^{1/2}-t)^\lambda |u_\varepsilon'|^2}{u_\varepsilon} \\
+(1+\varepsilon^{1/2}-t)^\lambda f(t) = 0, \qquad 0<t<1.
\end{gathered} \tag{2.2}
$$

Lemma 2.1. *Under the assumptions of Theorem 1.1, we have*

$$
\big|(1+\varepsilon^{1/2}-t)^\lambda u_\varepsilon'(t)\big| \leqslant \frac{(1+\varepsilon^{1/2})^{1+\lambda} \max_{[0,1]} f}{1+\lambda}.
$$

In particular, for any $\delta \in (0,1)$ there exists a positive constant C_δ independent of ε such that $|u_\varepsilon'(t)| \leqslant C_\delta$, for $0 \leqslant t \leqslant \delta$.

Proof. By $u_\varepsilon(1) = u_\varepsilon(0) = \varepsilon$ and $u_\varepsilon(t) \geqslant \varepsilon$ for all $t \in [0,1]$, we have

$$\begin{aligned} u_\varepsilon'(0) &= \lim_{t\to 0^+} \frac{u_\varepsilon(t)-\varepsilon}{t} \geqslant 0, \\ u_\varepsilon'(1) &= \lim_{t\to 1^-} \frac{u_\varepsilon(t)-\varepsilon}{t-1} \leqslant 0. \end{aligned} \tag{2.3}$$

From (2.2) we obtain

$$\big[(1+\varepsilon^{1/2}-t)^\lambda u_\varepsilon'\big]' + A(1+\varepsilon^{1/2}-t)^\lambda \geqslant 0, \quad 0<t<1,$$

where $A = \max_{[0,1]} f$, i.e.

$$\Big[(1+\varepsilon^{1/2}-t)^\lambda u_\varepsilon' - \frac{A(1+\varepsilon^{1/2}-t)^{1+\lambda}}{1+\lambda}\Big]' \geqslant 0, \quad 0<t<1.$$

Therefore, the function $(1+\varepsilon^{1/2}-t)^\lambda u_\varepsilon' - \frac{A(1+\varepsilon^{1/2}-t)^{1+\lambda}}{1+\lambda}$ is non-decreasing on $[0,1]$, and then, noticing $\lambda > -1$ and using (2.3), we have

$$\begin{aligned} 0 \geqslant& \varepsilon^{\lambda/2} u_\varepsilon'(1) - \frac{A\varepsilon^{(1+\lambda)/2}}{1+\lambda} \\ \geqslant& (1+\varepsilon^{1/2}-t)^\lambda u_\varepsilon'(t) - \frac{A(1+\varepsilon^{1/2}-t)^{1+\lambda}}{1+\lambda} \\ \geqslant& (1+\varepsilon^{1/2})^\lambda u_\varepsilon'(0) - \frac{A(1+\varepsilon^{1/2})^{1+\lambda}}{1+\lambda} \\ \geqslant& -\frac{A(1+\varepsilon^{1/2})^{1+\lambda}}{1+\lambda}, \quad t\in[0,1], \end{aligned}$$

and hence

$$\frac{A(1+\varepsilon^{1/2}-t)^{1+\lambda}}{1+\lambda} \geqslant (1+\varepsilon^{1/2}-t)^\lambda u_\varepsilon'(t) \geqslant -\frac{A(1+\varepsilon^{1/2})^{1+\lambda}}{1+\lambda}, \quad t\in[0,1].$$

This completes the proof of Lemma 2.1. □

Obviously, we have

$$-u_\varepsilon'' - \lambda \frac{u_\varepsilon'}{t-1-\varepsilon^{1/2}} + \gamma \frac{|u_\varepsilon'|^2}{u_\varepsilon} - \min_{[0,1]} f \geqslant 0, \quad t\in(0,1), \tag{2.4}$$

$$-u_\varepsilon'' - \lambda \frac{u_\varepsilon'}{t-1-\varepsilon^{1/2}} + \gamma \frac{|u_\varepsilon'|^2}{u_\varepsilon} - \max_{[0,1]} f \leqslant 0, \quad t\in(0,1). \tag{2.5}$$

To obtain uniform bounds of u_ε, the following comparison theorem will be proved to be very useful.

Proposition 2.2. *Let $u_i \in C^2(0,1)\cap C[0,1]$ and $u_i > 0$ on $[0,1] (i=1,2)$. If $u_2 \geqslant u_1$ for $t = 0,1$, and*

$$-u_2'' - \eta \frac{u_2'}{t-1-\rho} + \varrho \frac{|u_2'|^2}{u_2} - \theta \geqslant 0, \quad t\in(0,1), \tag{2.6}$$

$$-u_1'' - \eta \frac{u_1'}{t-1-\rho} + \varrho \frac{|u_1'|^2}{u_1} - \theta \leqslant 0, \quad t\in(0,1), \tag{2.7}$$

where $\rho, \varrho, \theta > 0, \eta\in\mathbb{R}$, then $u_2(t) \geqslant u_1(t)$, $t\in[0,1]$.

Proof. From (2.6) and (2.7), we have

$$\Big(\frac{u_2^{1-\varrho}}{1-\varrho}\Big)'' + \frac{\eta}{t-1-\rho}\Big(\frac{u_2^{1-\varrho}}{1-\varrho}\Big)' \leqslant -\frac{\theta}{u_2^{\varrho}}, \quad (\varrho \neq 1)$$
$$\Big(\ln(u_2)\Big)'' + \frac{\eta}{t-1-\rho}\Big(\ln(u_2)\Big)' \leqslant -\frac{\theta}{u_2}, \quad (\varrho = 1)$$

and

$$\Big(\frac{u_1^{1-\varrho}}{1-\varrho}\Big)'' + \frac{\eta}{t-1-\rho}\Big(\frac{u_1^{1-\varrho}}{1-\varrho}\Big)' \geqslant -\frac{\theta}{u_1^{\varrho}}, \quad (\varrho \neq 1)$$
$$\Big(\ln(u_1)\Big)'' + \frac{\eta}{t-1-\rho}\Big(\ln(u_1)\Big)' \geqslant -\frac{\theta}{u_1}. \quad (\varrho = 1)$$

Combining the above inequalities, we obtain

$$w'' + \frac{\eta}{t-1-\rho}w' \leqslant \theta\Big(\frac{1}{u_1^{\varrho}} - \frac{1}{u_2^{\varrho}}\Big), \quad 0<t<1, \tag{2.8}$$

where $w : [0,1] \to \mathbb{R}$ is defined by

$$w = \begin{cases} \frac{u_2^{1-\varrho}}{1-\varrho} - \frac{u_1^{1-\varrho}}{1-\varrho}, & (\varrho \neq 1) \\ \ln(u_2) - \ln(u_1) & (\varrho = 1) \end{cases}$$

Clearly, $w \in C^2(0,1) \cap C[0,1]$. To prove the proposition, we argue by contradiction and assume that there exists a point t_0 of $(0,1)$ such that $u_2(t_0) - u_1(t_0) < 0$. From the assumption, it is easy to see that w reaches a minimum at some point t_* of $(0,1)$ such that

$$w(t_*) = \min_{t\in[0,1]} w(t) < 0, \tag{2.9}$$
$$w'(t_*) = 0, \quad w''(t_*) \geqslant 0. \tag{2.10}$$

Combining (2.10) with (2.8) we have

$$\theta\Big(\frac{1}{u_1^{\varrho}(t_*)} - \frac{1}{u_2^{\varrho}(t_*)}\Big) \geqslant 0.$$

This implies $u_2(t_*) \geqslant u_1(t_*)$. However, from (2.9) we find that $u_2(t_*) < u_1(t_*)$, a contradiction. Thus the proof of Proposition 2.2 is completed. □

Lemma 2.3. *Under the assumptions of Theorem 1.1, for all $\varepsilon \in (0,1)$ there exist positive constants C_i $(i = 1,2)$ independent of ε such that*

$$C_2(1+\varepsilon^{1/2}-t)^2 \geqslant u_\varepsilon(t) \geqslant C_1[t(1-t)+\varepsilon^{1/2}]^2, \quad t\in[0,1].$$

Proof. We shall first show the right-hand side of the above inequalities. Let $w_\varepsilon = C[t(1-t)+\varepsilon^{1/2}]^2$, where $C \in (0,1]$ will be determined later. By Proposition 2.2 and noticing (2.4), it suffices to show that

$$-w_\varepsilon'' - \lambda\frac{w_\varepsilon'}{t-1-\varepsilon^{1/2}} + \gamma\frac{|w_\varepsilon'|^2}{w_\varepsilon} - \min_{[0,1]} f \leqslant 0, \quad t\in(0,1), \tag{2.11}$$

for some sufficiently small positive constant C independent of ε. Simple calculations show that

$$w_\varepsilon' = 2C[t(1-t)+\varepsilon^{1/2}](1-2t),$$
$$w_\varepsilon'' = 2C(1-2t)^2 - 4C[t(1-t)+\varepsilon^{1/2}],$$

and

$$\begin{aligned}
&-w_\varepsilon'' - \lambda \frac{w_\varepsilon'}{t-1-\varepsilon^{1/2}} + \gamma \frac{|w_\varepsilon'|^2}{w_\varepsilon} - \min_{[0,1]} f \\
&= -2C(1-2t)^2 + 4C[t(1-t)+\varepsilon^{1/2}] \\
&\quad - 2C\lambda \frac{[t(1-t)+\varepsilon^{1/2}](1-2t)}{t-1-\varepsilon^{1/2}} + 4C\gamma(1-2t)^2 - \min_{[0,1]} f \\
&\leqslant 4C(2+|\lambda|+\gamma) - \min_{[0,1]} f, \quad 0<t<1.
\end{aligned}$$

Choosing a positive constant C such that

$$C \leqslant \min\Big\{1, \frac{\min_{[0,1]} f}{4(2+|\lambda|+\gamma)}\Big\},$$

we find that (2.11) holds.

Next we show the left-hand side of the inequalities. Let $v_\varepsilon = C(1+\varepsilon^{1/2}-t)^2$, where $C \geqslant 1$ will be determined later. A calculation shows that

$$-v_\varepsilon'' - \lambda \frac{v_\varepsilon'}{t-1-\varepsilon^{1/2}} + \gamma \frac{|v_\varepsilon'|^2}{v_\varepsilon} - \max_{[0,1]} f = 2C(2\gamma-1-\lambda) - \max_{[0,1]} f, \quad 0<t<1.$$

Choosing a positive constant C such that

$$C \geqslant \max\Big\{1, \frac{\max_{[0,1]} f}{2(2\gamma-1-\lambda)}\Big\}$$

and noticing $\gamma > \frac{1}{2}(1+\lambda)$, we find that

$$-v_\varepsilon'' - \lambda \frac{v_\varepsilon'}{t-1-\varepsilon^{1/2}} + \gamma \frac{|v_\varepsilon'|^2}{v_\varepsilon} - \max_{[0,1]} f \geqslant 0, \quad 0<t<1,$$

and then, by Proposition 2.2 and noticing (2.5), we obtain $u_\varepsilon \leqslant v_\varepsilon$ on $[0,1]$. This completes the proof of Lemma 2.3. $\square$

From (2.4), (2.5), Lemma 2.1 and Lemma 2.3, we derive that for any $\delta \in (0, \frac{1}{2})$ there exists a positive constant C_δ independent of ε such that

$$|u_\varepsilon''(t)| \leqslant C_\delta, \quad \delta \leqslant t \leqslant 1-\delta. \tag{2.12}$$

Differentiating (2.1) with respect to t we get

$$u_\varepsilon''' = \lambda \frac{u_\varepsilon' - (t-1-\varepsilon^{1/2})u_\varepsilon''}{(t-1-\varepsilon^{1/2})^2} + \gamma \frac{2u_\varepsilon u_\varepsilon' u_\varepsilon'' - (u_\varepsilon')^3}{u_\varepsilon^2} - f'(t), \quad 0<t<1.$$

By (2.12), Lemma 2.1 and Lemma 2.3, we derive that for any $\delta \in (0, \frac{1}{2})$, there exists a positive constant C_δ independent of ε such that

$$|u_\varepsilon'''(t)| \leqslant C_\delta, \quad \delta \leqslant t \leqslant 1-\delta.$$

From this and Lemma 2.1 and using Arzelá-Ascoli theorem and diagonal sequential process, we see that there exist a subsequence $\{u_{\varepsilon_j}\}$ of $\{u_\varepsilon\}$ and a function $u \in C^2(0,1) \cap C[0,1)$ such that, as $\varepsilon_j \to 0$,

$$\begin{gathered}
u_{\varepsilon_j} \to u, \quad \text{uniformly in } C[0,1-\delta], \\
u_{\varepsilon_j} \to u, \quad \text{uniformly in } C^2[\delta,1-\delta].
\end{gathered}$$

Combining this with (2.1) and $u_{\varepsilon_j}(0)=\varepsilon_j$, we find that u satisfies (1) and $u(0)=0$. By Lemma 2.3, we have

$$C_2(1-t)^2 \geqslant u(t) \geqslant C_1[t(1-t)]^2, \quad t\in[0,1), \tag{2.13}$$

therefore $u>0$ in $(0,1)$, and $\lim_{t\to 1^-} u(t)=0$. Define $u(1)=0$. Thus u is a solution to the Dirichlet problem (1.1), (1.2), and it follows from (2.13) that $u'(1)=0$.

It remains to show that for $\lambda\geqslant 0$, u satisfies $u'(0)=0$. Let $h_{\varepsilon_j}=C(t+\varepsilon_j^{1/2})^2$, where $C\geqslant \max\left\{1,\frac{\max_{[0,1]} f}{2(2\gamma-1)}\right\}$. Noticing $\lambda\geqslant 0$ and $\gamma>\frac{1}{2}(1+\lambda)$, we have

$$\begin{aligned}
&-h''_{\varepsilon_j}-\lambda\frac{h'_{\varepsilon_j}}{t-1-\varepsilon_j^{1/2}}+\gamma\frac{|h'_{\varepsilon_j}|^2}{h_{\varepsilon_j}}-\max_{[0,1]} f\\
&=2C(2\gamma-1)-2C\lambda\frac{t+\varepsilon_j^{1/2}}{t-1-\varepsilon_j^{1/2}}-\max_{[0,1]} f\\
&\geqslant 2C(2\gamma-1)-\max_{[0,1]} f\geqslant 0, \quad 0<t<1.
\end{aligned}$$

By Proposition 2.2 and noticing (2.5), we obtain $u_{\varepsilon_j}\leqslant h_{\varepsilon_j}$ on $[0,1]$. Letting $\varepsilon_j\to 0$, we have

$$u(t)\leqslant Ct^2, \quad t\in[0,1].$$

Combining this with the right-hand side of (2.13), we obtain $u'(0)=0$. This completes the proof of Theorem 1.1.

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