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BOUNDARY STABILIZATION OF A COUPLED SYSTEM OF NONDISSIPATIVE SCHRODINGER EQUATIONS

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ABSTRACT. We use the multiplier method and the approach in [1] to study the problem of exponential stabilization of a coupled system of two Schrodinger equations.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$ with smooth boundary Γ, and let (Γ_0, Γ_1) be a partition of Γ. Consider the boundary feedback system:

$$iy_t + \Delta y + F_1(y, \nabla y) + P_1(z) = 0 \quad \text{in } Q, \tag{1.1}$$

$$iz_t + \Delta z + F_2(z, \nabla z) + P_2(y) = 0 \quad \text{in } Q, \tag{1.2}$$

$$y = z = 0 \quad \text{on } \Sigma_0, \tag{1.3}$$

$$\frac{\partial y}{\partial \upsilon} + g(y_t) = 0, \quad \frac{\partial z}{\partial \upsilon} + h(z_t) = 0 \quad \text{on } \Sigma_1, \tag{1.4}$$

$$y(0) = y_0, \quad z(0) = z_0 \quad \text{in } \Omega. \tag{1.5}$$

where $T > 0$, $Q = \Omega \times]0, T[$, $\Sigma_l = \Gamma_l \times]0, T[$ $(l = 0, 1)$, υ is the outward unit normal to Γ, $\frac{\partial}{\partial \upsilon}$ denotes the normal derivative, y_t and z_t the time derivative, ∇ and Δ are the Gradient and the Laplacian in the space variables. Operators F_l, P_l $(l = 1, 2)$ and the functions g and h are defined in §2.

Our goal in this paper is to obtain, under a suitable geometrical conditions on $(\Omega, \Gamma_0, \Gamma_1)$, the exponential stability of the system (1.1)–(1.5): For any initial data (y_0, z_0) in some space, the energy E (see (2.13)) of the solution tends to zero exponentially as $t \to +\infty$.

In the case of one equation and where $F = 0$ and g is linear of the form $g(s) = (h.\upsilon)s$, h represents a vector field satisfying some geometrical conditions, a lot of work have been (see for example [5, 6]). Recently, in [3] they have obtain a result of stabilization in the space $L^2(\Omega)$ of one equation where $F = 0$ and linear feedback in the form iy. In these situations the system is dissipative (the usual energy is decreasing). Notice, that this property (the decrease of energy) specially $E(T) \le E(0)$ has a crucial role in studying the stability of the solution. So, the approach in [2] can not be applied to the non dissipative system.

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To the best of our best knowledge, nonlinear boundary stabilization for the coupled system of two Schrodinger equations with lower order terms has not been considered in the literature. In this case the energy is not decreasing; we do not have any information about the sign of the derivative of the energy (see (4.1)). This requires a careful treatment.

Our paper is divided in 3 sections. In §2, we give notation and assumptions. In §3, we give the result of existence of solution. In §4 we give some formulas witch are needed for proving our result. In §5 we use the multiplier method to show, that the energy of the solution of system (1.1)–(1.5) satisfies the inequalities:

$$\begin{gathered}\int_0^T E(t) \leq \lambda_1(E(T)+E(0)) + \lambda_2(E(0)-E(T)), \quad \text{for all } T>0 \\ E'(t) \leq \lambda_3 E(t), \quad \text{for all } t>0\,,\end{gathered} \tag{1.6}$$

where $\lambda_1 \leq \lambda_2$. E' is the time derivative of the energy. As in [1], we have

$$E(t) \leq cE(0)e^{-\omega t}, \quad \text{for all } t>0. \tag{1.7}$$

2. Notation and Assumptions

Let x_0 be a fixed point in of $\mathbb{R}^n$. Set $h=h(x)=x-x_0$ and $R=\sup\{|h(x)| : x\in\overline{\Omega}\}$. Assume that for some constant $h_0>0$, we have

$$\Gamma_0 = \{x\in\Gamma : h\upsilon \leq 0\}\,, \quad \Gamma_1 = \{x\in\Gamma : h\upsilon \geq h_0\}\,. \tag{2.1}$$

Assume that F_1, F_2 are linear differential operators of order one in the space variables with $L_\infty(\overline{Q})$-coefficients and P_1, P_2 are linear operators of zero order with $L_\infty(\overline{Q})$-coefficients:

$$F_1 = F_1(y,\nabla y) = v_1(x).\nabla y - q_1(x)y, \tag{2.2}$$

$$F_2 = F_2(z,\nabla z) = v_2(x).\nabla z - q_2(x)z, \tag{2.3}$$

$$P_1 = P_1(z) = \alpha_1(x)z, \tag{2.4}$$

$$P_2 = P_2(y) = \alpha_2(x)y. \tag{2.5}$$

Where in (2.4)–(2.5), α_1 and α_2 are a bounded complex valued functions such that

$$\operatorname{Im}(\overline{\alpha_1\alpha_2}) \geq 0\,, \tag{2.6}$$

$$|\alpha_1| = |\alpha_1(x)|_{L^\infty(\Omega)} \leq \frac{\min(1,\frac{n}{2})}{(n+R)C+R} \tag{2.7}$$

where q_l $(l=1,2)$ are a positives functions satisfying

$$\max_{l=1,2}(|q_l|) \leq \frac{\min(1,\frac{n}{2}) - |\alpha_1|((n+R)C+R)}{2R(C+1)}. \tag{2.8}$$

Where C is the positive constant satisfying

$$\int_\Omega |y|^2 \leq C\int_\Omega |\nabla y|^2.$$

We assume that for all $l=1,2$ $\;v_l = v_l(x)$ is a complex n-vector field with $|v_l| \in L_\infty(\overline{Q})$, the Hessian matrix V_l of v_l satisfies $|V_l| \in L_\infty(\overline{Q})$ and the following properties for $\operatorname{Im} v_l$ are achieved:

$$|\operatorname{Im} v_l.\mu| \leq \operatorname{Im} v_l.\upsilon \quad \text{on } \Gamma_1, \tag{2.9}$$

$$\operatorname{Im} v_l.\upsilon = 0 \quad \text{on } \Gamma_0, \tag{2.10}$$

where μ represent the tangential unit vector on Γ. Put

$$\beta = \max_{l=1,2} \big\{ |\alpha_2 - \overline{\alpha_1}|, |v_l|, |\nabla(\alpha_2 - \overline{\alpha_1})|, |V_l|, |\operatorname{div}(\operatorname{Im} v_l)| \big\}.$$

Assume that g and h are a complex valued functions such that there exists a constants $g_*, h_*, g^*, h^* > 0$ such that for all $s \in \mathbb{C}$ we have

$$g_*|s| \le |g(s)| \le g^*|s| \quad \text{and} \quad g(s)\overline{s} \ge 0, \tag{2.11}$$

$$h_*|s| \le |h(s)| \le h^*|s| \quad \text{and} \quad h(s)\overline{s} \ge 0. \tag{2.12}$$

Put

$$H^1_{\Gamma_0}(\Omega) = \big\{ u \in H^1(\Omega) : u = 0 \quad \text{on } \Gamma_0 \big\}.$$

We define the energy E of (1.1)–(1.5) by

$$E(t) = \frac{1}{2}\Big(\int_\Omega |\nabla y|^2 + \int_\Omega |\nabla z|^2 + \int_\Omega q_1|y|^2 + \int_\Omega q_2|z|^2 dx \Big) - \operatorname{Re} \int_\Omega \overline{\alpha_1} y \overline{z}. \tag{2.13}$$

Remark 2.1. We note that, because (2.7), we have

$$\zeta_1 \Big(\int_\Omega |\nabla y|^2 + \int_\Omega |\nabla z|^2 \Big) \le \|(y,z)\|^2_{H^1_{\Gamma_0}(\Omega)} \le \zeta_2 \Big(\int_\Omega |\nabla y|^2 + \int_\Omega |\nabla z|^2 \Big),$$

where

$$\|(y,z)\|^2_{H^1_{\Gamma_0}(\Omega)} = \int_\Omega |\nabla y|^2 + \int_\Omega |\nabla z|^2 + \int_\Omega q_1|y|^2 + \int_\Omega q_2|z|^2 dx - 2\operatorname{Re} \int_\Omega \alpha_1 y \overline{z}.$$

3. Existence of solutions

We define an operator

$$A(u_1,u_2) = (i\Delta u_1 + iF_1(u_1,\nabla u_1) + iP_1(u_2), i\Delta u_2 + iF_2(u_2,\nabla u_2) + iP_2(u_1))$$

with domain

$$\begin{aligned} D(A) = \big\{ &(u_1,u_2) \in V \times V : \text{for } l = 1,2\ \Delta u_l \in H^1_{\Gamma_0}(\Omega), \\ &[\frac{\partial u_1}{\partial \upsilon} + g(i\Delta u_1 + iF_1 + iP_1)]_{\Gamma_1} = 0, \\ &[\frac{\partial u_2}{\partial \upsilon} + h(i\Delta u_2 + iF_2 + iP_2)]_{\Gamma_1} = 0 \big\} \end{aligned}$$

Where $V = H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$. Let $U = (u_1,u_2)$, we may rewrite system (1.1)–(1.5) as

$$\begin{gathered} U_t = AU \\ U(0) = U_0. \end{gathered} \tag{3.1}$$

then the solvability of (1.1)–(1.5) is equivalent to the one of (3.1).

We can prove that, if g and h are globally lipschitz, there exists a positive constant k such that the operator $A - kI$ is maximal dissipative on $(H^1_{\Gamma_0}(\Omega))^2$. Therefore, we have the following theorem [2, 4].

Theorem 3.1. *1. For every $(y_0,z_0) \in D(A)$ the system (1.1)–(1.5) has a unique solution $(y,z) \in L^\infty([0,+\infty); D(A)) \cap W^{1,\infty}([0,+\infty); (H^1_{\Gamma_0}(\Omega))^2)$.*
2. For every $(y_0,z_0) \in (H^1_{\Gamma_0}(\Omega))^2$ the system (1.1)–(1.5) has a unique solution $(y,z) \in C([0,+\infty); (H^1_{\Gamma_0}(\Omega))^2)$.

4. Preliminary results

We begin by establishing a formula concerning the derivative of the energy of the system (1.1)–(1.5) and two estimates concerning $\int_{\Sigma_1}(g(y_t)\overline{y_t}+h(z_t)\overline{z_t})$ and $\operatorname{Re}\int_\Omega(v_1.\nabla y\overline{y_t}+(v_2.\nabla z+(\alpha_2-\overline{\alpha_1})y)\overline{z_t})$.

Lemma 4.1. *For each solution y of system (1.1)–(1.5) we have*

$$E'(t)=\operatorname{Re}\int_\Omega(v_1.\nabla y\overline{y_t}+(v_2.\nabla z+(\alpha_2-\overline{\alpha_1})y)\overline{z_t})-\int_{\Sigma_1}(g(y_t)\overline{y_t}+h(z_t)\overline{z_t}). \tag{4.1}$$

Proof. We multiply both side (1.1) by $\overline{y}_t$ and use Green's formula to obtain

$$i\int_\Omega|y_t|^2+\int_\Gamma\frac{\partial y}{\partial \upsilon}\overline{y_t}-\int_\Omega\nabla y.\nabla\overline{y_t}+\int_\Omega(F_1+P_1)\overline{y_t}=0.$$

Taking the real part, we obtain

$$\operatorname{Re}\int_\Omega\nabla y.\nabla\overline{y_t}+\operatorname{Re}\int_\Omega q_1y\overline{y_t}=\operatorname{Re}\int_\Gamma\frac{\partial y}{\partial\upsilon}\overline{y_t}+\operatorname{Re}\int_\Omega(v_1.\nabla y+\alpha_1z)\overline{y_t}.$$

But, by (1.3)–(1.4) and (2.11) we find that

$$\operatorname{Re}\int_\Gamma\frac{\partial y}{\partial\upsilon}\overline{y_t}=-\operatorname{Re}\int_{\Gamma_1}g(y_t)\overline{y_t}=-\int_{\Gamma_1}g(y_t)\overline{y_t}$$

and

$$\operatorname{Re}\int_\Omega\nabla y.\nabla\overline{y_t}+\operatorname{Re}\int_\Omega q_1y\overline{y_t}=-\int_{\Gamma_1}g(y_t)\overline{y_t}+\operatorname{Re}\int_\Omega(v_1.\nabla y+\alpha_1z)\overline{y_t}. \tag{4.2}$$

We obtain similar identity for z:

$$\operatorname{Re}\int_\Omega\nabla z.\nabla\overline{z_t}+\operatorname{Re}\int_\Omega q_2z\overline{z_t}=-\int_{\Gamma_1}h(z_t)\overline{z_t}+\operatorname{Re}\int_\Omega(v_2.\nabla z+\alpha_2y)\overline{z_t}. \tag{4.3}$$

However,

$$\operatorname{Re}\int_\Omega(\nabla y.\nabla\overline{y_t}+\nabla z.\nabla\overline{z_t}+q_1y\overline{y_t}+q_2z\overline{z_t}-\overline{\alpha_1}(\overline{z}y_t+\overline{z_t}y))=E'(t)\,.$$

Then by (4.2)–(4.3), we find (4.1). $\square$

Lemma 4.2. *There exists a constant $\overline{C}>0$ such that we have for all $t>0$,*

$$\begin{aligned}&\operatorname{Re}\int_\Omega(v_1.\nabla y\overline{y_t}+(v_2.\nabla z+(\alpha_2-\overline{\alpha_1})y)\overline{z_t})\\&\le 2\beta\max(g^*,h^*)\Big(\int_{\Gamma_1}g(y_t)\overline{y_t}+\int_{\Gamma_1}h(z_t)\overline{z_t}\Big)\\&\quad+\frac{\beta\overline{C}}{2}\Big(\int_\Omega|\nabla z|^2+\int_\Omega|\nabla y|^2+\int_\Omega q_1|y|^2+\int_\Omega q_2|z|^2\Big).\end{aligned} \tag{4.4}$$

Moreover, if

$$\beta<\frac{1}{4\max(h^*,g^*)} \tag{4.5}$$

then, for all $T>0$, we have

$$\begin{aligned}&\int_{\Sigma_1}g(y_t)\overline{y_t}+\int_{\Sigma_1}h(z_t)\overline{z_t}\\&\le 2(E(0)-E(T))+\frac{\beta\overline{C}}{2}\Big(\int_\Omega|\nabla z|^2+\int_\Omega|\nabla y|^2+\int_\Omega q_1|y|^2+\int_\Omega q_2|z|^2\Big).\end{aligned} \tag{4.6}$$

Proof. We start with the proof of (4.4). Using (1.1), we have

$$\operatorname{Re}\int_\Omega v_1.\nabla y\overline{y_t} = -\operatorname{Re}\int_\Omega iv_1.\nabla y(\Delta\overline{y} + \overline{v_1.\nabla y} - q\overline{y} + \overline{\alpha_1 z})\,.$$

Then

$$\operatorname{Re}\int_\Omega v_1.\nabla y\overline{y_t} = \operatorname{Im}\int_\Omega v_1.\nabla y\Delta\overline{y} - \operatorname{Im}\int_\Omega v_1.\nabla y q_1\overline{y} + \operatorname{Im}\int_\Omega v_1.\nabla y\overline{\alpha_1 z}.$$

First, we consider the term $\operatorname{Im}\int_\Omega v_1.\nabla y\Delta\overline{y}$. Applying Green's formula, we obtain:

$$\operatorname{Im}\int_\Omega v_1.\nabla y\Delta\overline{y} = \operatorname{Im}\int_\Gamma \frac{\partial\overline{y}}{\partial\upsilon}v_1.\nabla y - \operatorname{Im}\int_\Omega \nabla\overline{y}.\nabla(v_1.\nabla y).$$

Indeed, with $s = v_1$, we recall the identity

$$\nabla(s.\nabla u)\nabla\overline{u} = S\nabla u.\nabla\overline{u} + \frac{1}{2}s.\nabla(|\nabla u|^2), \tag{4.7}$$

where S denotes the Hessian of s. Then

$$\operatorname{Im}\int_\Omega v_1.\nabla y\Delta\overline{y} = \operatorname{Im}\int_{\Gamma_1} \frac{\partial\overline{y}}{\partial\upsilon}v_1.\nabla y - \operatorname{Im}\int_\Omega V_1\nabla y.\nabla\overline{y} - \frac{1}{2}\int_\Omega \operatorname{Im} v_1\nabla(|\nabla y|)^2.$$

We invoke the divergence identity:

$$\int_\Omega s.\nabla\psi = \int_\Gamma \psi s.\upsilon - \int_\Omega \psi divs, \tag{4.8}$$

with $\psi = |\nabla y|^2$ and $s = v_1$. Then

$$\begin{aligned}
&\operatorname{Im}\int_\Omega v_1.\nabla y\Delta\overline{y}\\
&= \operatorname{Im}\int_{\Gamma_1} \frac{\partial\overline{y}}{\partial\upsilon}v_1.\nabla y - \operatorname{Im}\int_\Omega V_1\nabla y.\nabla\overline{y} - \frac{1}{2}\int_\Gamma \operatorname{Im} v_1.\upsilon|\nabla y|^2 + \frac{1}{2}\int_\Omega \operatorname{Im}(divv_1)|\nabla y|^2.
\end{aligned}$$

Since, $s = (s.\upsilon).\upsilon + (s.\mu).\mu$ for all vector s on Γ,

$$|\nabla y|^2 = |\frac{\partial y}{\partial\upsilon}|^2 + |\frac{\partial y}{\partial\mu}|^2.$$

Moreover,

$$\frac{\partial y}{\partial\mu}\big|_{\Gamma_0} = \nabla y.\mu\big|_{\Gamma_0} = 0, \quad \text{or} \quad |\nabla y| = |\frac{\partial y}{\partial\upsilon}| \quad \text{on } \Gamma_0,.$$

Then

$$\operatorname{Re}\int_\Omega v_1.\nabla y\overline{y_t} = R_{\Gamma_0} + R_{\Gamma_1} + R_\Omega, \tag{4.9}$$

where

$$\begin{aligned}
R_{\Gamma_0} &= \operatorname{Im}\int_{\Gamma_0} v_1.\upsilon|\nabla y|^2 - \frac{1}{2}\int_{\Gamma_0}(\operatorname{Im} v_1.\upsilon)|\nabla y|^2\\
&= \frac{1}{2}\int_{\Gamma_0}(\operatorname{Im} v_1.\upsilon)|\nabla y|^2 \le 0 \quad \text{(by (2.10))}
\end{aligned}$$

and

$$\begin{aligned}
R_{\Gamma_1} &= \int_{\Gamma_1}\operatorname{Im} v_1.\upsilon|\frac{\partial y}{\partial\upsilon}|^2 + \operatorname{Im}\int_{\Gamma_1} v_1.\mu\frac{\partial y}{\partial\mu}\frac{\partial\overline{y}}{\partial\upsilon} - \frac{1}{2}\int_{\Gamma_1}(\operatorname{Im} v_1.\upsilon)|\nabla y|^2\\
&= \frac{1}{2}\int_{\Gamma_1}\operatorname{Im} v_1.\upsilon|\frac{\partial y}{\partial\upsilon}|^2 + \operatorname{Im}\int_{\Gamma_1} v_1.\mu\frac{\partial y}{\partial\mu}\frac{\partial\overline{y}}{\partial\upsilon} - \frac{1}{2}\int_{\Gamma_1}\operatorname{Im} v_1.\upsilon|\frac{\partial y}{\partial\mu}|^2\,.
\end{aligned}$$

Then

$$R_{\Gamma_1} \le \frac{1}{2}\int_{\Gamma_1} \operatorname{Im} v_1.\upsilon|\frac{\partial y}{\partial \upsilon}|^2 + \frac{1}{2}\int_{\Gamma_1} |\operatorname{Im} v_1.\mu||\frac{\partial y}{\partial \upsilon}|^2 + \frac{1}{2}\int_{\Gamma_1}(|\operatorname{Im} v_1.\mu| - \operatorname{Im} v_1.\upsilon)|\frac{\partial y}{\partial \mu}|^2.$$

By (2.9), $R_{\Gamma_1} \le \beta\int_{\Gamma_1}|\frac{\partial y}{\partial \upsilon}|^2$. Then, by (1.4) and (2.11), we obtain

$$R_{\Gamma_1} \le \beta g^* \int_{\Gamma_1} g(y_t)\overline{y_t}$$

and (see (4.9))

$$R_\Omega = -\operatorname{Im}\int_\Omega V_1\nabla y.\nabla\overline{y} + \frac{1}{2}\int_\Omega \operatorname{Im}(divv_1)|\nabla y|^2 - \operatorname{Im}\int_\Omega v_1.\nabla y q_1\overline{y} + \operatorname{Im}\int_\Omega v_1.\nabla y\overline{\alpha_1 z}.$$

Then

$$R_\Omega \le \frac{5}{2}\beta\int_\Omega|\nabla y|^2 + \beta\frac{|q_1|}{2}\int_\Omega q_1|y|^2 + \beta|\alpha_1|^2\frac{C}{2}\int_\Omega|\nabla z|^2.$$

We insert R_{Γ_l} $(l = 0, 1)$ and R_Ω in (4.9) to obtain

$$\begin{aligned} &\operatorname{Re}\int_\Omega v_1.\nabla y\overline{y_t} \\ &\le \beta g^*\int_{\Gamma_1} g(y_t)\overline{y_t} + \frac{5}{2}\beta\int_\Omega|\nabla y|^2 + \beta\frac{|q_1|}{2}\int_\Omega q_1|y|^2 + \beta|\alpha_1|^2\frac{C}{2}\int_\Omega|\nabla z|^2. \end{aligned} \tag{4.10}$$

Using the same argument and by (2.6), we obtain

$$\begin{aligned} &\operatorname{Re}\int_\Omega(v_2.\nabla z + (\alpha_2 - \overline{\alpha_1})y)\overline{z_t} \\ &\le 2\beta h^*\int_{\Gamma_1} h(z_t)\overline{z_t} + 4\beta\int_\Omega|\nabla z|^2 + |q_2|\beta\int_\Omega q_2|z|^2 \\ &\quad + \frac{\beta}{2}\big((|\alpha_2|^2+2)C + C' + 1\big)\int_\Omega|\nabla y|^2, \end{aligned} \tag{4.11}$$

where we have put $\beta = \max(\beta, \beta^2)$. However, (4.10)–(4.11) gives

$$\begin{aligned} &\operatorname{Re}\int_\Omega(v_1.\nabla y\overline{y_t} + (v_2.\nabla z + (\alpha_2 - \overline{\alpha_1})y)\overline{z_t}) \\ &\le 2\beta\max(g^*, h^*)(\int_{\Gamma_1} g(y_t)\overline{y_t} + \int_{\Gamma_1} h(z_t)\overline{z_t}) \\ &\quad + \frac{\beta\overline{C}}{2}(\int_\Omega|\nabla z|^2 + \int_\Omega|\nabla y|^2 + \int_\Omega q_2|z|^2 + \int_\Omega q_1|y|^2), \end{aligned}$$

where $\overline{C} = \max\big\{|\alpha_1|^2C + 8, (|\alpha_2|^2+2)C + C' + 6, |q_1|, 2|q_2|\big\}$. Using (4.1), we have

$$\begin{aligned} &\int_{\Sigma_1} g(y_t)\overline{y_t} + \int_{\Sigma_1} h(z_t)\overline{z_t} \\ &= E(0) - E(T) + \operatorname{Re}\int_\Omega(v_1.\nabla y\overline{y_t} + (v_2.\nabla z + (\alpha_2 - \overline{\alpha_1})y)\overline{z_t}). \end{aligned}$$

By (4.4), we find

$$\begin{aligned}&\int_{\Sigma_1} g(y_t)\overline{y_t} + \int_{\Sigma_1} h(z_t)\overline{z_t}\\&\le E(0) - E(T) + 2\beta \max(g^*, h^*)\Big(\int_{\Gamma_1} g(y_t)\overline{y_t} + \int_{\Gamma_1} h(z_t)\overline{z_t}\Big)\\&\quad + \frac{\beta\overline{C}}{2}\Big(\int_\Omega |\nabla z|^2 + \int_\Omega |\nabla y|^2 + \int_\Omega q_2|z|^2 + \int_\Omega q_1|y|^2\Big).\end{aligned}$$

Finally, by (4.5), we obtain (4.6). □

5. Main result

We begin by stating an identity given by the multiplier method.

Lemma 5.1. *Every solution y of (1.1)–(1.5) satisfies*

$$\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2 = I_\Omega + I_{\Sigma_0} + I_{\Sigma_1} + I_Q, \tag{5.1}$$

where

$$\begin{aligned}I_\Omega &= -\frac{1}{2}\operatorname{Im}\int_\Omega (yh.\nabla\overline{y} + zh.\nabla\overline{z})\Big|_0^T,\\ I_{\Sigma_0} &= \frac{1}{2}\int_{\Sigma_0}\Big(|\frac{\partial y}{\partial \upsilon}|^2 + |\frac{\partial z}{\partial \upsilon}|^2\Big)h.\upsilon,\\ I_{\Sigma_1} &= \frac{1}{2}\operatorname{Im}\int_{\Sigma_1} h.\upsilon(y\overline{y_t} + z\overline{z_t}) + \frac{n}{2}\operatorname{Re}\int_{\Sigma_1}\Big(\frac{\partial\overline{y}}{\partial\upsilon}y + \frac{\partial\overline{z}}{\partial\upsilon}z\Big)\\ &\quad + \frac{1}{2}\int_{\Sigma_1}\Big(|\frac{\partial y}{\partial \upsilon}|^2 + |\frac{\partial z}{\partial \upsilon}|^2\Big)h.\upsilon + \operatorname{Re}\int_{\Sigma_1} h.\mu\Big(\frac{\partial\overline{y}}{\partial\mu}\frac{\partial y}{\partial\upsilon} + \frac{\partial\overline{z}}{\partial\mu}\frac{\partial z}{\partial\upsilon}\Big)\\ &\quad - \frac{1}{2}\int_{\Sigma_1} h.\upsilon\Big(|\frac{\partial y}{\partial \mu}|^2 + |\frac{\partial z}{\partial \mu}|^2\Big),\\ I_Q &= \operatorname{Re}\int_Q \big[(F_1+P_1)h.\nabla\overline{y} + (F_2+P_2)h.\nabla\overline{z}\big]\\ &\quad + \frac{n}{2}\operatorname{Re}\int_Q [(F_1+P_1)\overline{y} + (F_2+P_2)\overline{z}].\end{aligned}$$

Proof. Multiply (1.1) by $h.\nabla\overline{y}$ and integrate over Q to obtain

$$\begin{aligned}0 &= \int_Q (iy_t + \Delta y + (F_1+P_1))h.\nabla\overline{y}\\ &= i\int_Q y_t h.\nabla\overline{y} + \int_Q \Delta y h.\nabla\overline{y} + \int_Q (F_1+P_1)h.\nabla\overline{y}.\end{aligned} \tag{5.2}$$

Taking the real part,

$$0 = -\operatorname{Im}\int_Q y_t h.\nabla\overline{y} + \operatorname{Re}\int_Q \Delta y h.\nabla\overline{y} + \operatorname{Re}\int_Q (F_1+P_1)h.\nabla\overline{y}. \tag{5.3}$$

Integrating by parts,

$$\int_Q y_t h.\nabla\overline{y} = \int_\Omega yh.\nabla\overline{y}\Big|_0^T - \int_Q yh.\nabla\overline{y_t}\,.$$

Using the divergence identity (4.8) with $\operatorname{div} h = n$, we obtain

$$\int_Q y_t h.\nabla\overline{y} = \int_\Omega yh.\nabla\overline{y}\Big|_0^T - \int_\Sigma y\overline{y_t}h.\nu + n\int_Q y\overline{y_t} + \int_Q \overline{y_t}h.\nabla y.$$

Then

$$2i\operatorname{Im}\int_Q y_t h.\nabla\overline{y} = \int_\Omega yh.\nabla\overline{y}\Big|_0^T - \int_\Sigma y\overline{y_t}h.\nu + n\int_Q y\overline{y_t},$$

so

$$2\operatorname{Im}\int_Q y_t h.\nabla\overline{y} = \operatorname{Im}\int_\Omega yh.\nabla\overline{y}\Big|_0^T - \operatorname{Im}\int_\Sigma y\overline{y_t}h.\nu + n\operatorname{Im}\int_Q y\overline{y_t}. \tag{5.4}$$

However,

$$\begin{aligned}\int_Q y\overline{y_t} &= i\int_Q y(-\Delta\overline{y} - \overline{F_1 + P_1})\\ &= -i\int_\Sigma \frac{\partial\overline{y}}{\partial\upsilon}y + i\int_Q |\nabla y|^2 - i\int_Q y\overline{F_1 + P_1}.\end{aligned}$$

Then

$$\operatorname{Im}\int_Q y\overline{y_t} = -\operatorname{Re}\int_\Sigma \frac{\partial\overline{y}}{\partial\upsilon}y + \int_Q |\nabla y|^2 - \operatorname{Re}\int_Q y\overline{F_1 + P_1}\,.$$

We substitute this expression in (5.4) and we use (1.3) to obtain

$$\begin{aligned}-\operatorname{Im}\int_Q y_t h\nabla\overline{y} &= -\frac{1}{2}\operatorname{Im}\int_\Omega yh\nabla\overline{y}\Big|_0^T + \frac{1}{2}\operatorname{Im}\int_{\Sigma_1} y\overline{y_t}h.\nu\\ &\quad + \frac{n}{2}\operatorname{Re}\int_{\Sigma_1}\frac{\partial\overline{y}}{\partial\upsilon}y - \frac{n}{2}\int_Q |\nabla y|^2 + \frac{n}{2}\operatorname{Re}\int_Q \overline{F_1 + P_1}y.\end{aligned} \tag{5.5}$$

Concerning the term $\int_Q \Delta yh\nabla\overline{y}$, if we use Green's Formula, identity (4.7), we find

$$\begin{aligned}\int_Q \Delta yh\nabla\overline{y} &= \int_\Sigma \frac{\partial y}{\partial\upsilon}h\nabla\overline{y} - \int_Q \nabla y\nabla(h\nabla\overline{y})\\ &= \int_\Sigma \frac{\partial y}{\partial\upsilon}h\nabla\overline{y} - \int_Q |\nabla y|^2 - \frac{1}{2}\int_Q h\nabla(|\nabla y|^2)\,.\end{aligned}$$

We use the divergence identity (4.8) to find

$$\operatorname{Re}\int_Q \Delta yh\nabla\overline{y} = \operatorname{Re}\int_\Sigma \frac{\partial y}{\partial\upsilon}h\nabla\overline{y} - \frac{1}{2}\int_\Sigma |\nabla y|^2 h.\upsilon + (\frac{n}{2} - 1)\int_Q |\nabla y|^2. \tag{5.6}$$

However, in Γ,

$$\begin{gathered}|\nabla y|^2 = |\frac{\partial y}{\partial\upsilon}|^2 + |\frac{\partial y}{\partial\mu}|^2,\\ \frac{\partial y}{\partial\upsilon}h\nabla\overline{y} = |\frac{\partial y}{\partial\upsilon}|^2 h.\upsilon + \frac{\partial y}{\partial\upsilon}\frac{\partial\overline{y}}{\partial\mu}h.\mu\end{gathered}$$

and in Γ_0, $\frac{\partial y}{\partial\mu} = 0$. Then from (5.6),

$$\begin{aligned}\operatorname{Re}\int_Q \Delta yh\nabla\overline{y} &= \frac{1}{2}\int_{\Sigma_0}|\frac{\partial y}{\partial\upsilon}|^2 h.\upsilon + \frac{1}{2}\int_{\Sigma_1}|\frac{\partial y}{\partial\upsilon}|^2 h.\upsilon\\ &\quad + \operatorname{Re}\int_{\Sigma_1}\frac{\partial y}{\partial\upsilon}\frac{\partial\overline{y}}{\partial\mu}h.\mu - \frac{1}{2}\int_{\Sigma_1}|\frac{\partial y}{\partial\mu}|^2 h.\upsilon + (\frac{n}{2} - 1)\int_Q |\nabla y|^2.\end{aligned} \tag{5.7}$$

Finally, we insert (5.5) and (5.7) in (5.3) to obtain

$$\begin{aligned}
\int_Q |\nabla y|^2 = &-\frac{1}{2}\operatorname{Im}\int_\Omega yh\nabla\overline{y}\Big|_0^T + \frac{1}{2}\int_{\Sigma_0}|\frac{\partial y}{\partial \upsilon}|^2 h.\upsilon + \frac{1}{2}\operatorname{Im}\int_{\Sigma_1} y\overline{y_t}h.\nu \\
&+ \frac{n}{2}\operatorname{Re}\int_{\Sigma_1}\frac{\partial \overline{y}}{\partial \upsilon}y + \frac{1}{2}\int_{\Sigma_1}|\frac{\partial y}{\partial \upsilon}|^2 h.\upsilon + \operatorname{Re}\int_{\Sigma_1}\frac{\partial y}{\partial \upsilon}\frac{\partial \overline{y}}{\partial \mu}h.\mu \\
&- \frac{1}{2}\int_{\Sigma_1}|\frac{\partial y}{\partial \mu}|^2 h.\upsilon + \frac{n}{2}\operatorname{Re}\int_Q \overline{F_1 + P_1}y + \operatorname{Re}\int_Q \overline{F_1 + P_1}h.\nabla\overline{y}.
\end{aligned} \tag{5.8}$$

We obtain a similar identity for z. From such inequality and (5.8), we deduce (5.1). □

Our main result is the following.

Theorem 5.2. *The solution of* (1.1)–(1.5) *is exponentially stable.*

Proof. From (4.1), (2.11), (2.12), (4.4), (4.5) and Remark 2.1, we have

$$E'(t) \le \lambda_3 E(t), \quad \text{for all } t > 0.$$

Now, we prove that there are two constants λ_1 and λ_2, $(\lambda_1 \le \lambda_2)$, such that

$$\int_0^T E(t) \le \lambda_1(E(T) + E(0)) + \lambda_2(E(0) - E(T)), \quad \text{for all } T > 0.$$

We bound the terms: I_Ω, $I_{\Sigma_l}(l = 0, 1)$ and I_Q in (5.1). For all $\varepsilon_1 > 0$,

$$I_\Omega \le R\big(\frac{C}{\varepsilon_1} + \varepsilon_1\big)\frac{1}{\zeta_1}(E(T) + E(0)).$$

By Cauchy-Scwartz,

$$\begin{aligned}
I_Q \le &\frac{\beta}{2}(2R + \frac{n}{2}(C+1))\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big) \\
&+ \frac{\max_{l=1,2}|q_l|}{2}R(C+1)\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big) \\
&+ \frac{n}{2}\operatorname{Re}\int_Q(\alpha_1 z\overline{y} + \alpha_2 y\overline{z}) + \operatorname{Re}\int_Q(\alpha_1 zh.\nabla\overline{y} + \alpha_2 yh.\nabla\overline{z}) \\
&- \frac{n}{2}\Big(\int_Q q_1|y|^2 + \int_Q q_2|z|^2\Big).
\end{aligned}$$

However,

$$\begin{aligned}
&\frac{n}{2}\operatorname{Re}\int_Q(\alpha_1 z\overline{y} + \alpha_2 y\overline{z}) \\
&= n\operatorname{Re}\int_Q \alpha_1 z\overline{y} + \frac{n}{2}\operatorname{Re}\int_Q(\alpha_2 - \overline{\alpha_1})y\overline{z} \\
&\le n|\alpha_1|\frac{C}{2}\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big) + \frac{n}{2}\beta(C+1)\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Re}\int_Q (\alpha_1 zh.\nabla\overline{z} + \alpha_2 yh.\nabla\overline{y})\\
&= \operatorname{Re}\int_Q (\alpha_1(zh.\nabla\overline{z} + \overline{y}h.\nabla y) + (\alpha_2 - \overline{\alpha_1})yh.\nabla\overline{y})\\
&\le R|\alpha_1|\frac{(C+1)}{2}\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big) + \frac{\beta R(C+1)}{2}\int_Q |\nabla y|^2\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_Q \le{}& \frac{\beta}{2}\overline{\overline{C}}\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big) + \frac{\max_{l=1,2}(|q_l|)}{2}R(C+1)\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big)\\
&+ \frac{|\alpha_1|}{2}((n+R)C+R)\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big) - \frac{n}{2}(\int_Q q_1|y|^2 + \int_Q q_2|z|^2),
\end{aligned}$$

where $\overline{\overline{C}} = 2R + (\frac{3n}{2} + R)(C+1)$. By (2.8), we find

$$\begin{aligned}
I_Q \le{}& \frac{\beta}{2}\overline{\overline{C}}\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big)\\
&+ \Big(\frac{\min(1,\frac{n}{2})}{2} - \frac{|\alpha_1|}{2}((n+R)C+R)\Big)\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big)\\
&+ \frac{|\alpha_1|}{2}((n+R)C+R)\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big) - \frac{n}{2}\Big(\int_Q q_1|y|^2 + \int_Q q_2|z|^2\Big)\,.
\end{aligned}$$

So that

$$\begin{aligned}
I_Q \le{}& \Big(\frac{\min(1,\frac{n}{2})}{2} + \frac{\beta}{2}\overline{\overline{C}}\Big)\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2 + \int_\Omega q_1|y|^2 + \int_\Omega q_2|z|^2\Big)\\
&- \frac{n}{2}\Big(\int_Q q_1|y|^2 + \int_Q q_2|z|^2\Big).
\end{aligned}$$

Concerning the term I_{Σ_l}, we have $I_{\Sigma_0} \le 0$. Put

$$I_{\Sigma_1} = I_{\Sigma_1}(y) + I_{\Sigma_1}(z),$$

where

$$\begin{aligned}
I_{\Sigma_1}(y) ={}& \frac{1}{2}\operatorname{Im}\int_{\Sigma_1} h.\upsilon y\overline{y_t} + \frac{n}{2}\operatorname{Re}\int_{\Sigma_1}\frac{\partial\overline{y}}{\partial\upsilon}y + \frac{1}{2}\int_{\Sigma_1}|\frac{\partial y}{\partial\upsilon}|^2 h.\upsilon\\
&+ \operatorname{Re}\int_{\Sigma_1} h.\mu\frac{\partial\overline{y}}{\partial\mu}\frac{\partial y}{\partial\upsilon} - \frac{1}{2}\int_{\Sigma_1} h.\upsilon|\frac{\partial y}{\partial\mu}|^2
\end{aligned}$$

and

$$\begin{aligned}
I_{\Sigma_1}(z) ={}& \frac{1}{2}\operatorname{Im}\int_{\Sigma_1} h.\upsilon z\overline{z_t} + \frac{n}{2}\operatorname{Re}\int_{\Sigma_1}\frac{\partial\overline{z}}{\partial\upsilon}z + \frac{1}{2}\int_{\Sigma_1}|\frac{\partial z}{\partial\upsilon}|^2 h.\upsilon\\
&+ \operatorname{Re}\int_{\Sigma_1} h.\mu\frac{\partial\overline{z}}{\partial\mu}\frac{\partial z}{\partial\upsilon} - \frac{1}{2}\int_{\Sigma_1} h.\upsilon(|\frac{\partial y}{\partial\mu}|^2 + |\frac{\partial z}{\partial\mu}|^2)\,.
\end{aligned}$$

We start with $I_{\Sigma_1}(y)$. Using Cauchy-Scwartz and (2.1), for $\varepsilon_2 > 0$, we obtain

$$\begin{aligned} I_{\Sigma_1}(y) \le & RC'\varepsilon_2 \int_Q |\nabla y|^2 + R\frac{1}{\varepsilon_2}\int_{\Sigma_1} |y_t|^2 + n\varepsilon_2 C' \int_Q |\nabla y|^2 + n\frac{1}{\varepsilon_2}\int_{\Sigma_1} |\frac{\partial y}{\partial \upsilon}|^2 \\ & + \frac{R}{2}\int_{\Sigma_1} |\frac{\partial y}{\partial \upsilon}|^2 + \frac{R}{2}\Big(\frac{1}{\tau}\int_{\Sigma_1} |\frac{\partial y}{\partial \upsilon}|^2 + \tau\int_{\Sigma_1} |\frac{\partial y}{\partial \mu}|^2\Big) - h_0\int_{\Sigma_1} |\frac{\partial y}{\partial \mu}|^2. \end{aligned}$$

Choosing $\tau = \frac{2h_0}{R}$ and using (1.4), (2.11) we obtain

$$I_{\Sigma_1}(y) \le C'\varepsilon_2(R+n)\int_Q |\nabla y|^2 + \Big(\frac{R}{g_*\varepsilon_2} + \big(\frac{n}{\varepsilon_2} + R + \frac{R^2}{4h_0}\big)g^*\Big)\int_{\Sigma_1} g(y_t)\overline{y_t},$$

similarly for $I_{\Sigma_1}(z)$,

$$I_{\Sigma_1}(z) \le C'\varepsilon_2(R+n)\int_Q |\nabla z|^2 + \Big(\frac{R}{h_*\varepsilon_2} + \big(\frac{n}{\varepsilon_2} + R + \frac{R^2}{4h_0}\big)h^*\Big)\int_{\Sigma_1} h(z_t)\overline{z_t}.$$

Therefore,

$$\begin{aligned} I_\Sigma \le & C'\varepsilon_2(R+n)\Big(\int_Q |\nabla y|^2 + \int_Q |\nabla z|^2\Big) \\ & + \Big(\frac{R}{\min(g_*, h_*)\varepsilon_2} + \big(\frac{n}{\varepsilon_2} + R + \frac{R^2}{4h_0}\big)\max(g^*, h^*)\Big)\int_{\Sigma_1} (g(y_t)\overline{y_t} + h(z_t)\overline{z_t})\,. \end{aligned}$$

Using (4.6) we find

$$\begin{aligned} I_\Sigma \le & \lambda_2(E(0) - E(T)) \\ & + \big(C'\varepsilon_2(R+n) + \beta\overline{C}\lambda_2\big)\Big(\int_\Omega |\nabla z|^2 + \int_\Omega |\nabla y|^2 + \int_\Omega q_1|y|^2 + \int_\Omega q_2|z|^2\Big), \end{aligned}$$

where

$$\lambda_2 = 2\Big(\frac{R}{\min(g_*, h_*)\varepsilon_2} + \big(\frac{n}{\varepsilon_2} + R + \frac{R^2}{4h_0}\big)\max(g^*, h^*)\Big).$$

We insert this in (5.1) to obtain

$$\begin{aligned} & \frac{1}{2}\Big(\min(1, \frac{n}{2}) - \beta(\overline{C} + \lambda_2\overline{\overline{C}}) - 2C'\varepsilon_2(R+n)\Big) \\ & \times \Big(\int_\Omega |\nabla z|^2 + \int_\Omega |\nabla y|^2 + \int_\Omega q_2|z|^2 + \int_\Omega q_1|y|^2\Big) \\ & \le \lambda_1\big(E(T) + E(0)\big)) + \lambda_2\big(E(0) - E(T)\big), \end{aligned}$$

where, we have put

$$\lambda_1 = \frac{R}{\zeta_1}(\frac{C}{\varepsilon_1} + \varepsilon_1).$$

We choose

$$\beta \le \frac{\min(1, \frac{n}{2})}{\overline{C} + \lambda_2\overline{\overline{C}}}$$

and

$$\varepsilon_2 < \min\Big(\frac{\min(1, \frac{n}{2}) - \beta(\overline{C} + \lambda_2\overline{\overline{C}})}{2C'(R+n)}, \sqrt{\frac{2\zeta_1^2 n\max(g^*, h^*)}{RC\min(g_*, h_*)}}\Big).$$

On the other hand, if we choose ε_1 such that

$$\frac{\varepsilon_2 CR}{\zeta_1\max(g^*, h^*)} < \varepsilon_1 < \frac{2\zeta_1}{\min(g_*, h_*)\varepsilon_2},$$

then $\lambda_1 \le \lambda_2$. □

References

[1] A. Guesmia, *A New Approach of Stabilisation of Nondissipative Distributed Systems*, SIAM J. Control Optimi., 42 (2003), no. 1, 24-52.

[2] V. Komornick, *Exact Controlability and Stabilisation*. The multiplier method, Masson Paris (1994).

[3] I. Lasieka, R. Triggiani and X. Zhang, *Global Uniqueness, Observability and Stabilisation of Nonconservative Schrodinger Equations Via Pointwise Carleman Estimates*. J. INV. III-Posed Problems 11 (2003), no. 3, 1-96.

[4] J. L. Lions, *Quelque Methodes de Resolution des Problemes aux Limites Non Lineaire*. Gauthier Villars 1969.

[5] E. Machtyngier and E. Zuazua, *Stabilisation of Schrodinger Equation*, Portugaliae Mathematica 51 (1994), no. 2, 243-256.

[6] E. Machtyngier, *Exact Controllability for Schrodinger equation*. SIAM J. Cont. and Optim. 32 (1994), 24-34.

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