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INVERSE SPECTRAL ANALYSIS FOR SINGULAR DIFFERENTIAL OPERATORS WITH MATRIX COEFFICIENTS

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Abstract. Let L_α be the Bessel operator with matrix coefficients defined on $(0,\infty)$ by

$$L_\alpha U(t) = U''(t) + \frac{I/4-\alpha^2}{t^2}U(t),$$

where α is a fixed diagonal matrix. The aim of this study, is to determine, on the positive half axis, a singular second-order differential operator of $L_\alpha + Q$ kind and its various properties from only its spectral characteristics. Here Q is a matrix-valued function. Under suitable circumstances, the solution is constructed by means of the spectral function, with the help of the Gelfund-Levitan process. The hypothesis on the spectral function are inspired on the results of some direct problems. Also the resolution of Fredholm's equations and properties of Fourier-Bessel transforms are used here.

1. Introduction

By an inverse problem, physicists mean the derivation of forces from experimental data. A well-known solution of an inverse problem was the discovery of the gravitation law by Newton from the observations of Kepler. Inverse problems receive considerable attention in mathematics, physics, mechanics, meteorology and other branches of science. In spectral analysis, this consists in recovering operators from their spectral characteristics that means the bounded states and the scattering matrix or the spectral function. A procedure for explicitly constructing a potential for a boundary-problem without singularity from its spectral characteristics was formulated by Gelfand and Levitan in [6], they reduced the problem to a linear integral equation. The extension of the Gelfund-Levitan theory to higher waves ($l > 0$) are due first to Stashevskaya [16], Volk [17] and also to Jost and Kohn [8]. In the literature, in this direction, we have several other studies; see for example [1, 2, 5, 7, 13, 14]. For example, the inverse scattering problem for the radial Schrödinger equation with coupling between the l^{th} and the $(l+2)$ angular momentum, which reduces to a system of two singular second order differential equations is considered in [13]. Spectral problems associated with a generalization of a such system are studied in [3, 11, 12]. These papers deal with the equation

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defined, on $]0,\infty[$, by

$$U'' + \frac{I/4-\alpha^2}{t^2}U + Q(t)U = -\lambda^2 U, \tag{1.1}$$

where λ is a complex parameter, α is a diagonal $n\times n$ matrix, such that

$$[\alpha]_{ii} = \alpha_i, \quad \alpha_n \geq \dots \geq \alpha_1 > -1/2 \tag{1.2}$$

and Q is a real symmetric sufficiently smooth $n\times n$ matrix-valued function. For a such potential, (1.1) is solved and its various needed solutions are determined. Associated Fourier-Bessel transform is studied and properties of the spectral function are deduced. In the following, we make a brief recall of useful results. Let so be given the matrix Bessel operator L_α defined, for $t>0$, by

$$L_\alpha U(t) = U_{tt}(t) + \frac{I/4-\alpha^2}{t^2}U(t) \tag{1.3}$$

for which the $n\times n$ diagonal matrix-valued function given, for $\lambda\in\mathbb{C}$, by

$$\big[\mathcal{J}_\alpha(t,\lambda)\big]_{jj} = (2/\lambda)^{\alpha_j}\Gamma(\alpha_j+1)\sqrt{t}J_{\alpha_j}(\lambda t), \tag{1.4}$$

is the eigenfunction associated with the eigenvalue $-\lambda^2$ such that

$$\lim_{t\to 0^+} t^{-\alpha-I/2}\mathcal{J}_\alpha(t,\lambda) = I, \tag{1.5}$$

where J_ν is the Bessel function of the first kind. Under conditions on Q, the solution $\Phi(t,\lambda)$ of (1.1) satisfying (1.5) may have the form

$$\Phi(t,\lambda) = \mathcal{J}_\alpha(t,\lambda) + \int_0^t K(t,u)\mathcal{J}_\alpha(u,\lambda)du.$$

Properties of the kernel $K(t,u)$, as for example its twice differentiability on t and u, are deduced. Among other the following relation holds

$$(L_\alpha+Q)_t K(t,u) = \Big[(L_\alpha)_u K^*(t,u)\Big]^*,$$

where

$$\big[L_\alpha U^*(t)\big]^* = U_{tt}(t) + U(t)\frac{I/4-\alpha^2}{t^2}.$$

We have also the useful relation

$$K(t,t) = -\frac{1}{2}\int_0^t Q(s)ds, \quad t>0\,.$$

Since $Q(t)$ is usually taken integrable at zero, so obviously $K(t,t)$ vanish at zero. Finally let $S_0(\lambda)$ be the spectral function associated with L_α and let $S(\lambda)$ be the portion of the spectral function associated with the continuous spectrum of $L_\alpha+Q$, we show among other that for λ large we have

$$S(\lambda) - S_0(\lambda) = 2^{-2\alpha}\mathbf{\Gamma}^{-2}(\alpha+I)\lambda^\alpha O(1)\lambda^\alpha$$

and that, for $\alpha_1 \geq 1$ (see [3]), $S(\lambda)$ is integrable at zero. Here $\mathbf{\Gamma}(\alpha)$ is the diagonal constant matrix defined by $[\mathbf{\Gamma}(\alpha)]_{jj} = \Gamma(\alpha_j)$, $1\leq j\leq n$. In [12] and for $t,u>0$, we consider the function

$$\Omega(t,u) = \int_0^\infty \mathcal{J}_\alpha(t,\lambda)(S-S_0)(\lambda)\mathcal{J}_\alpha(u,\lambda)d\lambda + \sum_{j=1}^m \mathcal{J}_\alpha(t,\lambda_j)C_j\mathcal{J}_\alpha(u,\lambda_j),$$

where the C_j are spectral parameters associated with the finite discrete spectrum λ_j, $1 \le j \le m$, of the considered operator. The function $\Omega(t,u)$ is related to the kernel $K(t,u)$, for $0 < u \le t$, by the Gelfand-Levitan equation:

$$K(t,u) + \Omega(t,u) + \int_0^t K(t,s)\Omega(s,u)ds = 0. \tag{1.6}$$

The previous equation is solved in [12] as a Volterra one, where $\Omega(t,u)$ is seen as its unknown component. Properties of differentiability and estimates on this function are thus obtained. Among other properties, we have

$$\lim_{u\to 0^+} \Omega(t,u) = \lim_{u\to 0^+} \Omega_u(t,u) = 0,$$
$$(L_\alpha)_t\Omega(t,u) = \Big[(L_\alpha)_u\Omega^*(t,u)\Big]^*.$$

Conversely, would it be possible to construct a system of singular differential operators from only its spectral characteristics? This question is fairly obvious since the used functions are measurable quantities. Thing which allows scientists to be very interested by a such subject, usually called inverse spectral problem.

In the present paper we are concerned with the resolution of such problem for a singular second order differential operator $L_\alpha + Q$, with matrix coefficients, for which α and L_α are given respectively by (1.2), (1.3) and where the potential Q is to recover from the measured spectral properties. Analogous processes to those handled in the references above are used here. The main mean is the resolution of the Gelfand-Levitan equation (1.6) where $K(t,u)$ is taken as an unknown function. Properties of symmetry for $\Omega(t,u)$ and conditions on the spectral function allow to solve (1.6) as a Fredholm's equation which give $K(t,u)$ and its useful properties. This enables us to set

$$Q(t) = -2\frac{d}{dt}K(t,t), \quad t > 0. \tag{1.7}$$

Let us give a brief outline of the plan and basic ideas of this survey. ¿From the hypothesis below, we first obtain in the second section useful properties of $\Omega(t,u)$. Then in the third we construct a function $K(t,u)$ related to $\Omega(t,u)$ by the Gelfand-Levitan equation (1.6). Properties of differentiability and estimates on $K(t,u)$ deduced from those of $\Omega(t,u)$ are also obtained. This allows to construct, in the forth section, a potential Q by the relation (1.7) and a function $\Phi(t,\lambda)$ which should be an eigenfunction of the operator $L_\alpha + Q$. In the fifth section, the symmetry of Q is proved and its asymptotic behavior is obtained in special cases. The case where the spectrum differs from which of L_α by a finite discrete one is finally studied in the sixth section.

The final thing to be said here is that recovering the properties of $Q(t)$ from those of $S(\lambda)$ turns out to be difficult. This is due to the fact that the kernel $\Omega(t,u)$ of (1.6) is a matrix-valued one expressed in terms of Bessel functions for which there are no simple addition formulas such as exist for the trigonometric functions in the scalar case. These difficulties appear in solving this equation as well as in searching properties of its solution and yield us to look for the asymptotic behavior of $Q(t)$ in restricted cases.

2. Preliminaries

For the case where the operator L_α is the matrix Bessel operator, with α given by (1.2) and whose spectrum reduces to the continuous one associated with the spectral function

$$S_0(\lambda) = 2^{-2\alpha}\Gamma^{-2}(\alpha+I)\lambda^{2\alpha+I}, \quad \lambda > 0,$$

we require a singular differential operator which takes the form $L_\alpha + Q$. We assume given a finite system of discrete eigenvalues $\lambda_j = -i\mu_j$, $\mu_j > 0$ for $1 \le j \le m$, that these parameters are associated with hermitian normalizing factors C_j, the latest being positive defined and hermitian matrices. We suppose also given a prescribed $n \times n$ matrix-valued function $S(\lambda)$, defined for $\lambda \in \mathbb{R}^*$, seen as the portion of the spectral function associated with the continuous spectrum satisfying some regularity conditions. The goal of this study is to construct a function $K(t,u)$ which allows to deduce, for the required operator, the potential Q as well as an associated eigenfunction and to show some of their classical properties. The key of this problem is the resolution of the Gelfund-Levitan equation (1.6), where $\Omega(t,u)$ is given, for $t,u > 0$, by

$$\Omega(t,u) = \int_0^\infty \mathcal{J}_\alpha(t,\lambda)(S-S_0)(\lambda)\mathcal{J}_\alpha(u,\lambda)d\lambda + \sum_{j=1}^m \mathcal{J}_\alpha(t,\lambda_j)C_j\mathcal{J}_\alpha(u,\lambda_j). \tag{2.1}$$

Notation and hypotheses. First we suppose obviously that C_j and $S(\lambda)$ induce a tempered measure where especially we have

$$S(\lambda) = S^*(\lambda), \quad \lambda > 0\,. \tag{2.2}$$

Then further notation and hypothesis are needed.

Notation. Under the assumption (2.2) a Hilbert space should be constructed.

$$L_s^2 = \big\{ f :]0,\infty[\to \mathbb{C}^n : \|f\|_s^2 = \int_0^\infty f^*(\lambda)S(\lambda)f(\lambda)d\lambda < +\infty \big\}.$$

We set also, for $t > 0$ and $u > 0$,

$$\Omega^1(t,u) = t^{\alpha+I/2}\Omega(t,u)u^{-\alpha-I/2} \tag{2.3}$$

This function used in (1.6) yields the equation

$$K^1(t,u) + \Omega^1(t,u) + \int_0^t K^1(t,s)\Omega^1(s,u)ds = 0. \tag{2.4}$$

where

$$K^1(t,u) = t^{\alpha+I/2}K(t,u)u^{-\alpha-I/2} \tag{2.5}$$

For any $n \times n$ matrix A, we denote

$$\|A\| = \max_j \sum_k |A_{jk}|$$

recall that for a such norm we have $\|AB\| \le \|A\|.\|B\|$.

Hypotheses. For simplicity of computations, we assume the given function $S(\lambda)$ sufficiently regular such that $\Omega(t,u)$ is well defined. Then further hypothesis built from the results obtained in [12] are assumed. Thus, for $t>0$, we suppose that:

(A0) The function $u \to \Omega(t,u)$ is of class C^2 on $]0,\infty[$.

(A1) (i) $\lim_{u\to 0^+}\Omega(t,u)=0$, (ii) $\lim_{u\to 0^+}\Omega_u(t,u)=0$

(A2) $(L_\alpha)_t\Omega(t,u)=\big[(L_\alpha)_u\Omega^*(t,u)\big]^*$, $u>0$.

We assume also that, for any real $R>0$ and for $k=0,1$, there exist functions F_k^R, measurable and bounded on $(0,R)$, such that:

(B0) $\sup_{0<s,\,u\le t}|[\Omega_u^1(s,u)]_{ij}|\le F_0^R(t)$, $1\le i\le j\le n$

(B1) $\sup_{0<s,u\le t}|s^{2(\alpha_j-\alpha_i)}[\Omega_u^1(s,u)]_{ij}|\le F_1^R(t)$, $1\le j\le i\le n$

(B2) There exists a function F_2^R, integrable on $(0,R)$, such that

$$\sup_{0<s,u\le t}\|\Omega_{uu}^1(s,u)\|\le F_2^R(t),$$

Remark 2.1. *(i) The hypothesis above are coherent with the results of* [12].
(ii) The function $\Omega(t,u)$ satisfies the enumerated hypothesis in the case where there exists a some small real $\delta>0$ such that

$$\lambda^{-\alpha}\Big(S(\lambda)-S_0(\lambda)\Big)\lambda^{-\alpha}=\frac{O(1)}{\lambda^{2+\delta}},\quad (\lambda\to+\infty)$$

Remark 2.2. *Given an operator L_1 with a potential $Q_1\not\equiv 0$ and with no discrete spectrum, such that $S_1(\lambda)$ is its spectral function. By the technics below and under conditions analogous to those given above, it is possible to construct an operator L for a prescribed spectral function $S(\lambda)$. Both L and L_1 are considered in the class of singular differential operators of type $L_\alpha+Q$.*

Further properties of $\Omega(t,u)$. Additional properties of $\Omega(t,u)$ are needed to deduce the existence of the solution of (1.6) and its useful properties.

Remark 2.3. (i) The Hypothesis (A1) implies $\lim_{t\to 0^+}\Omega(t,t)=0$.
(ii) By means of the properties (2.2) of $S(\lambda)$ and those of C_j, $1\le j\le m$, the relation (2.1) yields $\Omega(t,u)=\Omega^*(u,t)$.
(iii) The above property shows easily that if $\Omega(t,u)$ is derivable on u, then it is also derivable on t. Moreover, for $R>t>0$, we have

$$\sup_{0<s,u\le t}\|\Omega_u(s,u)\|=\sup_{0<s,u\le t}\|\Omega_s(s,u)\|.$$

(iv) For $0<s,u\le t\le R$ and for $i\le j$, the Assumption (B0) implies

$$|[\Omega^1(s,u)]_{ij}|\le\int_0^u|[\Omega_v^1(s,v)]_{ij}|dv\le C(R)tF_0^R(t)$$

and so, using (B1), we deduce that the functions $\Omega^1(t,u)$, $\Omega(t,u)$ as well as

$$\omega(t,u)=t^{-\alpha-I/2}\Omega(t,u)u^{\alpha+I/2}$$

are bounded on $(0,R)\times(0,R)$.

Remark 2.3 (ii) enables us to manipulate (1.6) as a Fredholm's equation associated with the countable Hilbert space $E=L^2\big((0,R),M_n(\mathbb{C})\big)$. This space is supplied with the norm $\|.\|_2$, associated with the scalar product

$$\langle f,g\rangle=\sum_{j=1}^n\int_0^R g_j^*(s)f_j(s)ds.$$

where f_j are the columns vectors of the $n \times n$ matrix-valued function f. Let $E_1 = L^2\Big((0,R), \mathbb{C}^n\Big)$ equipped with its usual scalar product and let $b > 0$. Then, for $f \in E_1$, we set

$$L(f)(u) = \int_0^b \Omega(u,s) f(s) ds, \quad 0 < u \le b.$$

For further use, we recall the following results.

Lemma 2.4. *Under the Hypothesis (A0), (A1), (B0), (B1), the operator L defined above is compact and self-adjoint on the Hilbert space E_1.*

Proof. By the Remark 2.3 (ii), we see easily that L is self-adjoint in the countable space E_1. Therefore since by the Remark 2.3 (vi), we have

$$\int_{(0,b)\times(0,b)} \|\Omega(t,u)\|^2 dt du < +\infty,$$

so the components of $L(f)$ given by

$$[L(f)]_i(u) = \sum_{k=1}^{n} \int_0^b \Omega_{ik}(u,s) f_k(s) ds, \ 1 \le i \le n$$

are compact on E_1 and so is L. This allows to deduce that it's possible to construct a Hilbert basis φ_j, $j = 1, 2, \dots$, of E_1 which are eigenfunctions of L whose eingenvalues denoted λ_j are real. Furthermore, for $u \in (0,b)$ and $f \in E_1$,

$$L(f)(u) = \sum_{j=1}^{+\infty} \lambda_j < f, \varphi_j > \varphi_j(u)$$

where the previous series converges uniformly on $(0,b)$. □

3. Existence and differentiability of $K(t,u)$

The main objective of this section is the resolution of the Gelfund-Levitan equation (1.6) associated with $\Omega(t,u)$. Thus by mean of the general theory of compact self-adjoint operators (see [9, 19]) and the Lemma 2.4, we conclude that, for a fixed $t > 0$, (1.6) is with respect to $K(t,u)$ of Fredholm's.

Existence of $K(t,u)$.

Lemma 3.1. *Let $t_0 > 0$ be fixed. Then if the rows of $\Omega(t,u)$ satisfy the conditions of the Lemma 2.4, for $0 < u \le t_0$, the only solution of*

$$h_0(u) + \int_0^{t_0} h_0(s)\Omega(s,u) ds = 0 \tag{3.1}$$

in $L^2(0,t_0)$ is the trivial solution.

Proof. To solve this problem ,we shall use properties of Fourier-Bessel transforms. We assume that the (3.1) has a solution h_0 in $L^2(0,t_0)$ and we denote

$$h(u) = \begin{cases} h_0(u) & \text{for } u \le t_0 \\ 0 & \text{for } u > t_0 \end{cases}$$

By construction this function is square integrable on $(0,\infty)$. Multiplying the equation (3.1) by $h^*(u)$ at right and integrating with respect to u then substituting to $\Omega(s,u)$ its expression given by (2.1), we obtain

$$\begin{aligned}\int_0^\infty h(u)h^*(u)du + \sum_{j=1}^m\Big(\int_0^\infty \mathcal{J}_\alpha(u,\lambda_j)h^*(u)du\Big)^* C_j\Big(\int_0^\infty \mathcal{J}_\alpha(u,\lambda_j)h^*(u)du\Big)\\ + \int_0^\infty\Big(\int_0^\infty \mathcal{J}_\alpha(u,\lambda)h^*(u)du\Big)^*(S(\lambda)-S_0(\lambda))\Big(\int_0^\infty \mathcal{J}_\alpha(u,\lambda)h^*(u)du\Big)d\lambda = 0.\end{aligned}$$

By the Plancherel's Formula for L_α, this may be simplified to

$$\begin{aligned}\int_0^\infty\Big(\int_0^\infty \mathcal{J}_\alpha(u,\lambda)h^*(u)du\Big)^* S(\lambda)\Big(\int_0^\infty \mathcal{J}_\alpha(u,\lambda)h^*(u)du\Big)d\lambda\\ + \sum_{j=1}^m\Big(\int_0^\infty \mathcal{J}_\alpha(u,\lambda_j)h^*(u)du\Big)^* C_j\Big(\int_0^\infty \mathcal{J}_\alpha(u,\lambda_j)h^*(u)du\Big) = 0.\end{aligned}$$

The hypothesis on $S(\lambda)$ and C_j yield that the latest expression can be seen as a scalar product of $\mathcal{F}_\alpha\big(u^{-\alpha-I/2}h^*(u)\big)$ by it self in the Hilbert space $L^2_S\bigoplus(\mathbb{C}^n)^m$ (see [4, 11]) and so it vanishes. Since $\mathcal{F}_\alpha$ is a bijection on the space L^2_G where, for $t>0$, $G(t)=t^{2\alpha+I}$. The proof complete. ☐

Theorem 3.2. *Let $R>0$ and let $t\in]0,R]$ be fixed, then under the hypothesis of the Lemma 2.4, the Gelfund-Levitan equation* (1.6) *has a unique solution square integrable on* $(0,t)$. *Furthermore, there exists a measurable function $\mu_1(t)$, bounded on $(0,R)$, such that*

$$\|K(t,u)\| \le c(R)\mu_1(t).$$

Proof. To have the unicity of the solution $K(t,u)$ of the equation (1.6), it suffices to recall that, by mean of the Lemma 3.1, the associated homogeneous equation has for any fixed $t>0$, a trivial solution, square integrable on $(0,t)$. To prove its existence, we construct first by mean of (1.6) and the Remark 2.3 (ii) a new Gelfund-Levitan equation, given by

$$K^*(t,u) + \Omega(u,t) + \int_0^t \Omega(u,s)K^*(t,s)ds = 0. \tag{3.2}$$

The Lemma 2.4 says that, for $0\le u\le t$, each column of (3.2) is of Fredholm's, explicitly given by

$$[K^*(t,u)]_k + [\Omega(u,t)]_k + \int_0^t \Omega(u,s)[K^*(t,s)]_k ds = 0,\quad 1\le k\le n.$$

Then using results of Lemma 2.4, we deduce that their solutions in E_1 exist and that in the case where (-1) is not an eigenvalue, we have

$$[K^*(t,u)]_k = -[\Omega(u,t)]_k - \sum_{j=1}^\infty \frac{\lambda_j(t)}{\lambda_j(t)+1}\langle[\Omega(u,.)]_k,\varphi_j(t,.)\rangle\varphi_j(t,u). \tag{3.3}$$

where φ_j is a particular eigenfunction associated with the eigenvalue λ_j, $j=1,2,\dots$ defined in the previous Lemma. We recall that

$$\langle[\Omega(.,u)]_k,\varphi_j(t,.)\rangle = \int_0^t \varphi_j^*(t,s)[\Omega(s,u)]_k\,ds.$$

and that the series in expression (3.3) converges uniformly on $[0,t]$. Estimates on the solution are obtained by use of this relation, the Remark 2.3 (iv) and Cauchy-Schawrz's inequality. Furthermore the unicity of the solution deduced from the Lemma 3.1 yields inevitably that (-1) is not an eigenvalue of the operator in question and so the results above are sufficient to conclude. □

Corollary 3.3. *Under assumptions of the Lemma 2.4, we have*

$$\lim_{t\to 0^+} K(t,t) = 0.$$

For the proof of the above corollary, we use the previous proposition, Remark 2.3 (i), and (1.6).

Proposition 3.4. *Under assumptions of the Lemma 2.4, the function* $K^1(t,u)$ *given by* (2.5) *is bounded on* $0 < u \le t \le R$.

Proof. The Theorem 3.2 and the relation (2.5) yield that the function $K^1(t,u)$ is well defined and that it's a solution of the equation (2.4), moreover for a fixed $\epsilon > 0$, it's bounded on $\epsilon < u \le t \le R$. Since the kernel $\Omega^1(t,u)$ is not hermitian, we can not use the technics of the previous proposition to study the behavior of this solution elsewhere. At zero and since $\Omega^1(t,u)$ is square integrable on $(0,R)\times(0,R)$, there exists an $\epsilon > 0$ such that

$$\int_0^\epsilon \int_0^\epsilon \|\Omega^1(t,u)\|^2 dt du < 1. \tag{3.4}$$

Thus for a fixed $t = b \le \epsilon$ and for $0 < u \le b$, we set $K^1(b,u) = \varphi(u)$ and $\Omega^1(b,u) = -f(u)$, the equation (2.4) becomes

$$\varphi(u) + \int_0^b \varphi(s)\Omega^1(s,u)ds = f(u). \tag{3.5}$$

By analogy with the scalar case (see [19, p. 121]), we solve this equation by means of the resolvent method. For this we set

$$\begin{gathered} \gamma_1(t,u) = -\Omega^1(t,u), \\ \gamma_n(t,u) = -\int_0^b \gamma_{n-1}(t,s)\Omega^1(s,u)ds, \quad n \ge 2 \end{gathered} \tag{3.6}$$

and we show recursively that

$$\gamma_{n+m}(t,u) = \int_0^b \gamma_n(t,s)\gamma_m(s,u)ds, \quad n,m \ge 1. \tag{3.7}$$

The relation (3.6) and Cauchy's inequality show that, for $0 < t,\ u \le b$ and by iteration, we have

$$\int_0^b \int_0^b \|\gamma_n(s,u)\|^2 ds du \le \Big[\int_0^b \int_0^b \|\Omega^1(s,u)\|^2 ds du\Big]^n.$$

Therefore, for $n \ge 3$, we deduce that

$$\|\gamma_n(t,u)\|^2 \le \Big[\int_0^b \int_0^b \|\gamma_{n-2}(s,u)\|^2 ds\, du\Big] \int_0^b \int_0^b \|\Omega^1(t,s)\Omega^1(r,u)\|^2 ds\, dr.$$

Accordingly, it follows that

$$\begin{aligned}&\|\gamma_n(t,u)\|^2\\&\le\Big[\int_0^b\int_0^b\|\Omega^1(s,u)\|^2dsdu\Big]^{n-2}\Big\{\int_0^b\|\Omega^1(t,s)\|^2ds\int_0^b\|\Omega^1(r,u)\|^2dr\Big\}.\end{aligned}$$

The term between braces is bounded by (3.4), the series $\Gamma(t,u)=\sum_{n\ge 1}\gamma_n(t,u)$ converges uniformly on the domain in question. Using (3.7), we deduce that

$$\Gamma(t,u)=-\Omega^1(t,u)-\int_0^b\Gamma(t,r)\Omega^1(r,u)dr$$

and so that the solution of (3.5) is given by

$$\varphi(u)=f(u)+\int_0^b f(r)\Gamma(r,u)dr.$$

Estimates on $\Gamma(t,u)$ and properties of $\Omega^1(t,u)$ show that the solution $K^1(t,u)$ of (2.4) behaves regularly at zero. For the case where $u\to 0^+$, and where t is sufficiently large so that the assumption (3.4) is not satisfied, we use further results on Fredholm's equations which we do not detail here. □

Remark 3.5. By analogous computations to those done in the Proposition 3.4, we show that the function $k(t,u)=t^{-\alpha-I/2}K(t,u)u^{\alpha+I/2}$ is bounded on the domain $0<u\le t\le R$.

Differentiability of $K(t,u)$.

Lemma 3.6. *Under the Hypothesis (A0), (B0), (B1), (B2), the function $u\mapsto K(t,u)$ is of class C^2 on $]0,t]$. By differentiation of (1.6) with respect to u, integral equations associated with K_u and K_{uu} are determined; i.e.,*

$$K_u(t,u)=-\Omega_u(t,u)-\int_0^t K(t,s)\Omega_u(s,u)ds \tag{3.8}$$

$$K_{uu}(t,u)=-\Omega_{uu}(t,u)-\int_0^t K(t,s)\Omega_{uu}(s,u)ds. \tag{3.9}$$

For which we have the estimates

$$\begin{gathered}\|K_u(t,u)\|\le c_1(R)\theta_1(t)\\\|K^1_{uu}(t,u)\|\le c_2(R)\theta_2(t)\end{gathered}$$

where the θ_k, $k=1,2$ are integrable functions on $(0,R)$.

Proof. The continuity of $K(t,u)$, with respect to u, is obtained by means of the properties of its expression (3.3). To have (3.8), we use results of Theorem 3.2 which yield that, for a fixed t such that $R\ge t>0$, the function $u\mapsto K(t,s)\Omega(s,u)$, $0<s,u\le t$, satisfies the Derivation Theorem hypothesis at the first order. Furthermore, the relation (3.8) and the hypothesis gives the estimate on K_u. Analogous equation to (3.8) is obtained by means of (2.3)-(2.5); we have

$$K^1_u(t,u)=-\Omega^1_u(t,u)-\int_0^t K^1(t,s)\Omega^1_u(s,u)ds$$

The process above and the results of the Proposition 3.4, applied to the previous equation, are then used to obtain an equation on $K^1_{uu}(t,u)$, analogous to (3.9), estimates about are then deduced. □

Lemma 3.7. *Under the assumptions of Lemma 3.6 and for $u > 0$, the function $t \mapsto K(t,u)$ is of class C^2 on $[u,R]$ and by differentiation of (1.6) with respect to t, we obtain :*

$$K_t(t,u) = -\Omega_t(t,u) - K(t,t)\Omega(t,u) - \int_0^t K_t(t,s)\Omega(s,u)ds, \tag{3.10}$$

$$\begin{aligned} K_{tt}(t,u) = -\Omega_{tt}(t,u) - \big[\frac{d}{dt}K(t,t)\big]\Omega(t,u) - K(t,t)\Omega_t(t,u) \\ - K_t(t,t)\Omega(t,u) - \int_0^t K_{tt}(t,s)\Omega(s,u)ds. \end{aligned} \tag{3.11}$$

Furthermore there exist functions ν_1 and ν_2, integrable on $(0,R)$, such that:

$$\|K_t(t,u)\| \le \nu_1(t) \quad \text{and} \quad \|K^1_{tt}(t,u)\| \le \nu_2(t)\,.$$

Proof. For $t \ge u > 0$ some fixed parameters and for h sufficiently small, we consider the difference quantity $\delta^t_h K(t,u) = K(t+h,u) - K(t,u)$. Used in (1.6) it yields

$$\delta^t_h K(t,u) + \int_0^t \delta^t_h K(t,s)\Omega(s,u)ds = -\delta^t_h\Omega(t,u) - \int_t^{t+h} K(t+h,s)\Omega(s,u)ds.$$

We obtain hence a Fredholm's equation with a second member uniformly estimated in h, vanishing when $h \to 0$. The technics of Theorem 3.2 allow to have the same behavior for $\delta^t_h K(t,u)$ and so the continuity of $t \to K(t,u)$ is deduced. Its twice differentiability will be proved by similar arguments. Indeed for the first derivatives, the difference quotient $\Delta^t_h K(t,u) = (\delta^t_h K(t,u)/h)$ and (1.6) again give the relation

$$\Delta^t_h K(t,u) + \int_0^t \Delta^t_h K(t,s)\Omega(s,u)ds = -\Delta^t_h\Omega(t,u) - \int_t^{t+h} \frac{K(t+h,s)}{h}\Omega(s,u)ds.$$

Then since the free term $\Delta^t_h\Omega(t,u) + \int_t^{t+h}\frac{K(t+h,s)}{h}\Omega(s,u)ds$ is also estimated uniformly in h because of the differentiability of $\Omega(t,u)$ with respect to t and since, the last equation is of Fredholm's kind, we can so estimate $\Delta^t_h K(t,u)$. As $h \to 0$, the result (3.10) follows and estimates on $K_t(t,u)$ are obtained. An analogous equation to (3.10) is deduced for $K^1_t(t,u)$ for which we apply the above process. Finally a similar result to (3.11) is deduced for $K^1_{tt}(t,u)$. ☐

Further properties of $K(t,u)$. Lemmas 3.6 and 3.7, imply that the function $K(t,t)$ is differentiable for $t > 0$; therefore, we can set

$$Q(t) = -2\frac{d}{dt}K(t,t), \quad t > 0. \tag{3.12}$$

Remark 3.8. For a class U, of $\mathcal{C}^2$ function defined on $]0,+\infty[$, let

$$\begin{gathered} \Delta_\alpha U = U_{tt} + \frac{2\alpha+I}{t}U_t, \\ \tilde\Delta_\alpha U = U_{tt} - \frac{2\alpha+I}{t}U_t + \frac{2\alpha+I}{t^2}U\,. \end{gathered}$$

Then simple computations yield

$$\begin{gathered} (L_\alpha)_t\Omega(t,u) = t^{-\alpha-I/2}\big[(\tilde\Delta_\alpha)_t\Omega^1(t,u)\big]u^{\alpha+I/2}, \\ \big[(L_\alpha)_u\Omega^*(t,u)\big]^* = t^{-\alpha-I/2}\big[(\Delta_\alpha)_u(\Omega^1)^*(t,u)\big]^*u^{\alpha+I/2}. \end{gathered}$$

Proposition 3.9. *Under the hypotheses (A0), (A1), (A2), (B0), (B1), (B2), the function $K(t,u)$ satisfies the following two assertions:*

$$\lim_{u\to 0^+} K(t,u) = \lim_{u\to 0^+} K_u(t,u) = 0, \tag{3.13}$$

$$(L_\alpha + Q)_t K(t,u) = \big[(L_\alpha)_u K^*(t,u)\big]^*. \tag{3.14}$$

Proof. The hypothesis (A1)(i), the relation (1.6) as well as the Remark 2.3 (iv) and the Theorem 3.2 imply that $K(t,u)$ vanish as $u\to 0^+$. The same arguments and (3.8) complete the proof of the first assertion. To have (3.14) we show that for a fixed $\epsilon>0$, (A2) and integrations by parts yield

$$\begin{aligned}
&t^{-\alpha-I/2}\Big(\int_\epsilon^t \big[(\tilde\Delta_\alpha)_s (K^1)^*(t,s)\big]^*\Omega^1(s,u)ds\Big)u^{\alpha+I/2}\\
&= -K(t,t)\Omega_t(t,u) + K_u(t,t)\Omega(t,u) + K(t,\epsilon)\Omega_t(\epsilon,u) - K_u(t,\epsilon)\Omega(\epsilon,u)\\
&\quad + t^{-\alpha-I/2}\Big(\int_\epsilon^t K^1(t,s)\big[(\Delta_\alpha)_u(\Omega^1)^*(s,u)\big]^* ds\Big)u^{\alpha+I/2}.
\end{aligned}$$

Thanks to (A1) and (3.13), the second member of the last identity converges as $\epsilon\to 0^+$, so we have

$$\begin{aligned}
&\int_0^t K(t,s)\big[(L_\alpha)_u\Omega^*(s,u)\big]^* ds\\
&= \int_0^t \big[(L_\alpha)_s K^*(t,s)\big]^*\Omega(s,u)ds + K(t,t)\Omega_t(t,u) - K_u(t,t)\Omega(t,u).
\end{aligned}\tag{3.15}$$

Then Lemmas 3.6, 3.7 and the relation (1.6), yield

$$\begin{aligned}
&(L_\alpha+Q)_t K(t,u) - \big[(L_\alpha)_u K^*(t,u)\big]^*\\
&= -(L_\alpha)_t\Omega(t,u) + \big[(L_\alpha)_u\Omega^*(t,u)\big]^*\\
&- \int_0^t \Big\{(L_\alpha+Q)_t K(t,s)\Omega(s,u) - K(t,s)\big[(L_\alpha)_u\Omega^*(s,u)\big]^*\Big\}ds\\
&- \Big[\frac{d}{dt}K(t,t) + K_t(t,t) + Q(t)\Big]\Omega(t,u) - K(t,t)\Omega_t(t,u).
\end{aligned}$$

Finally the condition (A2), the relations (3.12) and (3.15) assert that

$$\begin{aligned}
&\Big[(L_\alpha+Q)_t K(t,u) - \big[(L_\alpha)_u K^*(t,u)\big]^*\Big]\\
&+ \int_0^t \Big[(L_\alpha+Q)_t K(t,s) - \big[(L_\alpha)_s K^*(t,s)\big]^*\Big]\Omega(s,u)ds = 0\,.
\end{aligned}$$

Remark 3.8 yields

$$\begin{aligned}
&\Big[(\tilde\Delta_\alpha+Q^1)_t K^1(t,u) - \big[(\Delta_\alpha)_u (K^1)^*(t,u)\big]^*\Big]u^{\alpha+I/2}\\
&+ \int_0^t \Big[(\tilde\Delta_\alpha+Q^1)_t K^1(t,s) - \big[(\Delta_\alpha)_s (K^1)^*(t,s)\big]^*\Big]s^{\alpha+I/2}\Omega(s,u)ds = 0
\end{aligned}$$

where $Q^1(t) = t^{\alpha+I/2}Q(t)t^{-\alpha-I/2}$, $t>0$. Since $\alpha_1>1$, the properties of the kernel $\Omega(t,u)$ and those of the solution $K^1(t,u)$, obtained in the Lemmas 3.6 and 3.7 show that the mapping

$$s \mapsto \Big[(\tilde\Delta_\alpha+Q^1)_t K^1(t,s) - \big[(\Delta_\alpha)_s (K^1)^*(t,s)\big]^*\Big]s^{\alpha+I/2}$$

is in $L^2(0,t)$, then by the Lemma 3.1 the proof is complete. $\square$

4. Derivation of the differential operator

For $t>0$ and $\lambda\in\mathbb{C}$, we set

$$\Phi(t,\lambda)=\mathcal{J}_\alpha(t,\lambda)+\int_0^t K(t,u)\mathcal{J}_\alpha(u,\lambda)du, \tag{4.1}$$

where $\mathcal{J}_\alpha$ is given by (1.3) and $K(t,u)$ is the solution of Fredholm's equation (1.6). In this section we plan to show important properties of $\Phi(t,\lambda)$. First we remark that the regularity of $K(t,u)$ yields that this function is well defined. Then, by the Remark 3.5 and the relation (4.1) we deduce that

$$t^{-\alpha-I/2}\Phi(t,\lambda)=t^{-\alpha-I/2}\mathcal{J}_\alpha(t,\lambda)+\int_0^t k(t,u)u^{-\alpha-I/2}\mathcal{J}_\alpha(u,\lambda)du. \tag{4.2}$$

We have so, for the potential Q defined by (3.12), the results below.

Theorem 4.1. *For $\lambda\in\mathbb{C}$ and under the hypothesis (A0), (A1), (A2), (B0), (B1), (B2), the function $\Phi(.,\lambda)$ is, on $]0,\infty[$, the solution of the singular second order differential equation with matrix coefficients given by*

$$U''+\frac{I/4-\alpha^2}{t^2}U+Q(t)U=-\lambda^2U$$

such that

$$\lim_{t\to0^+}t^{-\alpha-I/2}\Phi(t,\lambda)=I.$$

Proof. The second derivatives of $K(t,u)$ with respect to t obtained in Lemma 3.7 and its estimates imply that, for a fixed $\lambda\in\mathbb{C}$, the mapping $t\mapsto\Phi(t,\lambda)$ is twice differentiable on $]0,+\infty[$. By the expressions of these derivatives and since $\mathcal{J}_\alpha(\lambda,.)$ is an eigenfunction of the operator L_α associated with the eigenvalue $-\lambda^2$. Then justified integrations by parts and technics, used in the proof of the Proposition 3.9, show that

$$\begin{aligned}
&\big[L_\alpha+Q+\lambda^2I\big]\Phi(t,\lambda)\\
&=\int_0^t\Big[(L_\alpha+Q)_tK(t,u)-\big\{(L_\alpha)_uK^*(t,u)\big\}^*\Big]\mathcal{J}_\alpha(\lambda,u)du\\
&\quad+\Big[\frac{d}{dt}K(t,t)+\Big(\frac{\partial}{\partial t}K(t,u)\Big)_{u=t}+\Big(\frac{\partial}{\partial u}K(t,u)\Big)_{u=t}+Q(t)\Big]\mathcal{J}_\alpha(\lambda,t)\\
&\quad+\lim_{u\to0^+}\Big[K(t,u)\frac{\partial}{\partial u}\mathcal{J}_\alpha(\lambda,u)-K_u(t,u)\mathcal{J}_\alpha(\lambda,u)\Big].
\end{aligned}$$

Proposition 3.9 again and the relation (3.12) allow us to deduce that the last expression vanishes. Then relations (1.5) and (4.2) as well as the Remark 3.5 show that $\lim_{t\to0^+}t^{-\alpha-I/2}\Phi(t,\lambda)=I$. $\square$

Corollary 4.2. *Under the hypothesis of the Theorem 4.1, the mapping $\lambda\mapsto\Phi(t,\lambda)$ is even and analytic on $\mathbb{C}$.*

Proof. For this we recall only that the mapping $\lambda\mapsto\mathcal{J}_\alpha(t,\lambda)$ is even and analytic on $\mathbb{C}$. Then by the relation (4.2) and the properties of $k(t,u)$ given in the Remark 3.5, the result is easily deduced. $\square$

5. Properties of the potential Q

We look in this section for common properties of the potential Q. We recall for example that from the properties of $K(t,u)$, the function $Q(t)$, $t>0$, is well defined by the relation (3.12). Furthermore, by means of the Corollary 3.3, we can set

$$K(t,t) = -\frac{1}{2}\int_0^t Q(s)ds, \quad t>0$$

so the locally integrability of $Q(t)$ is simply deduced. Next we will look for its further classical properties as symmetry and integrability at infinity.

Symmetry of $Q(t)$.

Theorem 5.1. *Under the hypothesis (B1), (B2), and since $S(\lambda)$, $\lambda>0$, is an hermitian matrix-valued function, then so is the potential $Q(t)$, for $t>0$.*

Proof. The Gelfand-Levitan equation (1.6) and the Remark 2.3 (ii), yield that the both relations below hold

$$\Omega(t,t) + K(t,t) + \int_0^t K(t,s)\Omega(s,t)ds = 0,$$
$$\Omega(t,t) + K^*(t,t) + \int_0^t \Omega(t,s)K^*(t,s)ds = 0.$$

To have $Q^* = Q$ it is sufficient to show that the integrals in the two preceding expressions are equal. Now by the same arguments as before, we have

$$\Omega(u,t) = \begin{cases} -K^*(t,u) - \int_0^t \Omega(u,s)K^*(t,s)ds, & t\ge u>0 \\ -K(u,t) - \int_0^u K(u,s)\Omega(s,t)ds, & u\ge t>0. \end{cases}$$

Therefore, we deduce that

$$\int_0^t \Omega(t,s)K^*(t,s)ds = -\int_0^t K(t,s)K^*(t,s)ds - \int_0^t\int_0^t K(t,s)\Omega(s,u)K^*(t,u)dsdu$$

and that also

$$\int_0^t K(t,s)\Omega(s,t)ds = -\int_0^t K(t,s)K^*(t,s)ds - \int_0^t\int_0^t K(t,s)\Omega(s,u)K^*(t,u)dsdu.$$

This suffices to prove the required result. □

Remark 5.2. *In order that the potential $Q(t)$, $t>0$, to be real, it suffices to assume that the matrix-valued function $S(\lambda)$, $\lambda\in\mathbb{R}^*$, and the matrices C_j, $1\le j\le m$, are so.*

Behavior of $Q(t)$. Recall that $\frac{d}{dt}\Omega(t,t)$ is a well defined function, on the positive half axis. In the following, the behavior of $Q(t)$ at zero and at infinity will be studied by mean of its relation with this function. Because of all the difficulties mentioned in the introduction, the relation between the both as well as the behavior of the latest at infinity will be obtained under strong conditions some of them are satisfied in the regular case (see [1]). In this aim we introduce the following assumptions.

- (H1) For a fixed $R>0$, there exists a function G which is integrable on $]0,2R[$ and such that $\|\Omega_u(t,u)\| \le G(t+u)$.
- (H2) Moreover, we suppose that $\int_0^{2R} sG(s)ds < 1$.

Remark 5.3. By Remark 2.2, we deduce that the second assumption is not as restrictive as it appears.

For the following results, we denote

$$\sigma(t) = \int_t^{2t} G(s)ds, \quad \sigma_1(t) = \int_0^t sG(s)ds, \quad \tilde{\sigma}_1(t) = [1 - \sigma_1(t)]^{-1} \tag{5.1}$$

Lemma 5.4. *For $0 < u, t \leq R$ and under the hypothesis of the Lemma 3.6, (H1) and (H2), we have*

$$\|\Omega(t,u)\| \leq \int_{t\vee u}^{t+u} G(s)ds. \tag{5.2}$$

Moreover, for $0 < u \leq t \leq R$, we have

$$\|K(t,u)\| \leq \sigma(t)\Big[1 + \tilde{\sigma}_1(t)\int_u^{t+u} wG(w)dw\Big], \tag{5.3}$$

$$\|K_t(t,u)\| \leq G(t+u) + \sigma^2(t)\delta_0(t) + \sigma(t)\tilde{\sigma}_1(2t)\int_u^{t+u} G(s)ds. \tag{5.4}$$

where $t_\vee u = sup(t,u)$ and where δ_0 is a function of σ_1.

Proof. From Remark 2.3 (ii) and (H2), we deduce easily that $\|\Omega_t(t,u)\| \leq G(t+u)$, hence the result 5.2 is obtained. To obtain 5.3, we use successive approximations on (1.6). Thus we set

$$K^{(0)}(t,u) = -\Omega(t,u),$$
$$K^{(n)}(t,u) = -\int_0^t K^{(n-1)}(t,s)\Omega(s,u)ds,$$

and we show recursively that

$$\|K^{(n)}(t,u)\| \leq \sigma(t)\sigma_1^{n-1}(2t)\int_u^{t+u} wG(w)dw, \quad n \geq 1.$$

This result is justified by means of 5.2 and simple permutation of integrals. To have estimates on $K_t(t,u)$, we use the same process as above applied to the relation (i) of the Lemma 3.7. We set

$$K_t^{(0)}(t,u) = -\Omega_t(t,u) - K(t,t)\Omega(t,u),$$
$$K_t^{(n)}(t,u) = -\int_0^t K_t^{(n-1)}(t,s)\Omega(s,u)ds$$

then by (H1), the results 5.2 and 5.3, we have

$$\|K_t^{(0)}(t,u)\| \leq G(t+u) + \sigma^2(t)\Big[1 + \tilde{\sigma}_1(2t)\int_t^{2t} wG(w)dw\Big]$$

and recursively again this yields that, for $n \geq 1$,

$$\begin{aligned}&\|K_t^{(n)}(t,u)\| \\ &\leq \sigma(t)\sigma_1^{n-1}(2t)\int_u^{t+u} G(w)dw + \sigma^2(t)\Big(1 + \tilde{\sigma}_1(2t)\sigma_1(2t)\Big)\sigma_1^n(2t)\int_u^{t+u} wG(w)dw,\end{aligned}$$

so the last estimate follows. □

We have then the following useful results.

Corollary 5.5. *Under the hypothesis of Lemma 5.4, we have*

$$\int_0^t \|K(t,s)\Omega_u(s,t)\|ds \le \sigma^2(t)\Big[1+\tilde{\sigma}_1(2t)\sigma_1(2t)\Big],$$

$$\int_0^t \|K_t(t,s)\Omega(s,t)\|ds \le \sigma^2(t)\delta_1(t),$$

where δ_1 is a bounded function expressed by mean of σ_1.

Theorem 5.6. *For any fixed $R>0$ such that the assumptions of Lemma 5.4 hold, there exists a positive constant $c(R)$ such that*

$$\|2\frac{d}{dt}\Omega(t,t)-Q(t)\| \le c(R)\Big(\int_t^{2t} G(s)ds\Big)^2, \quad 0<t<R.$$

In particular if these assumptions are satisfied for $R=+\infty$, then

$$\int_0^\infty (1+t)\|Q(t)\|dt < \infty\,.$$

Moreover the function $Q(t)$ has the same asymptotic behavior as $2\frac{d}{dt}\Omega(t,t)$.

Proof. By the (3.8) and (3.10), we have

$$\begin{aligned}&\Omega_t(t,u)+\Omega_u(t,u)+K_t(t,u)+K_u(t,u)\\&=-K(t,t)\Omega(t,u)-\int_0^t K_t(t,s)\Omega(s,u)ds-\int_0^t K(t,s)\Omega_u(s,u)ds.\end{aligned}$$

Therefore, the relation (3.12) and properties of derivatives allow us to have

$$\begin{aligned}&\frac{d}{dt}\Omega(t,t)-\frac{1}{2}Q(t)\\&=-K(t,t)\Omega(t,t)-\int_0^t K_t(t,s)\Omega(s,t)ds-\int_0^t K(t,s)\Omega_u(s,t)ds.\end{aligned}$$

By this relation, Corollary 5.5, and since Lemma 5.4 implies

$$\|K(t,t)\Omega(t,t)\| \le \sigma^2(t)\Big[1+\tilde{\sigma}_1(2t)\sigma_1(2t)],$$

we deduce easily that

$$\|\frac{d}{dt}\Omega(t,t)-\frac{1}{2}Q(t)\| \le 2\big(1+\tilde{\sigma}_1(2t)\sigma_1(2t)+\delta_1(t)\big)\sigma^2(t). \tag{5.5}$$

By noticing that the function $\tilde{\sigma}_1(2t)\sigma_1(2t)+\delta_1(t)$ is bounded on $(0,R)$, the first assertion of the theorem is proved. Furthermore since

$$t\sigma(t) \le \int_t^{2t} sG(s)ds,$$

it follows that, for any $R>0$,

$$\int_0^R t\sigma^2(t)dt \le \int_0^R \Big(\int_t^{2t} sG(s)ds\Big)\sigma(t)dt \le \sigma_1(2R)\int_0^R \sigma(t)dt \le \sigma_1^2(2R) < +\infty.$$

Therefore, if (H2) is satisfied for $R=+\infty$, we deduce that

$$\int_0^\infty (1+t)\|Q(t)\|dt < \infty.$$

By these assumptions, we can remark also that

$$\int_t^{2t} G(s)ds = o(1)$$

as $t \to 0^+$, or $t \to +\infty$. Therefore, the relation (5.5) yields that the functions $2\frac{d}{dt}\Omega(t,t)$ and $Q(t)$ are equivalent in this sense and the proof is complete. □

6. Inverse problem and discrete spectrum

We consider here the simplest case where the required operator L has, associated with the continuous spectrum, the same spectral function $S_0(\lambda)$ as L_α. We assume also that the discrete spectrum reduces to an only one eigenvalue $\lambda_0 = -i\mu_0$, $\mu_0 > 0$ with a corresponding normalizing factor C_0, which is a positive definite hermitian constant matrix not necessary diagonal. We remark that in this case, for $t > 0$ and $u > 0$, we have

$$\Omega(t,u) = \mathcal{Y}_\alpha^*(t) C_0 \mathcal{Y}_\alpha(t),$$

where $\mathcal{Y}_\alpha(t) = \mathcal{J}_\alpha^*(t, -i\mu_0)$, is the real valued function deduced from (1.4). Our purpose in this section is to study the behavior at zero and at infinity of the potential ΔQ, associated with this problem. We recall that in the third section, we have shown the existence and the unicity of a square integrable solution of (1.6). In this special case we will solve it rather algebraically. We try to look for its solution in the form

$$K(t,u) = K(t)\mathcal{Y}_\alpha(u).$$

This allows to replace (1.6) by

$$\Big[K(t) + \mathcal{Y}_\alpha^*(t)C_0 + K(t)\Big(\int_0^t \mathcal{Y}_\alpha(s)\mathcal{Y}_\alpha^*(s)ds\Big)C_0\Big]\mathcal{Y}_\alpha(u) = 0.$$

The location of the zeros for the Bessel function of the first kind yields that necessarily that

$$K(t)\Big[I + R(t)C_0\Big] = -\mathcal{Y}_\alpha^*(t)C_0,$$

where

$$R(t) = \int_0^t \mathcal{Y}_\alpha(s)\mathcal{Y}_\alpha^*(s)ds. \tag{6.1}$$

To obtain $K(t)$, we need the following result.

Lemma 6.1. *For a fixed $t > 0$, the $n \times n$ matrix valued function $I + R(t)C_0$, $t > 0$ is positive defined and so it is invertible.*

Proof. For $X \in \mathbb{C}^n$ and $t > 0$, we have

$$X^* R(t) X = \int_0^t [\mathcal{Y}_\alpha^*(s)X]^*[\mathcal{Y}_\alpha^*(s)X]ds \geq 0$$

and if this quantity vanish then $X = 0$. It results that for any $t > 0$, $R(t)$ is a positive defined matrix and since C_0 satisfies yet this property, then $I + R(t)C_0$ is positive defined too and so it is invertible. □

From the result above, we deduce that the $n \times n$ matrix valued function

$$V(t) = C_0^{-1} + R(t)$$

is invertible and so that $K(t) = -\mathcal{Y}_\alpha^*(t)V^{-1}(t)$. Consequently, for $0 < u \le t$, the function

$$K(t,u) = -\mathcal{Y}_\alpha^*(t)V^{-1}(t)\mathcal{Y}_\alpha(u)$$

is a solution of (1.6). In particular, we have

$$\Delta Q(t) = 2\frac{d}{dt}\big[\mathcal{Y}_\alpha^*(t)V^{-1}(t)\mathcal{Y}_\alpha(t)\big] \tag{6.2}$$

and the relation (4.1) above takes the form

$$\Phi(t,\lambda) = \mathcal{J}_\alpha(t,\lambda) - \mathcal{Y}_\alpha^*(t)V^{-1}(t)\int_0^t \mathcal{Y}_\alpha(u)\mathcal{J}_\alpha(u,\lambda)du. \tag{6.3}$$

The study of the asymptotic behavior of $\Phi(t,\lambda)$ is possible from the estimates below, but our main interest will be the asymptotic behavior of ΔQ. The relation (6.2) shows that it suffices to have those of $\mathcal{Y}_\alpha(t)$, $\mathcal{Y}_\alpha'(t)$ and $V^{-1}(t)$ there. In this aim, we set

$$N_\alpha(t) = \frac{\Gamma(\alpha+I)}{2\sqrt{\pi}}\Big(\frac{2}{\mu_0}\Big)^{\alpha+I/2}e^{\mu_0 t} \tag{6.4}$$

and

$$(\alpha,k) = \frac{1}{k!}\big(\alpha^2 - I/4\big)\dots\big(\alpha^2 - I(k-1/2)^2\big), \quad k = 1,2,\dots$$

Remark 6.2. The asymptotic behavior of the Bessel functions (see [15, 18]) yield that as $t \to 0^+$,

$$\mathcal{Y}_\alpha(t) = t^{\alpha+I/2}\Big[I + (\alpha+I)^{-1}(\frac{\mu_0 t}{2})^2 + O(t^4)\Big],$$
$$\mathcal{Y}_\alpha'(t) = t^{\alpha-I/2}\Big[(\alpha+I/2) + (\alpha+5I/2)(\alpha+I)^{-1}(\frac{\mu_0 t}{2})^2 + O(t^4)\Big].$$

As t approaches infinity, we have

$$\mathcal{Y}_\alpha(t) = N_\alpha(t)\Big[I - \frac{(\alpha,1)}{2\mu_0 t} + \frac{(\alpha,2)}{(2\mu_0 t)^2} - \frac{(\alpha,3)}{(2\mu_0 t)^3} + O(\frac{1}{t^4})\Big],$$
$$\mathcal{Y}_\alpha'(t) = \mu_0 N_\alpha(t)\Big[I - \frac{(\alpha,1)}{2\mu_0 t} + \frac{(\alpha,2)+2(\alpha,1)}{(2\mu_0 t)^2} - \frac{(\alpha,3)+4(\alpha,2)}{(2\mu_0 t)^3} + O(\frac{1}{t^4})\Big].$$

The study of the asymptotic behavior of the function $R(t)$ must be done too.

Lemma 6.3. *For $t > 0$, $R(t)$ is a diagonal matrix-valued function and it can be expressed as*

$$R(t) = \frac{1}{2\mu_0^2}\Big\{t\Big(\mu_0^2\mathcal{Y}_\alpha^2(t) - \mathcal{Y}_\alpha'^2(t)\Big) + \mathcal{Y}_\alpha(t)\mathcal{Y}_\alpha'(t) + \frac{(\alpha,1)}{t}\mathcal{Y}_\alpha^2(t)\Big\} \tag{6.5}$$

Its asymptotic behavior, at zero and at infinity, are respectively

$$R(t) = \frac{1}{2}(\alpha+I)^{-1}t^{2\alpha+2I}\big[I + O(t^2)\big], \tag{6.6}$$

$$R(t) = \frac{N_\alpha^2(t)}{2\mu_0}\Big[I - \frac{(\alpha,1)}{\mu_0 t} + \frac{2(\alpha,2)}{(2\mu_0 t)^2} + O(\frac{1}{t^3})\Big], \tag{6.7}$$

where N_α is defined by (6.4).

Proof. It is easy to see that by (1.4) and (6.1),

$$R(t) = (\frac{2}{\mu_0})^{2\alpha} e^{i\alpha\pi}\mathbf{\Gamma}^2(\alpha+I)\int_0^t sJ_\alpha^2(s\lambda_0)ds. \tag{6.8}$$

Manipulating Bessel equations we show, for λ and ν in $\mathbb{C}$ distinct complex parameters, that (see [10, p. 128])

$$\int_0^a tJ_\mu(\lambda t)J_\mu(\nu t)dt = \frac{a\nu J_\mu(\lambda a)J'_\mu(\nu a) - a\lambda J'_\mu(\lambda a)J_\mu(\nu a)}{\lambda^2-\nu^2}, \quad a>0.$$

Taking the limit of this quantity as $\nu \to \lambda$, we obtain

$$\int_0^a tJ_\mu^2(\lambda t)dt = \frac{a^2}{2}\Big[(J'_\mu)^2(\lambda a) + (1-\frac{\mu^2}{\lambda^2a^2})J_\mu^2(\lambda a)\Big].$$

This result yields that

$$R(t) = \Big(\frac{2}{\mu_0}\Big)^{2\alpha} e^{i\alpha\pi}\mathbf{\Gamma}^2(\alpha+I)\frac{t^2}{2}\Big[(J'_\alpha)^2(-i\mu_0 t) + (1+\frac{\alpha^2}{t^2\mu_0^2})J_\alpha^2(-i\mu_0 t)\Big].$$

The relation between $J_\alpha(t)$ and $\mathcal{Y}_\alpha(t)$ completes the proof of assertion 6.5. To prove 6.6 and 6.7, we use 6.5 and Remark 6.2. $\square$

Proposition 6.4. *The function $\Delta Q(t)$ has the behavior*

$$\Delta Q(t) = \begin{cases} 2t^\alpha\Big[C_0(\alpha+I/2)+(\alpha+I/2)C_0+O(t^2)\Big]t^\alpha, & \text{as } t\to 0^+\\ \frac{(\alpha,2)}{2t^2}[I+O(\frac{1}{t^2})] & \text{as } t\to+\infty \end{cases}$$

Proof. On the one hand, by definition and from the Lemma 6.3, we show that at infinity,

$$V(t) = \frac{N_\alpha^2(t)}{2\mu_0}\Big[I - \frac{(\alpha,1)}{\mu_0 t} + \frac{2(\alpha,2)}{(2\mu_0 t)^2} + O(\frac{1}{t^3})\Big].$$

So that, when $t\to+\infty$,

$$V^{-1}(t) = 2\mu_0 N_\alpha^{-2}(t)\Big[I + \frac{(\alpha,1)}{\mu_0 t} + \frac{2(\alpha,1)^2-(\alpha,2)}{2(\mu_0 t)^2} + O(\frac{1}{t^3})\Big].$$

On the other hand and by means of (6.2), the potential ΔQ, defined by (3.12), takes the form

$$\begin{aligned}\Delta Q(t) = 2\Big[&(\mathcal{Y}_\alpha^*)'(t)V^{-1}(t)\mathcal{Y}_\alpha(t) + \mathcal{Y}_\alpha^*(t)V^{-1}(t)\mathcal{Y}'_\alpha(t)\\ &- \mathcal{Y}_\alpha^*(t)V^{-1}(t)\mathcal{Y}_\alpha(t)\mathcal{Y}_\alpha^*(t)V^{-1}(t)\mathcal{Y}_\alpha(t)\Big].\end{aligned}$$

Then, by the behavior at infinity of $\mathcal{Y}_\alpha(t)$, $\mathcal{Y}'_\alpha(t)$, and $V^{-1}(t)$, the result is deduced. For the behavior at zero, we use an analogous approach. $\square$

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