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# REDUCTION OF INFINITE DIMENSIONAL EQUATIONS

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Abstract. In this paper, we use the general Legendre transformation to show the infinite dimensional integrable equations can be reduced to a finite dimensional integrable Hamiltonian system on an invariant set under the flow of the integrable equations. Then we obtain the periodic or quasi-periodic solution of the equation. This generalizes the results of Lax and Novikov regarding the periodic or quasi-periodic solution of the KdV equation to the general case of isospectral Hamiltonian integrable equation. And finally, we discuss the AKNS hierarchy as a special example.

## 1. Introduction

Soliton equations emerged about 40 years ago [1, 2]. C.W. Cao discovered the nonlinearization method [3]-[6] to obtain the finite dimensional integrable systems [7, 8] associated with soliton equations. This way works well for many soliton equations [9]-[17]. Its main drawback, however, is that there is no single approach for finding the Lax pair [18]-[19] of a soliton equation. More precisely, different soliton equations require very different ways of finding their Lax pairs. Furthermore, this method does not work for every soliton equation, and for some equations, we have both the Bargmann and Neumann systems, for some others, we only have the Bargmann systems. So, it is natural to seek how to explain this drawback and to ask whether there is a single method that works for every infinite dimensional system (i.e., soliton equation). We have been trying to answer these questions for the last few years. Even through we have been unable to characterize the conditions that ensure the existence of both the Bargmann and Neumann systems, we have found, however, a new method which works for every soliton equation. More specifically, for every existing infinite dimensional integrable Hamiltonian system, we can obtain the associated finite dimensional integrable Hamiltonian system without knowing its Lax pairs for the corresponding higher order soliton equations.

Let $J$ be a Hamiltonian operator, $u_t = J\frac{\delta H_1}{\delta u}$ be an infinite dimensional integrable Hamiltonian equation ($u = (u_1, \dots, u_N)^T$), and $\{H_m\}_{m=0}^{\infty}$ be the first integrals of $u_t = J\frac{\delta H_1}{\delta u}$. Its higher order equations are $u_{t_m} = J\frac{\delta I}{\delta u}$ (here $I = \sum_{l=0}^{m} C_{m-l} H_l$,

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$C_0 = 1$, $C_{m-l}$ are constants, $m = 0, 1, 2, \dots$). We use the general Legendre transformation to show that the infinite dimensional integrable equation $u_{t_m} = J\frac{\delta H_m}{\delta u}$ can be reduced to a finite dimensional integrable Hamiltonian system on an invariant set $S$. Then we obtain the periodic or quasi-periodic solution of equations $u_{t_m} = J\frac{\delta I}{\delta u}$ ($m = 0, 1, 2, \dots$). This generalizes the results of Lax ([18, 19]) and Novikov [20] regarding the periodic or quasi-periodic solution of the KdV [21, 24] equation to the general case of isospectral Hamiltonian integrable equations. As a special example, we will discuss the AKNS [1] hierarchy.

Generally, looking for the periodic or quasi-periodic solution of infinite dimensional integrable equations is very difficult. In [3], the nonlinear Schrödinger equation is investigated and its periodic solution is obtained. Flaschka [22] and Lax [18] discussed the algebraic structure of the KdV equation and obtained its periodic or quasi-periodic solution. Novikov [20] studied in details the relationship between the KdV equation and its stationary equation and obtained its periodic solution. Cao [3] used the nonlinearization of Lax pairs and obtained the periodic or quasi-periodic solutions (involutive solutions) of the AKNS, the KdV, and the Harry Dym [12, 23] equations. Now we synthesize their results and generalize them to the (general) isospectral integrable equations, and obtain a (general) method to solve general infinite dimensional integrable equations $u_{t_m} = J\frac{\delta H_m}{\delta u}$ for periodic or quasi-periodic solutions. We also discuss the algebraic and geometric properties of the vector field of the Hamiltonian integrable equations $u_{t_m} = J\frac{\delta H_m}{\delta u}$ ($m \ge 0$), and prove that they can be reduced on an invariant subset $S$ to a finite dimensional integrable Hamiltonian system

$$\dot{q}_i = \frac{\partial T_m}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial T_m}{\partial q_i}, \quad i = 1, 2, \dots, n$$

where $\dot{q}_i = \frac{\partial q_i}{\partial t_m}$, $\dot{p}_i = \frac{\partial p_i}{\partial t_m}$, $T_m$ is determined by $\frac{dT_m}{dx} = -\frac{\delta I}{\delta u} J \frac{\delta H_m}{\delta u}$ and is a function of $(q, p)$ ($m = 0, 1, 2, \dots$).

## 2. The general Legendre transformation

Let $L = L(u, u', u'', \dots, u^{(n)})$ be the Lagrangian, which depends only on $u = (u_1, \dots, u_N)^T$ and its derivatives with respect to x : $u^{(j)} = \frac{d^j u}{dx^j}$, $j = 1, 2, \dots, n$. Let $I = \int_\Omega L dx$. The Euler-Lagrange equation is

$$\frac{\delta I}{\delta u} = \sum_{l=0}^{n} (-1)^l \frac{d^l}{dx^l} \frac{\partial L}{\partial u^{(l)}} = 0 \tag{2.1}$$

where, when $\Omega = (-\infty, +\infty)$, $u$ and $u^{(j)} (j = 1, 2, \dots)$ decrease rapidly as $x \to \infty$ and when $\Omega = [\alpha, \alpha + T]$, $u(x + T) = u(x)$ for $T > 0$ and $\alpha$ a constant.

We introduce the following canonical coordinates $q_i$, $p_i$ ($i = 1, 2, \dots, n$).

$$\begin{aligned} q_i &= (q_{1i}, q_{2i}, \dots, q_{Ni})^T = u^{(i-1)}, \\ p_i &= (p_{1i}, p_{2i}, \dots, p_{Ni})^T = \sum_{l=0}^{n-i} (-1)^l \frac{d^l}{dx^l} \frac{\partial L}{\partial u^{(i+l)}} \\ &= \sum_{l=0}^{n-i} (-1)^l \frac{d^l}{dx^l} \Big( \frac{\partial L}{\partial u_1^{(i+l)}}, \frac{\partial L}{\partial u_2^{(i+l)}}, \dots, \frac{\partial L}{\partial u_N^{(i+l)}} \Big)^T, \end{aligned} \tag{2.2}$$

where $i = 1, 2, \dots, n$. Let

$$H(q,p) = \sum_{i=1}^{n} q_i' p_i - L = \sum_{i=1}^{n}\sum_{l=1}^{N} q_{li}' p_{li} - L \tag{2.3}$$

where $q = (q_1, q_2, \dots, q_n)^T$, $p = (p_1, p_2, \dots, p_n)^T$. Equation (2.3) is called the general Legendre transformation.

**Definition 2.1.** A Lagrangian $L$ is said to be non-singular if equation (2.2) can be uniquely solved in the form

$$u^{(i)} = u^{(i)}(q,p), \quad i = 0, 1, \dots, 2n-1. \tag{2.4}$$

**Lemma 2.2.** *If the Lagrange function $L$ satisfies the condition*

$$\det Q = \det\Big(\frac{\partial^2 L}{\partial u_\alpha^{(n)} \partial u_\beta^{(n)}}\Big) \neq 0 \tag{2.5}$$

*where $\alpha = 1, 2, \dots, N$, $\beta = 1, 2, \dots, N$, then $L$ is non-singular.*

*Proof.* Since $L = L(u, u', \dots, u^{(n)}) = L(q, u^{(n)})$,

$$p_n = \frac{\partial L}{\partial u^{(n)}} = f(q, u^{(n)}),$$

and the Jacobi determinant

$$J(u^{(n)}) = \Big|\frac{\partial p_n}{\partial u^{(n)}}\Big| = \Big|\frac{\partial^2 L}{\partial u_\alpha^{(n)} \partial u_\beta^{(n)}}\Big| = \det Q \neq 0,$$

we obtain

$$u^{(n)} = f_n(q, p_n). \tag{2.6}$$

Next,

$$\begin{aligned} p_{n-1} &= \frac{\partial L}{\partial u^{(n-1)}} - \frac{d}{dx}\frac{\partial L}{\partial u^{(n)}} \\ &= \frac{\partial L}{\partial u^{(n-1)}} - Q \cdot u^{(n+1)} - \sum_{j=0}^{n-1} \frac{\partial^2 L}{\partial u^{(n)} \partial u^{(j)}} \cdot u^{(j+1)}. \end{aligned}$$

So $\det Q \neq 0$ yields

$$u^{(n+1)} = f_{n-1}(q, p_{n-1}, p_n). \tag{2.7}$$

Similarly, we obtain

$$u^{(n+k)} = f_{n-k}(q, p_{n-k}, p_{n-k+1}, \dots, p_n), \quad k = 0, 1, \dots, n-1. \tag{2.8}$$

Thus, the Lagrangian is non-singular. □

**Lemma 2.3.** *If the Lagrangian $L$ has the form*

$$L = a(u^{(n)})^2 + L_0(u, u', \dots, u^{(n-1)})$$

*where $a \neq 0$ is a constant, then $L$ is non-singular.*

For the proof of the above lemma, use $\det Q = 2aN \neq 0$ and Lemma 2.2.

**Remark 2.4.** If the Lagrangian $L$ is non-singular, then the general Legendre transformation
$$(u, u', \dots, u^{(2n-1)})^T \longrightarrow (q, p)^T$$
satisfies the relations
$$q_i' = \frac{\partial H}{\partial p_i}, \quad i = 1, 2, \dots, n, \tag{2.9}$$
$$p_1' = -\frac{\partial H}{\partial q_1} - \frac{\delta I}{\delta u}, \quad p_i' = -\frac{\partial H}{\partial q_i}, \quad i = 2, 3, \dots, n. \tag{2.10}$$

For calculating the above expressions vote that $q_i' = q_{i+1}$ $(i = 1, 2, \dots, n-1)$. we have
$$H = \sum_{i=1}^{n} q_i' p_i - L = q_2 p_1 + q_3 p_2 + \dots + q_n p_{n-1} + q_n' p_n - L,$$
which implies (2.9). For (2.10), we have
$$\begin{aligned} p_1' &= \frac{d}{dx} p_1 = \frac{d}{dx}\Big(\sum_{l=0}^{n-1} (-1)^l \frac{d^l}{dx^l} \frac{\partial L}{\partial u^{(1+l)}}\Big) \\ &= \sum_{l=0}^{n-1} (-1)^l \frac{d^{l+1}}{dx^{l+1}} \frac{\partial L}{\partial u^{(1+l)}} = -\sum_{k=1}^{n} (-1)^k \frac{d^k}{dx^k} \frac{\partial L}{\partial u^{(k)}} \\ &= \frac{\partial L}{\partial u} - \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} \frac{\partial L}{\partial u^{(k)}} \\ &= -\frac{\partial H}{\partial q_1} - \frac{\delta I}{\delta u}\,, \end{aligned}$$
which is the first formula of (2.10). Next,
$$\begin{aligned} p_i' &= \frac{d}{dx} p_i = \frac{d}{dx}\left(\sum_{l=0}^{n-i} (-1)^l \frac{d^l}{dx^l} \frac{\partial L}{\partial u^{(i+l)}}\right) = \sum_{l=0}^{n-i} (-1)^l \frac{d^{l+1}}{dx^{l+1}} \frac{\partial L}{\partial u^{(i+l)}} \\ &= -\sum_{k=1}^{n-i} (-1)^k \frac{d^k}{dx^k} \frac{\partial L}{\partial u^{(i+k-1)}} \\ &= -\sum_{k=0}^{n-(i-1)} (-1)^k \frac{d^k}{dx^k} \frac{\partial L}{\partial u^{(i-1+k)}} + \frac{\partial L}{\partial u^{(i-1)}} \\ &= -p_{i-1} + \frac{\partial L}{\partial u^{(i-1)}} \\ &= -p_{i-1} + \frac{\partial L}{\partial q_i} \\ &= -\frac{\partial H}{\partial q_i}, \end{aligned}$$
which is the second formula of (2.10).

**Theorem 2.5.** *For non-singular Lagrangian $L$, the Euler-Lagrange equation* (2.1) *is equivalent to the Hamiltonian system*
$$q_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2, \dots, n) \tag{2.11}$$

*where $H$ is given by* (2.3), *and*

$$\frac{dH}{dx} = -u' \frac{\delta I}{\delta u} = -\sum_{l=1}^{N} u'_l \frac{\delta I}{\delta u_l},$$

*where the symbol ' ' indicates $\frac{\partial}{\partial x}$.*

*Proof.* Since (2.1) is $\frac{\delta I}{\delta u} = 0$, (2.9) and (2.10) are equivalent to (2.11). By direct calculation, we obtain $\frac{dH}{dx} = -u' \frac{\delta I}{\delta u}$. □

**Remark 2.6.** The non-singular general Legendre transformation is invertible, i.e., we can determine $u^{(i)}$ from (2.2):

$$u^{(k)} = h_k(q, p), \quad k = 0, 1, 2, \dots, 2n-1. \tag{2.12}$$

## 3. The reduction

Suppose

$$u_t = K(u) = J\frac{\delta H}{\delta u} \tag{3.1}$$

is an infinite dimensional integrable Hamiltonian equation,

$$H_m = \int_\Omega L_m dx \quad (m = -1, 0, 1, 2, \dots)$$

are its infinitely many involutive first integrals in pair, where $H = H_1$. Its $m$th-order equations are defined by

$$u_{t_m} = J\frac{\delta H_m}{\delta u} \tag{3.2}$$

where $J$ is a differential operator for the corresponding soliton equations. For examples, for the KdV equation, $J = \frac{\partial}{\partial x}$, for the AKNS equation, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The general higher-order stationary equations are defined by

$$\sum_{l=0}^{n} C_{n-l} J \frac{\delta H_l}{\delta u} = 0 \tag{3.3}$$

or

$$\frac{\delta I}{\delta u} = 0 \tag{3.4}$$

where

$$I = \sum_{l=-1}^{n} C_{n-l} H_l = \int_\Omega L(u, \dots, u^{(n)}) dx, \quad J\frac{\delta H_{-1}}{\delta u} = 0, \quad n \geq 0,$$

$C_0 = 1$, $C_i$ $(i = -1, 1, 2, \dots, n+1)$ are constants. Suppose the functional space is

$$E = \{F : F = \int_\Omega P(u, u', \dots, u^{(m)}) dx\}, \quad (m \geq 0).$$

The Poisson bracket on space $E$ is defined as

$$\{H, F\}(x) = \frac{d}{dt}\Big|_{t=0} H(g_F^t u(x)) = \int_\Omega \frac{\delta H}{\delta u} \cdot J\frac{\delta F}{\delta u} dx = \Big(\frac{\delta H}{\delta u}, J\frac{\delta F}{\delta u}\Big) \tag{3.5}$$

where $H, F \in E$, $g_F^t$ is the solution operator of equation $u_t = J\frac{\delta F}{\delta u}$, and $(\cdot,\cdot)$ is the standard inner product in $L_2(\Omega)$. Hence,

$$\{H_m, H_n\} = 0, \quad m, n = -1, 0, 1, 2, \dots. \tag{3.6}$$

**Definition 3.1.** In the space $E$, the Hamiltonian vector field $\vec{F}$ of a functional $F$ is defined by

$$L_{\vec{F}}(H) = \frac{d}{dt}\Big|_{t=0} H(g_F^t u(x)) = \{H, F\}, \tag{3.7}$$

and for any $H \in E$,

$$L_{\lambda\vec{F}}(H) = \lambda\{H, F\} = \lambda L_{\vec{F}}(H) \quad (\lambda \text{ is constant}).$$

**Lemma 3.2.** *All the Hamiltonian vector fields form a Lie algebra. Its Lie bracket* $[\vec{H}, \vec{F}]$ *is defined by*

$$L_{[\vec{H},\vec{F}]} = L_{\vec{F}}L_{\vec{H}} - L_{\vec{H}}L_{\vec{F}}. \tag{3.8}$$

*Proof.* The bilinearity and anti-symmetry are obvious. From the definition of Lie bracket we have

$$L_{[[\vec{H},\vec{F}],\vec{A}]} = L_{\vec{A}}L_{\vec{F}}L_{\vec{H}} - L_{\vec{A}}L_{\vec{H}}L_{\vec{F}} + L_{\vec{H}}L_{\vec{F}}L_{\vec{A}} - L_{\vec{F}}L_{\vec{H}}L_{\vec{A}}$$

and $L_{[[\vec{H},\vec{F}],\vec{A}]} + L_{[[\vec{F},\vec{A}],\vec{H}]} + L_{[[\vec{A},\vec{H}],\vec{F}]}$ has 12 terms in total, and every term appears twice with opposite signs. So the Jacobi identity holds. ☐

**Lemma 3.3.** *The vector field* $\vec{F}$ *of* $F = \{F_1, F_2\}$ *can be represented by*

$$\vec{F} = [\vec{F}_1, \vec{F}_2]. \tag{3.9}$$

*Proof.* By the Jacobi identity of the Poisson bracket of functionals on the space $E$ we have

$$L_{\vec{F}}(H) = \{H, F\} = \{H, \{F_1, F_2\}\} = -\{F_1, \{F_2, H\}\} - \{F_2, \{H, F_1\}\}.$$

On the other hand,

$$\begin{aligned}
L_{[\vec{F}_1,\vec{F}_2]}(H) &= (L_{\vec{F}_2}L_{\vec{F}_1} - L_{\vec{F}_1}L_{\vec{F}_2})(H)\\
&= L_{\vec{F}_2}L_{\vec{F}_1}(H) - L_{\vec{F}_1}L_{\vec{F}_2}(H)\\
&= L_{\vec{F}_2}(\{H, F_1\}) - L_{\vec{F}_1}(\{H, F_2\})\\
&= \{\{H, F_1\}, F_2\} - \{\{H, F_2\}, F_1\}\\
&= -\{F_2, \{H, F_1\}\} + \{F_1, \{H, F_2\}\}\\
&= -\{F_1, \{F_2, H\}\} - \{F_2, \{H, F_1\}\}.
\end{aligned}$$

Comparing the above two equations we obtain

$$L_{\vec{F}}(H) = L_{[\vec{F}_1,\vec{F}_2]}(H)$$

for arbitrary $H \in E$. ☐

**Corollary 3.4.** *The map of the Lie algebra of functionals on E onto the Lie algebra of Hamiltonian vector fields is an algebra homomorphism.*

**Lemma 3.5.**

$$J\frac{\delta\{H, F\}}{\delta u} = \Big[J\frac{\delta H}{\delta u}, J\frac{\delta F}{\delta u}\Big] \tag{3.10}$$

*where*

$$[a, b] = a'[b] - b'[a], \quad a'[b] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} a(u + \varepsilon b).$$

*Proof.* For any $A \in E$, by the symmetry of $(\frac{\delta A}{\delta u})'$, we can show that

$$(\frac{\delta A}{\delta u}, J\frac{\delta\{H,F\}}{\delta u}) = \Big(\frac{\delta A}{\delta u}, (J\frac{\delta H}{\delta u})'\big[J\frac{\delta F}{\delta u}\big] - (J\frac{\delta F}{\delta u})'\big[J\frac{\delta H}{\delta u}\big]\Big).$$

Hence (3.10) holds. □

**Corollary 3.6.** *$\{J\frac{\delta H_m}{\delta u}\}$ are the symmetries of $u_{t_i} = J\frac{\delta H_i}{\delta u}$ $(i = 0, 1, 2, \dots)$.*

*Proof.* Since $\{H_m, H_i\} = 0$, Lemma 3.5 implies

$$\big[J\frac{\delta H_m}{\delta u}, \frac{\delta H_i}{\delta u}\big] = J\frac{\delta\{H_m, H_i\}}{\delta u} = 0.$$

So, by a property of symmetry: $\sigma$ is a symmetry of $u_t = K(u)$ if and only if $[K, \sigma] = 0$, this corollary is proved. □

**Theorem 3.7.** *The flows defined by* (3.2) *commute with each other.*

The above result follows from $[\vec{H}_m, \vec{H}_m] = 0$ and Lemma 3.3 or Lemma 3.5.

**Theorem 3.8.** *The solutions of the stationary equation* (3.4) *form an invariant manifold $S$ of the flows defined by equation* (3.2).

*Proof.* First we prove that $I = \sum_{l=-1}^{n} C_{n-l} H_l$ is a conserved functional of equation (3.2). It suffices to show $\{I, H_m\} = 0$. In fact,

$$\begin{aligned}\{I, H_m\} = (\frac{\delta I}{\delta u}, J\frac{\delta H_m}{\delta u}) &= \Big(\sum_{l=-1}^{n} \frac{\delta H_l}{\delta u}, J\frac{\delta H_m}{\delta u}\Big)\\ &= \sum_{l=0}^{n} C_{n-l}(\frac{\delta H_l}{\delta u}, J\frac{\delta H_m}{\delta u})\\ &= \sum_{l=0}^{n} C_{n-l}\{H_l, H_m\} = 0.\end{aligned}$$

By a theorem given by Lax (see [18, 19]), if $I$ is a conserved functional of (3.2), then the set of stationary points of $I$, i.e., the solution set of (3.4), forms an invariant set for the flow (3.2). □

Using Remark 2.6, we can reduce an arbitrary function

$$P = P(u, u', \dots, u^{(n)})$$

to a function $P_1 = P_1(q, p)$ where $u^{(k)} = h_k(q, p)$ is on the manifold $S$. We call $P_1$ the reduction of $P$ through equations (2.12) and (2.2), but we still use $P$ to denote $P_1$. Let $J\frac{\delta H_i}{\delta u}$ be the reduction of $J\frac{\delta H_i}{\delta u}$, and define $T_i$ by:

$$\frac{dT_i}{dx} = -\frac{\delta I}{\delta u} J\frac{\delta H_i}{\delta u}, \quad i = 0, 1, 2, \dots. \tag{3.11}$$

**Theorem 3.9.** *If the Lagrangian of equation* (3.4) *is non-singular, then via the Legendre transformation* (2.2)*, the stationary equation* (3.4) *is transformed into a classical integrable Hamiltonian system:*

$$q_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2, \dots, n) \tag{3.12}$$

*where the Hamiltonian function*

$$H(q,p)=\sum_{i=1}^{n} q_i' p_i - L(q_1,\dots,q_n,q_n'), \tag{3.13}$$

$$\frac{dH}{dx}=-u_x\frac{\delta I}{\delta u}, \tag{3.14}$$

*and the involutive first integrals are* $T_i$ *(*$i=0,1,2,\dots$*).*

To prove this theorem, we define the Poisson bracket $\{\cdot,\cdot\}$ in the symplectic space $(\mathbf{R}^{2n},\omega^2)$ and prove that $\{T_i\}$ is an involutive system.

**Definition 3.10.** In the symplectic space $(\mathbf{R}^{2n},\ \omega^2)$, we define the Poisson bracket as

$$\{A,B\}=\frac{d}{dt}\Big|_{t=0}A(g_B^t(q(x),p(x)))=\sum_{i=1}^{n}\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{3.15}$$

where the symplectic structure $\omega^2=\sum_{i=1}^{n} dp_i\wedge dq_i=\sum_{i=1}^{n}\sum_{l=1}^{N} dp_{li}\wedge dq_{li}$.

To prove $\{T_i,T_j\}=0$, $\{T_i,H\}=0$ $(i,j=1,2,\dots)$, we first give the following theorem.

**Theorem 3.11.** *Under the reduction through equations* (2.12) *and* (2.2), *the infinite dimensional Hamiltonian integrable system*

$$\frac{\partial u}{\partial t_m}=J\frac{\delta H_m}{\delta u} \tag{3.16}$$

*is transformed into the finite dimensional Hamiltonian system on* $S$,

$$\dot q_j=\frac{\partial T_m}{\partial p_j},\quad \dot p_j=-\frac{\partial T_m}{\partial q_j} \tag{3.17}$$

*where*

$$\dot q_j=\frac{\partial q_j}{\partial t_m},\quad \dot p_j=\frac{\partial p_j}{\partial t_m},\quad j=1,2,\dots,n\,.$$

*Proof.* By (3.13), we have

$$\frac{\partial^2 H}{\partial q_j\partial p_{j-1}}=1\quad (j=1,2,\dots,n-1), \tag{3.18}$$

$$\frac{\partial^2 H}{\partial q_l\partial p_s}=0\quad (l\le s<n \text{ or } 1\le s<l-1), \tag{3.19}$$

$$\frac{\partial^2 H}{\partial p_n\partial p_j}=0\quad (j<n). \tag{3.20}$$

From Remark 2.4 we see that $\frac{\delta I}{\delta u}=0$ is only one of the Hamiltonian equations (3.12). Same as [20], the other equations of (3.12) can be considered as a relation between $(q_1,\dots,q_n,p_1,\dots,p_n)$ and $(u,u',\dots,u^{(2n-1)})$. Hence, from the identity formulas (2.9) – (2.11) we obtain

$$\frac{dT_m}{dx}=-\frac{\delta I}{\delta u}J\frac{\delta H_m}{\delta u}=\big(\frac{\partial H}{\partial q_1}+p_1'\big)J\frac{\delta H_m}{\delta u}. \tag{3.21}$$

On the other hand,

$$\frac{dT_m}{dx}=\frac{\partial T_m}{\partial p_1}p_1'+\sum_{l=2}^{n}\frac{\partial T_m}{\partial p_l}p_l'+\sum_{l=1}^{n}\frac{\partial T_m}{\partial q_l}q_l'$$

$$= \frac{\partial T_m}{\partial p_1} p_1' - \sum_{l=2}^{n} \frac{\partial T_m}{\partial p_l} \frac{\partial H}{\partial q_l} + \sum_{l=1}^{n} \frac{\partial T_m}{\partial q_l} \frac{\partial H}{\partial p_l}.$$

The identity (2.10) is true for arbitrary $u(x)$. Hence in (3.21) we can consider that $p_1'$ is arbitrary. Thus, comparing the above two expressions, we obtain

$$J \frac{\delta H_m}{\partial u} = \frac{\partial T_m}{\partial p_1}. \tag{3.22}$$

By (3.16) and $q_1 = u$, on $S$ we have

$$\dot{q}_1 = \frac{dq_1}{dt_m} = \frac{du}{dt_m} = J \frac{\delta H_m}{\delta u} = \frac{\partial T_m}{\partial p_1},$$

i.e.,

$$\dot{q}_1 = \frac{\partial T_m}{\partial p_1}.$$

We prove this theorem by mathematical induction. Let us assume $\dot{q}_j = \frac{\partial T_m}{\partial p_j}$ $(j = 1, 2, \dots, k)$, then as $j = k + 1$, we have

$$\dot{q}_{k+1} = (\dot{q}_k)' = \frac{d}{dx} \Big( \frac{\partial T_m}{\partial p_k} \Big). \tag{3.23}$$

Let $A = (A_1, \dots, A_N)^T$, $B$ and $A_i$ are the functionals with respect to

$$(q, p) = (q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n) \in \mathbf{R}^{2n}.$$

We define

$$\{A, B\} \equiv (\{A_1, B\}, \dots, \{A_N, B\})^T.$$

Using of the Jacobi identity of the Poisson bracket (3.15) and $\{T_m, H\} = 0$ on $S$, we have

$$\begin{aligned} \dot{q}_{k+1} &= \frac{dq_{k+1}}{dt} = \frac{dq_k'}{dt} = \frac{d}{dx}(\dot{q}_k) \\ &= \frac{d}{dx}\Big(\frac{\partial T_m}{\partial p_k}\Big) = \frac{d}{dx}\{q_k, T_m\} = \{\{q_k, T_m\}, H\} \\ &= -\{\{T_m, H\}, q_k\} - \{\{H, q_k\}, T_m\} \\ &= \{\{q_k, H\}, T_m\} \\ &= \{q_k', T_m\} \\ &= \frac{\partial T_m}{\partial p_{k+1}}. \end{aligned} \tag{3.24}$$

Hence

$$\dot{q}_j = \frac{\partial T_m}{\partial p_j}, \quad j = 1, 2, \dots, n\,.$$

Next we prove the formulas

$$\dot{p}_j = -\frac{\partial T_m}{\partial q_j}, \quad j = 1, 2, \dots, n.$$

Since $(\dot{q}_n)' = (q_n')^{\cdot}$, we have

$$\Big(\frac{\partial T_m}{\partial p_n}\Big)' = \Big(\frac{\partial H}{\partial p_n}\Big)^{\cdot}. \tag{3.25}$$

By (3.18) – (3.20) and the Jacobi identity of the the Poisson bracket, we have on $S$

$$\Big(\frac{\partial H}{\partial p_n}\Big)^{\cdot} = \sum_{j=1}^{n} \frac{\partial^2 H}{\partial p_n \partial q_j} \frac{\partial T_m}{\partial p_j} + \frac{\partial^2 H}{\partial p_n^2} \dot{p}_n. \tag{3.26}$$

On the other hand,

$$\begin{aligned}
\Big(\frac{\partial T_m}{\partial p_n}\Big)' &= (\dot{q}_n)' = \{\dot{q}_n, H\} \\
&= \Big\{\frac{\partial T_m}{\partial p_n}, H\Big\} = \{\{q_n, T_m\}, H\} \\
&= -\{\{T_m, H\}, q_n\} - \{\{H, q_n\}, T_m\} \\
&= \{\{q_n, H\}, T_m\} = \Big\{\frac{\partial H}{\partial p_n}, T_m\Big\}
\end{aligned}$$

That is

$$\Big(\frac{\partial T_m}{\partial p_n}\Big)' = \sum_{j=1}^{n} \frac{\partial^2 H}{\partial p_n \partial q_j} \frac{\partial T_m}{\partial p_j} - \frac{\partial^2 H}{\partial p_n^2} \frac{\partial T_m}{\partial q_n}. \tag{3.27}$$

Comparing (3.26) with (3.27), we have

$$\dot{p}_n = -\frac{\partial T_m}{\partial q_n}.$$

Now we use mathematical induction again. Let us assume $\dot{p}_j = -\frac{\partial T_m}{\partial q_j}$ $(j = n, n-1, \dots, k)$. Then when $j = k-1$, similarly we obtain

$$(\dot{p}_k)' = -\sum_{j=1}^{n} \frac{\partial^2 H}{\partial q_k \partial q_j} \frac{\partial T_m}{\partial q_j} + \frac{\partial^2 H}{\partial q_k \partial p_n} \frac{\partial T_m}{\partial q_n} + \frac{\partial T_m}{\partial q_{k-1}}, \tag{3.28}$$

$$(p_k')^{\cdot} = -\sum_{j=1}^{n} \frac{\partial^2 H}{\partial q_k \partial q_j} \frac{\partial T_m}{\partial q_j} + \frac{\partial^2 H}{\partial q_k \partial p_n} \frac{\partial T_m}{\partial q_n} - \dot{p}_{k-1}. \tag{3.29}$$

Using (3.18) and $(p_k')^{\cdot} = (\dot{p}_k)'$ we obtain

$$\dot{p}_{k-1} = -\frac{\partial T_m}{\partial q_{k-1}} \quad (k = n, n-1, \dots, 3, 2) \tag{3.30}$$

which completes the proof. □

**Corollary 3.12.** *The flows defined by equation* (3.17) *commute with each other on* $S$.

*Proof.* From Theorem 3.7, the solution operators of $u_{t_m} = J\frac{\delta H_m}{\delta u}$ commute, and when $n = 1$, $H_1 = H$. Denoting $t_1$ by $x$, the solution operators of $u_{t_n} = J\frac{\delta H_n}{\delta u}$ and $u_x = J\frac{\delta H}{\delta u}$ commute. By the invertibility of the general Legendre transformation

$$(u, u', \dots, u^{(2n-1)}) \longrightarrow (q, p),$$

we obtain that the solution operators determined by (3.17) commute. □

**Theorem 3.13.** *System* $\{T_i\}$ *defined by* (3.11) *is an involutive system.*

*Proof.* By Corollary 3.12, the flows defined by equation (3.17) commute. By a theorem given by V.I. Arnold (see [7, page 211]), two flows commute if and only if the Poisson bracket of their corresponding vector fields is equal to zero. Thus, we obtain $\{T_i, T_j\} = 0$. □

The involutivity of $\{T_i\}$ implies the following theorem.

**Theorem 3.14.** *The Hamiltonian system defined by* (3.17) *is a FIH system in the symplectic space* $(\mathbf{R}^{2n}, \omega^2)$ *on* $S$.

*Proof of Theorem 3.9.* Since we have $\{T_i, H\} = 0$ $(i = 0, 1, 2, \dots)$ and $\{T_i, T_j\} = 0$ $(i, j = 0, 1, 2, \dots)$, the finite dimensional Hamiltonian system defined by (3.12) is a FIH system in the symplectic space $(\mathbf{R}^{2n}, \omega^2)$ on $S$. □

**Remark 3.15.** The first component $q_1 = u$ of system (3.17) is the solution of the higher order equation $u_{t_m} = J\frac{\delta H_m}{\delta u}$. When $\Omega = [\alpha, \alpha + T]$, we can obtain its periodic solution and when $\Omega = (-\infty, +\infty)$, we can obtain the rapid decreasing solution at infinity.

## 4. A special example: The AKNS hierarchy

By [1, page 54] the $m$th order AKNS equation can be written in the Hamiltonian form

$$u_{t_n} = J\frac{\delta H_n}{\delta u}\,, \tag{4.1}$$

where

$$\begin{aligned}
u &= (v, w)^T,\\
J &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\\
H_n &= H_n(v, w) = \int_\Omega \mu_n dx,\\
\mu_0 &= -vw,\\
\mu_1 &= -wv_x,\\
\mu_2 &= -wv_{xx} + (wv)^2,\\
\mu_3 &= -wv_{xxx} + 4vw^2v_x + wv^2w_x,\\
\mu_4 &= -wv_{xxxx} + 6vwv_xw_x + 5w^2v_x^2 + 6vw^2v_{xx} - 2(wv)^3 + wv^2w_{xx},\\
&\vdots\\
\mu_{n+1} &= w\Big(\frac{\mu_n}{w}\Big)_x + \sum_{k=0}^{n-1} \mu_k\mu_{n-1-k}.
\end{aligned}$$

Using integration by parts, we rewrite $H_n$ as follows.

$$\begin{aligned}
H_0 &= -\int_\Omega wv dx,\\
H_1 &= -\int_\Omega wv_x dx = \int_\Omega L_1 dx,\\
H_2 &= \int_\Omega [w_xv_x + (wv)^2] dx = \int_\Omega L_2 dx,\\
H_3 &= \int_\Omega [w_xv_{xx} + 4w^2vv_x + wv^2w_x] dx = \int_\Omega L_3 dx,\\
H_4 &= \int_\Omega [-w_{xx}v_{xx} - w^2v_x^2 - v^2w_x^2 - 8ww_xvv_x - 2(vw)^3] dx = \int_\Omega L_4 dx,
\end{aligned}$$

$$\vdots$$

$$H_n = \int_\Omega L_n(u, u', \dots, u^{(m)})dx, \quad n = 0, 1, 2, \dots$$

where

$$m = \begin{cases} k & \text{if } n = 2k, \\ k+1 & \text{if } n = 2k+1, \end{cases} \quad (k = 0, 1, 2, \dots).$$

**Lemma 4.1.** *The Lagrangian* $L_{2n} = L_{2n}(u, u', \dots, u^{(n)})$ *is non-singular.*

*Proof.* The result of this lemma follows from the following formulas

$$L_{2n} = (-1)^{n+1} v^{(n)} w^{(n)} + (\text{terms with order less than } n),$$

and

$$Q = (-1)^{(n+1)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, $\det Q \neq 0$, and $L_{2n}$ is non-singular. □

Consider the $2n$th order stationary equation

$$\sum_{l=0}^{2n} C_{2n-l} J \frac{\delta H_1}{\delta u} = J \frac{\delta I}{\delta u} = 0$$

or

$$\frac{\delta I}{\delta u} = 0 \tag{4.2}$$

where

$$I = \sum_{l=0}^{2n} C_{2n-l} H_l = \int_\Omega \sum_{l=0}^{2n} C_{2n-l} L_l dx = \int_\Omega L(u, u', \dots, u^{(n)})dx, \tag{4.3}$$

$C_0 = 1$, and $C_i$ are constants.

**Theorem 4.2.** *Under the reduction of* (2.2) *and* (2.12), *where* $N = 2$, *the infinite dimensional integrable AKNS hierarchy* (4.1) *can be transformed into the finite dimensional integrable Hamiltonian system* (3.17) *on* $S$. *Where* $S$ *is the solution set of equation* (4.2).

For example, let $n = 2$, $C_i = 0$ $(i \neq 0)$, $C_0 = 1$, the equation (4.2) has the form

$$\begin{aligned} \frac{\delta I}{\delta u} &= \begin{bmatrix} -w_{xxxx} + 6vw_x^2 + 8wvw_{xx} + 4ww_x v_x + 2w^2 v_{xx} - 6w^3 v^2 \\ -v_{xxxx} + 6wv_x^2 + 8wvv_{xx} + 4vv_x w_x + 2v^2 w_{xx} - 6v^3 w^2 \end{bmatrix} \\ &= \begin{bmatrix} c0 \\ 0 \end{bmatrix}. \end{aligned} \tag{4.4}$$

The corresponding Legendre transformation is

$$\begin{aligned} q_1 &= (v, w)^T = (q_{11}, q_{21})^T, \\ q_2 &= (v, w)_x^T = (q_{12}, q_{22})^T, \\ p_1 &= \frac{\partial L_4}{\partial u'} - (\frac{\partial L_4}{\partial u''})' = \begin{bmatrix} -2v_x w^2 - 8vww_x + w_{xxx} \\ -2w_x v^2 - 8vwv_x + v_{xxx} \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}, \\ p_2 &= \frac{\partial L_4}{\partial u''} = \begin{bmatrix} -w_{xx} \\ -v_{xx} \end{bmatrix} = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}. \end{aligned}$$

It can be solved for $u^{(i)}$:

$$\begin{aligned}(v,w) &= (q_{11},q_{21}),\\ (v,w)_x &= (q_{12},q_{22}),\\ (v,w)_{xx} &= -(p_{22},p_{12}),\\ (v,w)_{xxx} &= (p_{21}+2q_{11}^2q_{22}+8q_{11}q_{12}q_{21},\ p_{11}+2q_{21}^2q_{12}+8q_{11}q_{21}q_{22}).\end{aligned}$$

The Hamiltonian $H$ corresponding to the Lagrangian $L$ is

$$H = (q_{12}p_{11}+q_{22}p_{21}-p_{12}p_{22})+2(q_{11}q_{21})^3+q_{12}^2q_{21}^2+8q_{11}q_{12}q_{21}q_{22}+q_{11}^2q_{22}^2.$$

Thus, the Euler-Lagrange equation (4.4) is equivalent to the following classical integrable Hamiltonian equation:

$$q_j' = \frac{\partial H}{\partial p_j},\quad p_j' = -\frac{\partial H}{\partial q_j},\quad j=1,2. \tag{4.5}$$

And the involutive first integrals in pair are

$$T_i = -\int \frac{\delta I}{\delta u}J\frac{\delta H_i}{\delta u}dx,\quad i=0,1,2,\dots.$$

Here the integral constants are zeros. By direct calculations we have:

$$\begin{aligned}T_0 &= q_{11}p_{11}+q_{12}p_{12}-(q_{21}p_{21}+q_{22}p_{22}),\\ T_1 &= H = q_{12}p_{11}+q_{22}p_{21}-p_{12}p_{22}+2(q_{11}q_{21})^3\\ &\quad +q_{12}^2q_{21}^2+8q_{11}q_{12}q_{21}q_{22}+q_{11}^2q_{22}^2,\\ T_2 &= -p_{11}p_{22}+p_{21}p_{12}\\ &\quad +2(q_{21}q_{12}^2q_{22}-q_{11}q_{22}^2q_{12}+q_{11}q_{21}^2p_{21}-q_{21}q_{11}^2p_{11})\\ &\quad +4(-q_{11}q_{12}q_{21}p_{22}+q_{11}q_{22}q_{21}p_{12})+6(q_{11}^3q_{21}^2q_{22}-q_{21}^3q_{11}^2q_{12}),\\ T_3 &= (2q_{21}^2q_{12}+2q_{11}q_{21}q_{22}+p_{11})(2q_{11}^2q_{22}+2q_{11}q_{21}q_{12}+p_{21})\\ &\quad -(q_{21}p_{22}+q_{11}p_{12}+q_{12}q_{22}+3q_{11}^2q_{21}^2)^3.\end{aligned}$$

Moreover,

$$\frac{\partial}{\partial t_i}\begin{pmatrix}q_j\\ p_j\end{pmatrix} = \begin{pmatrix}\frac{\partial T_i}{\partial p_j}\\ -\frac{\partial T_i}{\partial q_j}\end{pmatrix} \tag{4.6}$$

are the constraining AKNS equations on $S$. Let $i=2$ and $i=3$ in (4.6), we obtain the following systems

$$\frac{\partial}{\partial t_2}\begin{bmatrix}q_{11}\\ q_{21}\end{bmatrix} = \begin{bmatrix}-p_{22}-2q_{21}q_{11}^2\\ p_{12}+2q_{11}q_{21}^2\end{bmatrix}, \tag{4.7}$$

$$\frac{\partial}{\partial t_3}\begin{bmatrix}q_{11}\\ q_{21}\end{bmatrix} = \begin{bmatrix}p_{21}+2q_{22}q_{11}^2+2q_{11}q_{21}q_{12}\\ p_{11}+2q_{12}q_{21}^2+2q_{11}q_{21}q_{22}\end{bmatrix}. \tag{4.8}$$

Or

$$\begin{aligned}\frac{\partial u}{\partial t_2} &= \begin{bmatrix}v\\ w\end{bmatrix}_{t_2} = \begin{bmatrix}v_{xx}-2wv^2\\ -w_{xx}+2vw^2\end{bmatrix} = J\frac{\delta H_2}{\delta u},\\ \frac{\partial u}{\partial t_3} &= \begin{bmatrix}v\\ w\end{bmatrix}_{t_3} = \begin{bmatrix}v_{xxx}-6vwv_x\\ w_{xxx}-6vww_x\end{bmatrix} = J\frac{\delta H_3}{\delta u}.\end{aligned}$$

**Remark 4.3.** Note that systems (4.5), (4.7) and (4.8) are new finite dimensional completely integrable Hamiltonian systems derived from the infinite dimensional integrable AKNS system using our new method.

Also, systems (4.5), (4.7) and (4.8) are perhaps the easiest ones to construct out of infinitely many finite dimensional completely integrable Hamiltonian systems. They can be obtained by taking different coefficients $C_{2n-l}$ or different values of $n$ in (4.3).

Contrary to the beliefs of experts about twenty years ago that the finite dimensional completely integrable Hamiltonian systems are very rare, we have constructed infinitely many of them.

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