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L^2-BOUNDEDNESS AND L^2-COMPACTNESS OF A CLASS OF FOURIER INTEGRAL OPERATORS

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Abstract. In this paper, we study the L^2-boundedness and L^2-compactness of a class of Fourier integral operators. These operators are bounded (respectively compact) if the weight of the amplitude is bounded (respectively tends to 0).

1. Introduction

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space), the integral operators

$$F\varphi(x) = \int e^{iS(x,\theta)} a(x,\theta)\mathcal{F}\varphi(\theta)\, d\theta \tag{1.1}$$

appear naturally in the expression of the solutions of the hyperbolic partial differential equations and in the expression of the C^∞ -solution of the associate Cauchy's problem (see [5, 10]).

If we write formally the Fourier transformation $\mathcal{F}\varphi(\theta)$ in (1.1), we obtain the following Fourier integral operators

$$F\varphi(x) = \iint e^{i(S(x,\theta)-y\theta)} a(x,\theta)\varphi(y) dy\, d\theta \tag{1.2}$$

in which appear two C^∞-functions, the phase function $\phi(x, y, \theta) = S(x, \theta) - y\theta$ and the amplitude a.

Since 1970, many efforts have been made by several authors in order to study these type of operators (see, e.g., [1, 4, 6, 7, 8, 15]). The first works on Fourier integral operators deal with local properties. On the other hand, Asada and Fujiwara have studied for the first time a class of Fourier integral operators defined on $\mathbb{R}^n$.

For the Fourier integral operators, an interesting question is under which conditions on a and S these operators are bounded on L^2 or are compact on L^2.

It has been proved in [1] by a very elaborated proof and with some hypothesis on the phase function ϕ and the amplitude a that all operators of the form (2.1) (see below) are bounded on L^2. The technique used there is based on the fact that the operators $I(a, \phi)I^*(a, \phi), I^*(a, \phi)I(a, \phi)$ are pseudodifferential and it uses Caldéron-Vaillancourt's theorem (here $I(a, \phi)^*$ is the adjoint of $I(a, \phi)$).

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In this work, we apply the same technique of [1] to establish the boundedness and the compactness of the operators (1.2). To this end we give a brief and simple proof for a result of [1] in our framework.

We mainly prove the continuity of the operator F on $L^2(\mathbb{R}^n)$ when the weight of the amplitude a is bounded. Moreover, F is compact on $L^2(\mathbb{R}^n)$ if this weight tends to zero. Using the estimate given in [12] for h-pseudodifferential (h-admissible) operators, we also establish an L^2-estimate of $\|F\|$.

We note that if the amplitude a is juste bounded, the Fourier integral operator F is not necessarily bounded on $L^2(\mathbb{R}^n)$. Recently, Hasanov [6] and Messirdi-Senoussaoui [11] constructed a class of unbounded Fourier integral operators with an amplitude in the Hörmander's class $S^0_{1,1}$ and in $\bigcap_{0<\rho<1} S^0_{\rho,1}$.

To our knowledge, this work constitutes a first attempt to diagonalize the Fourier integral operators on $L^2(\mathbb{R}^n)$ (relying on the compactness of these operators).

Let us now describe the plan of this article. In the second section we recall the continuity of some general class of Fourier integral operators on $\mathcal{S}(\mathbb{R}^n)$ and on $\mathcal{S}'(\mathbb{R}^n)$. The assumptions and preliminaries results are given in the third section. The last section is devoted to prove the main result.

2. A general class of Fourier integral operators

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we consider the following integral transformations

$$(I(a,\phi)\varphi)(x) = \iint_{\mathbb{R}^n_y \times \mathbb{R}^N_\theta} e^{i\phi(x,\theta,y)} a(x,\theta,y)\varphi(y)dy\,d\theta \tag{2.1}$$

where, $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ and $N \in \mathbb{N}$ (if $N = 0$, θ doesn't appear in (2.1)).

In general the integral (2.1) is not absolutely convergent, so we use the technique of the oscillatory integral developed by Hörmander in [8]. The phase function ϕ and the amplitude a are assumed to satisfy the following hypothesis:

(H1) $\phi \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y, \mathbb{R})$ (ϕ is a real function)

(H2) For all $(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n$, there exists $C_{\alpha,\beta,\gamma} > 0$ such that

$$|\partial_y^\gamma \partial_\theta^\beta \partial_x^\alpha \phi(x,\theta,y)| \le C_{\alpha,\beta,\gamma} \lambda^{(2-|\alpha|-|\beta|-|\gamma|)_+}(x,\theta,y)$$

where $\lambda(x,\theta,y) = (1+|x|^2+|\theta|^2+|y|^2)^{1/2}$ is called the weight and

$$(2-|\alpha|-|\beta|-|\gamma|)_+ = \max(2-|\alpha|-|\beta|-|\gamma|, 0)$$

(H3) There exist $K_1, K_2 > 0$ such that

$$K_1\lambda(x,\theta,y) \le \lambda(\partial_y\phi, \partial_\theta\phi, y) \le K_2\lambda(x,\theta,y), \quad \forall (x,\theta,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$$

(H3*) There exist $K_1^*, K_2^* > 0$ such that

$$K_1^*\lambda(x,\theta,y) \le \lambda(x, \partial_\theta\phi, \partial_x\phi) \le K_2^*\lambda(x,\theta,y), \quad \forall (x,\theta,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y.$$

For any open subset Ω of $\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, $\mu \in \mathbb{R}$ and $\rho \in [0,1]$, we set

$$\begin{aligned}\Gamma^\mu_\rho(\Omega) = \big\{ & a \in C^\infty(\Omega) : \forall(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n,\ \exists C_{\alpha,\beta,\gamma} > 0 : \\ & |\partial_y^\gamma \partial_\theta^\beta \partial_x^\alpha a(x,\theta,y)| \le C_{\alpha,\beta,\gamma} \lambda^{\mu-\rho(|\alpha|+|\beta|+|\gamma|)}(x,\theta,y) \big\}\end{aligned}$$

When $\Omega = \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, we denote $\Gamma^\mu_\rho(\Omega) = \Gamma^\mu_\rho$.

To give a meaning to the right hand side of (2.1), we consider $g \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y)$, $g(0) = 1$. If $a \in \Gamma^\mu_0$, we define

$$a_\sigma(x,\theta,y) = g(x/\sigma, \theta/\sigma, y/\sigma)a(x,\theta,y), \quad \sigma > 0.$$

Now we are able to state the following result.

Theorem 2.1. *If ϕ satisfies (H1), (H2), (H3) and (H3*), and if $a \in \Gamma^\mu_0$, then*

1. For all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\lim_{\sigma\to+\infty}[I(a_\sigma,\phi)\varphi](x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function g. We define

$$(I(a,\phi)\varphi)(x) := \lim_{\sigma\to+\infty}(I(a_\sigma,\phi)\varphi)(x)$$

2. $I(a,\phi) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ and $I(a,\phi) \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$ (here $\mathcal{L}(E)$ is the space of bounded linear mapping from E to E and $\mathcal{S}'(\mathbb{R}^n)$ the space of all distributions with temperate growth on $\mathbb{R}^n$).

The proof of the above theorem can be found in [7] or in [12, propostion II.2].

Example 2.2. Let us give two examples of operators of the form (2.1) which satisfy (H1)-(H3*):

(1) The Fourier transform $\mathcal{F}\psi(x) = \int_{\mathbb{R}^n} e^{-ixy}\psi(y)dy$, $\psi \in \mathcal{S}(\mathbb{R}^n)$,
(2) Pseudodifferential operators

$$A\psi(x) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\theta}a(x,y,\theta)\psi(y)dy\,d\theta,$$

with $\psi \in \mathcal{S}(\mathbb{R}^n)$, $a \in \Gamma^\mu_0(\mathbb{R}^{3n})$.

3. Assumptions and Preliminaries

In this paper we consider the special form of the phase function

$$\phi(x,y,\theta) = S(x,\theta) - y\theta \tag{3.1}$$

where S satisfies

(G1) $S \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\theta, \mathbb{R})$,
(G2) For each $(\alpha,\beta) \in \mathbb{N}^{2n}$, there exist $C_{\alpha,\beta} > 0$, such that

$$|\partial^\alpha_x\partial^\beta_\theta S(x,\theta)| \le C_{\alpha,\beta}\lambda(x,\theta)^{(2-|\alpha|-|\beta|)_+},$$

(G3) There exists $C_1 > 0$ such that $|x| \le C_1\lambda(\theta,\partial_\theta S)$, for all $(x,\theta) \in \mathbb{R}^{2n}$,
(G3*) There exists $C_2 > 0$, such that $|\theta| \le C_2\lambda(x,\partial_x S)$, for all $(x,\theta) \in \mathbb{R}^{2n}$.

Proposition 3.1. *Let's assume that S satisfies (G1), (G2), (G3) and (G3*). Then the function $\phi(x,y,\theta) = S(x,\theta) - y\theta$ satisfies (H1), (H2), (H3) and (H3*).*

Proof. (H1) and (H2) are trivially satisfied. The condition (G3) implies

$$\lambda(x,\theta,y) \le \lambda(x,\theta) + \lambda(y) \le C_3(\lambda(\theta,\partial_\theta S) + \lambda(y)), \quad C_3 > 0.$$

Also, we have $\partial_{y_j}\phi = -\theta_j$ and $\partial_{\theta_j}\phi = \partial_{\theta_j}S - y_j$ and so

$$\lambda(\theta,\partial_\theta S) = \lambda(\partial_y\phi, \partial_\theta\phi + y) \le 2\lambda(\partial_y\phi,\partial_\theta\phi,y),$$

which finally gives for some $C_4 > 0$,

$$\lambda(x,\theta,y) \le C_3(2\lambda(\partial_y\phi,\partial_\theta\phi,y) + \lambda(y)) \le \frac{1}{C_4}\lambda(\partial_y\phi,\partial_\theta\phi,y)$$

The second inequality in (H3) is a consequence of the assumption (G2). By a similar argument we can show (H3*). $\square$

We now introduce the assumption

(G4) There exists $\delta_0 > 0$ such that
$$\inf_{x,\theta\in\mathbb{R}^n} |\det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)| \geq \delta_0.$$
We note that if $\phi(x,y,\theta) = S(x,\theta) - y\theta$, then
$$D(\phi)(x,\theta,y) = \begin{pmatrix} \frac{\partial^2\phi}{\partial x\partial y}(x,\theta,y) & \frac{\partial^2\phi}{\partial x\partial\theta}(x,\theta,y) \\ \frac{\partial^2\phi}{\partial\theta\partial y}(x,\theta,y) & \frac{\partial^2\phi}{\partial\theta\partial\theta}(x,\theta,y) \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta) \\ -I_n & \frac{\partial^2 S}{\partial\theta\partial\theta}(x,\theta) \end{pmatrix}$$
and
$$\big|\det D(\phi)(x,\theta,y)\big| = \big|\det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)\big| \geq \delta_0.$$

Remark 3.2. By the global implicit function theorem (cf. [14], [3, theorem 4.1.7]) and using (G1), (G2) and (G4), we can easily see that the mappings h_1 and h_2 defined by
$$h_1 : (x,\theta) \to (x, \partial_x S(x,\theta)), \quad h_2 : (x,\theta) \to (\theta, \partial_\theta S(x,\theta))$$
are global diffeomorphism of $\mathbb{R}^{2n}$. Indeed,
$$h_1'(x,\theta) = \begin{pmatrix} I_n & \frac{\partial^2 S}{\partial x^2}(x,\theta) \\ 0 & \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta) \end{pmatrix}, \quad h_2'(x,\theta) = \begin{pmatrix} 0 & \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta) \\ I_n & \frac{\partial^2 S}{\partial\theta^2}(x,\theta) \end{pmatrix}.$$
and $|\det h_1'(x,\theta)| = |\det h_2'(x,\theta)| = |\det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)| \geq \delta_0 > 0$, for all $(x,\theta) \in \mathbb{R}^{2n}$. Then
$$\|(h_1'(x,\theta))^{-1}\| = \frac{1}{|\det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)|}\|{}^tA(x,\theta)\|$$
$$\|(h_2'(x,\theta))^{-1}\| = \frac{1}{|\det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)|}\|{}^tB(x,\theta)\|,$$
where $A(x,\theta)$, $B(x,\theta)$ are respectively the cofactor matrix of $h_1'(x,\theta)$, $h_2'(x,\theta)$. By (G2), we know that $\|{}^tA(x,\theta)\|$ and $\|{}^tB(x,\theta)\|$ are uniformly bounded.

Let's now assume that S satisfies the following condition which is stronger than (G2).

(G5) For all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exist $C_{\alpha,\beta} > 0$, such that
$$|\partial_x^\alpha \partial_\theta^\beta S(x,\theta)| \leq C_{\alpha,\beta}\lambda(x,\theta)^{(2-|\alpha|-|\beta|)}.$$

Lemma 3.3. *If S satisfies (G1), (G4) and (G5), then S satisfies (G3) and (G3*). Also there exists $C_5 > 0$ such that for all $(x,\theta), (x',\theta') \in \mathbb{R}^{2n}$,*
$$|x - x'| + |\theta - \theta'| \leq C_5\big[|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(x',\theta')| + |\theta - \theta'|\big] \tag{3.2}$$

Proof. The mappings
$$\mathbb{R}^n \ni \theta \to f_x(\theta) = \partial_x S(x,\theta), \quad \mathbb{R}^n \ni x \to g_\theta(x) = \partial_\theta S(x,\theta)$$
are global diffeomorphisms of $\mathbb{R}^n$. From (G4) and (G5), it follows that $\|(f_x^{-1})'\|$, $\|(g_\theta^{-1})'\|$ and $\|(h_2^{-1})'\|$ are uniformly bounded on $\mathbb{R}^{2n}$. Thus (G5) and the Taylor's theorem lead to the following estimates: There exist $M, N > 0$, such that for all $(x,\theta), (x',\theta') \in \mathbb{R}^{2n}$,
$$|\theta| = |\ f_x^{-1}(f_x(\theta)) - f_x^{-1}(f_x(0))| \leq M|\partial_x S(x,\theta) - \partial_x S(x,0)| \leq C_6\lambda(x,\partial_x S),$$

with $C_6 > 0$;

$$|x| = |\ g_\theta^{-1}(g_\theta(\theta)) - g_\theta^{-1}(g_\theta(0))| \leq N|\partial_\theta S(x,\theta) - \partial_\theta S(0,\theta)| \leq C_7\lambda(\partial_\theta S,\theta),$$

with $C_7 > 0$;

$$\begin{aligned}|(x,\theta)-(x',\theta')| &= |h_2^{-1}(h_2(x,\theta)) - h_2^{-1}(h_2(x',\theta'))| \\ &\leq C_5|(\theta,\partial_\theta S\ (x,\theta)) - (\theta',\partial_\theta S\ (x',\theta'))|\end{aligned}$$

□

When $\theta = \theta'$ in (3.2), there exists $C_5 > 0$, such that for all $(x,x',\theta) \in \mathbb{R}^{3n}$,

$$|x - x'| \leq C_5|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(x',\theta)|. \tag{3.3}$$

Proposition 3.4. *If S satisfies (G1) and (G5), then there exists a constant $\epsilon_0 > 0$ such that the phase function ϕ given in (3.1) belongs to $\Gamma_1^2(\Omega_{\phi,\epsilon_0})$ where*

$$\Omega_{\phi,\epsilon_0} = \big\{(x,\theta,y) \in \mathbb{R}^{3n};\ \ |\partial_\theta S(x,\theta) - y|^2 < \epsilon_0\ (|x|^2 + |y|^2 + |\theta|^2)\big\}.$$

Proof. We have to show that: There exists $\epsilon_0 > 0$, such that for all $\alpha,\beta,\gamma \in \mathbb{N}^n$, there exist $C_{\alpha,\beta,\gamma} > 0$:

$$|\partial_x^\alpha\partial_\theta^\beta\partial_y^\gamma\phi(x,\theta,y)| \leq C_{\alpha,\beta,\gamma}\lambda(x,\theta,y)^{(2-|\alpha|-|\beta|-|\gamma|)}, \quad \forall(x,\theta,y) \in \Omega_{\phi,\epsilon_0}. \tag{3.4}$$

If $|\gamma| = 1$, then

$$|\partial_x^\alpha\partial_\theta^\beta\partial_y^\gamma\phi(x,\theta,y)| = |\partial_x^\alpha\partial_\theta^\beta(-\theta)| = \begin{cases}0 & \text{if } |\alpha| \neq 0\\ |\partial_\theta^\beta(-\theta)| & \text{if } \alpha = 0;\end{cases}$$

If $|\gamma| > 1$, then $|\partial_x^\alpha\partial_\theta^\beta\partial_y^\gamma\phi(x,\theta,y)| = 0$.
Hence the estimate (3.4) is satisfied.

If $|\gamma| = 0$, then for all $\alpha,\beta \in \mathbb{N}^n$; $|\alpha| + |\beta| \leq 2$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha\partial_\theta^\beta\phi(x,\theta,y)| = |\partial_x^\alpha\partial_\theta^\beta S(x,\theta) - \partial_x^\alpha\partial_\theta^\beta(y\theta)| \leq C_{\alpha,\beta}\lambda(x,\theta,y)^{(2-|\alpha|-|\beta|)}.$$

If $|\alpha| + |\beta| > 2$, one has $\partial_x^\alpha\partial_\theta^\beta\phi(x,\theta,y) = \partial_x^\alpha\partial_\theta^\beta S(x,\theta)$. In Ω_{ϕ,ϵ_0} we have

$$|y| = |\partial_\theta S(x,\theta) - y - \partial_\theta S(x,\theta)| \leq \sqrt{\epsilon_0}(|x|^2 + |y|^2 + |\theta|^2)^{1/2} + C_8\lambda(x,\theta),$$

with $C_8 > 0$. For ϵ_0 sufficiently small, we obtain a constant $C_9 > 0$ such that

$$|y| \leq C_9\lambda(x,\theta), \quad \forall(x,\theta,y) \in \Omega_{\phi,\epsilon_0}\ . \tag{3.5}$$

This inequality leads to the equivalence

$$\lambda(x,\theta,y) \simeq \lambda(x,\theta) \quad \text{in } \Omega_{\phi,\epsilon_0} \tag{3.6}$$

thus the assumption $(G5)$ and (3.6) give the estimate (3.4). □

Using (3.6), we have the following result.

Proposition 3.5. *If $(x,\theta) \to a(x,\theta)$ belongs to $\Gamma_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n)$, then $(x,\theta,y) \to a(x,\theta)$ belongs to $\Gamma_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n \times \mathbb{R}_y^n) \cap \Gamma_k^m(\Omega_{\phi,\epsilon_0}\ )$, $k \in \{0,1\}$.*

4. L^2-boundedness and L^2-compactness of F

The main result is as follows.

Theorem 4.1. *Let F be the integral operator of distribution kernel*
$$K(x,y)=\int_{\mathbb{R}^n} e^{i(S(x,\theta)-y\theta)}a(x,\theta)\widehat{d\theta} \tag{4.1}$$
where $\widehat{d\theta}=(2\pi)^{-n}\,d\theta$, $a\in\Gamma_k^m(\mathbb{R}^{2n}_{x,\theta})$, $k=0,1$ and S satisfies (G1), (G4) and (G5). Then FF^* and F^*F are pseudodifferential operators with symbol in $\Gamma_k^{2m}(\mathbb{R}^{2n})$, $k=0,1$, given by
$$\sigma(FF^*)(x,\partial_x S(x,\theta))\equiv|a(x,\theta)|^2|(\det\frac{\partial^2 S}{\partial\theta\partial x})^{-1}(x,\theta)|$$
$$\sigma(F^*F)(\partial_\theta S(x,\theta),\theta)\equiv|a(x,\theta)|^2|(\det\frac{\partial^2 S}{\partial\theta\partial x})^{-1}(x,\theta)|$$
we denote here $a\equiv b$ for $a,b\in\Gamma_k^{2p}(\mathbb{R}^{2n})$ if $(a-b)\in\Gamma_k^{2p-2}(\mathbb{R}^{2n})$ and σ stands for the symbol.

Proof. If $u\in\mathcal{S}(\mathbb{R}^n)$, then $Fu(x)$ is given by
$$\begin{aligned} Fu(x)&=\int_{\mathbb{R}^n}K(x,y)u(y)\,dy\\ &=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(S(x,\theta)-y\theta)}a(x,\theta)u(y)dy\widehat{d\theta}\\ &=\int_{\mathbb{R}^n}e^{iS(x,\theta)}a(x,\theta)\Big(\int_{\mathbb{R}^n}e^{-iy\theta}u(y)dy\Big)\widehat{d\theta}\\ &=\int_{\mathbb{R}^n}e^{iS(x,\theta)}a(x,\theta)\mathcal{F}u(\theta)\widehat{d\theta}. \end{aligned} \tag{4.2}$$
Here F is a continuous linear mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ (by Theorem 2.1). Let $v\in\mathcal{S}(\mathbb{R}^n)$, then
$$\begin{aligned}\langle Fu,v\rangle_{L^2(\mathbb{R}^n)}&=\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}e^{iS(x,\theta)}a(x,\theta)\mathcal{F}u(\theta)\widehat{d\theta}\Big)\overline{v(x)}\,dx\\ &=\int_{\mathbb{R}^n}\mathcal{F}u(\theta)\Big(\int_{\mathbb{R}^n}\overline{e^{-iS(x,\theta)}\overline{a(x,\theta)}v(x)\,dx}\Big)\widehat{d\theta}\end{aligned}$$
thus
$$\langle Fu(x),v(x)\rangle_{L^2(\mathbb{R}^n)}=(2\pi)^{-n}\langle\mathcal{F}u(\theta),\mathcal{F}((F^*v))(\theta)\rangle_{L^2(\mathbb{R}^n)}$$
where
$$\mathcal{F}((F^*v))(\theta)=\int_{\mathbb{R}^n}e^{-iS(\widetilde{x},\theta)}\overline{a}(\widetilde{x},\theta)v(\widetilde{x})d\widetilde{x}. \tag{4.3}$$
Hence, for all $v\in\mathcal{S}(\mathbb{R}^n)$,
$$(FF^*v)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(S(x,\theta)-S(\widetilde{x},\theta))}a(x,\theta)\overline{a}(\widetilde{x},\theta)d\widetilde{x}\widehat{d\theta}. \tag{4.4}$$
The main idea to show that FF^* is a pseudodifferential operator, is to use the fact that $(S(x,\theta)-S(\widetilde{x},\theta))$ can be expressed by the scalar product $\langle x-\widetilde{x},\xi(x,\widetilde{x},\theta)\rangle$ after considering the change of variables $(x,\widetilde{x},\theta)\to(x,\widetilde{x},\xi=\xi(x,\widetilde{x},\theta))$.

The distribution kernel of FF^* is
$$K(x,\tilde{x})=\int_{\mathbb{R}^n}e^{i(S(x,\theta)-S(\tilde{x},\theta))}a(x,\theta)\overline{a}(\tilde{x},\theta)\widehat{d\theta}.$$

We obtain from (3.3) that if $|x-\widetilde{x}| \geq \frac{\epsilon}{2}\lambda(x,\widetilde{x},\theta)$ (where $\epsilon > 0$ is sufficiently small) then

$$|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(\widetilde{x},\theta)| \geq \frac{\epsilon}{2C_5}\lambda(x,\widetilde{x},\theta). \tag{4.5}$$

Choosing $\omega \in C^\infty(\mathbb{R})$ such that

$$\begin{gathered} \omega(x) \geq 0, \quad \forall x \in \mathbb{R} \\ \omega(x) = 1 \quad \text{if } x \in [-\frac{1}{2}, \frac{1}{2}] \\ \operatorname{supp}\omega \subset]-1,1[\end{gathered}$$

and setting

$$\begin{gathered} b(x,\tilde{x},\theta) := a(x,\theta)\overline{a}(\tilde{x},\theta) = b_{1,\epsilon}(x,\tilde{x},\theta) + b_{2,\epsilon}(x,\tilde{x},\theta) \\ b_{1,\epsilon}(x,\tilde{x},\theta) = \omega(\frac{|x-\tilde{x}|}{\epsilon\lambda(x,\tilde{x},\theta)})b(x,\tilde{x},\theta) \\ b_{2,\epsilon}(x,\tilde{x},\theta) = [1-\omega(\frac{|x-\tilde{x}|}{\epsilon\lambda(x,\tilde{x},\theta)})]b(x,\tilde{x},\theta). \end{gathered}$$

We have $K(x,\widetilde{x}) = K_{1,\epsilon}(x,\widetilde{x}) + K_{2,\epsilon}(x,\widetilde{x})$, where

$$K_{j,\epsilon}(x,\tilde{x}) = \int_{\mathbb{R}^n} e^{i(S(x,\theta)-S(\tilde{x},\theta))} b_{j,\epsilon}(x,\tilde{x},\theta)\widehat{d\theta}, \quad j=1,2.$$

We will study separately the kernels $K_{1,\epsilon}$ and $K_{2,\epsilon}$.

On the support of $b_{2,\epsilon}$, inequality (4.5) is satisfied and we have

$$K_{2,\epsilon}(x,\widetilde{x}) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n).$$

Indeed, using the oscillatory integral method, there is a linear partial differential operator L of order 1 such that

$$L\big(e^{i(S(x,\theta)-S(\tilde{x},\theta))}\big) = e^{i(S(x,\theta)-S(\tilde{x},\theta))}$$

where

$$L = -i|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(\widetilde{x},\theta)|^{-2} \sum_{l=1}^{n} \left[(\partial_{\theta_l} S)(x,\theta) - (\partial_{\theta_l} S)(\widetilde{x},\theta)\right] \partial_{\theta_l}.$$

The transpose operator of L is

$${}^tL = \sum_{l=1}^{n} F_l(x,\widetilde{x},\theta)\partial_{\theta_l} + G(x,\widetilde{x},\theta)$$

where $F_l(x,\widetilde{x},\theta) \in \Gamma_0^{-1}(\Omega_\epsilon)$, $G(x,\widetilde{x},\theta) \in \Gamma_0^{-2}(\Omega_\epsilon)$,

$$F_l(x,\widetilde{x},\theta) = i|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(\widetilde{x},\theta)|^{-2}((\partial_{\theta_l} S)(x,\theta) - (\partial_{\theta_l} S)(\widetilde{x},\theta)),$$

$$G(x,\widetilde{x},\theta) = i\sum_{l=1}^{n} \partial_{\theta_l}\left[|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(\widetilde{x},\theta)|^{-2}((\partial_{\theta_l} S)(x,\theta) - (\partial_{\theta_l} S)(\widetilde{x},\theta))\right],$$

$$\Omega_\epsilon = \big\{(x,\tilde{x},\theta) \in \mathbb{R}^{3n} : |\partial_\theta S(x,\theta) - \partial_\theta S(\tilde{x},\theta)| > \frac{\epsilon}{2C_5}\lambda(x,\tilde{x},\theta)\big\}.$$

On the other hand we prove by induction on q that

$$({}^tL)^q b_{2,\epsilon}(x,\tilde{x},\theta) = \sum_{|\gamma|\leq q,\, \gamma\in\mathbb{N}^n} g_{\gamma,q}(x,\tilde{x},\theta)\partial_\theta^\gamma b_{2,\epsilon}(x,\tilde{x},\theta),\ g_\gamma^{(q)} \in \Gamma_0^{-q}(\Omega_\epsilon),$$

and so,

$$K_{2,\epsilon}(x,\tilde{x}) = \int_{\mathbb{R}^n} e^{i(S(x,\theta)-S(\tilde{x},\theta))}({}^tL)^q b_{2,\epsilon}(x,\tilde{x},\theta)\widehat{d\theta}.$$

Using Leibnitz's formula, (G5) and the form $({}^tL)^q$, we can choose q large enough such that for all $\alpha,\alpha',\beta,\beta' \in \mathbb{N}^n$, $\exists C_{\alpha,\alpha',\beta,\beta'} > 0$,

$$\sup_{x,\widetilde{x}\in\mathbb{R}^n} |x^\alpha \widetilde{x}^{\alpha'} \partial_x^\beta \partial_{\widetilde{x}}^{\beta'} K_{2,\epsilon}(x,\widetilde{x})| \le C_{\alpha,\alpha',\beta,\beta'}.$$

Next, we study K_1^ϵ: this is more difficult and depends on the choice of the parameter ϵ. It follows from Taylor's formula that

$$S(x,\theta) - S(\widetilde{x},\theta) = \langle x-\widetilde{x}, \xi(x,\widetilde{x},\theta)\rangle_{\mathbb{R}^n},$$
$$\xi(x,\widetilde{x},\theta) = \int_0^1 (\partial_x S)(\widetilde{x}+t(x-\widetilde{x}),\theta)dt.$$

We define the vectorial function

$$\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta) = \omega\big(\frac{|x-\tilde{x}|}{2\epsilon\lambda(x,\tilde{x},\theta)}\big)\xi(x,\widetilde{x},\theta) + \big(1-\omega(\frac{|x-\tilde{x}|}{2\epsilon\lambda(x,\tilde{x},\theta)})\big)(\partial_x S)(\widetilde{x},\theta).$$

We have

$$\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta) = \xi(x,\widetilde{x},\theta) \text{ on } \operatorname{supp} b_{1,\epsilon}.$$

Moreover, for ϵ sufficiently small,

$$\lambda(x,\theta) \simeq \lambda(\widetilde{x},\theta) \simeq \lambda(x,\widetilde{x},\theta) \text{ on } \operatorname{supp} b_{1,\epsilon}. \tag{4.6}$$

Let us consider the mapping

$$\mathbb{R}^{3n} \ni (x,\widetilde{x},\theta) \to (x,\widetilde{x},\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta)) \tag{4.7}$$

for which Jacobian matrix is

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ \partial_x\widetilde{\xi}_\epsilon & \partial_{\widetilde{x}}\widetilde{\xi}_\epsilon & \partial_\theta\widetilde{\xi}_\epsilon \end{pmatrix}.$$

We have

$$\begin{aligned}
&\frac{\partial\widetilde{\xi}_{\epsilon,j}}{\partial\theta_i}(x,\widetilde{x},\theta)\\
&= \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta) + \omega\big(\frac{|x-\tilde{x}|}{2\epsilon\lambda(x,\tilde{x},\theta)}\big)\big(\frac{\partial\xi_j}{\partial\theta_i}(x,\widetilde{x},\theta) - \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\big)\\
&\quad - \frac{|x-\tilde{x}|}{2\epsilon\lambda(x,\tilde{x},\theta)}\frac{\partial\lambda}{\partial\theta_i}(x,\tilde{x},\theta)\lambda^{-1}(x,\tilde{x},\theta)\omega'\big(\frac{|x-\tilde{x}|}{2\epsilon\lambda(x,\tilde{x},\theta)}\big)\big(\xi_j(x,\widetilde{x},\theta) - \frac{\partial S}{\partial x_j}(\widetilde{x},\theta)\big).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\big|\frac{\partial\widetilde{\xi}_{\epsilon,j}}{\partial\theta_i}(x,\widetilde{x},\theta) - \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\big|\\
&\le \big|\omega(\frac{|x-\tilde{x}|}{2\epsilon\lambda(x,\tilde{x},\theta)})\big|\big|\frac{\partial\xi_j}{\partial\theta_i}(x,\widetilde{x},\theta) - \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\big|\\
&\quad + \lambda^{-1}(x,\tilde{x},\theta)\big|\omega'(\frac{|x-\tilde{x}|}{2\epsilon\lambda(x,\tilde{x},\theta)})\big|\big|\xi_j(x,\widetilde{x},\theta) - \frac{\partial S}{\partial x_j}(\widetilde{x},\theta)\big|.
\end{aligned}$$

Now it follows from (G5), (4.6) and Taylor's formula that

$$\begin{aligned}\Big|\frac{\partial \xi_j}{\partial \theta_i}(x,\widetilde{x},\theta)-\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\Big| &\leq \int_0^1 \Big|\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x}+t(x-\widetilde{x}),\theta)-\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\Big|dt\\ &\leq C_{10}|x-\widetilde{x}|\lambda^{-1}(x,\tilde{x},\theta),\quad C_{10}>0\end{aligned}\tag{4.8}$$

$$\begin{aligned}\big|\xi_j(x,\widetilde{x},\theta)-\frac{\partial S}{\partial x_j}(\widetilde{x},\theta)\big| &\leq \int_0^1 \Big|\frac{\partial S}{\partial x_j}(\widetilde{x}+t(x-\widetilde{x}),\theta)-\frac{\partial S}{\partial x_j}(\widetilde{x},\theta)\Big|dt\\ &\leq C_{11}|x-\widetilde{x}|,\quad C_{11}>0\,.\end{aligned}\tag{4.9}$$

From (4.8) and (4.9), there exists a positive constant $C_{12}>0$ such that

$$\Big|\frac{\partial\widetilde{\xi}_{\epsilon,j}}{\partial\theta_i}(x,\widetilde{x},\theta)-\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\Big|\leq C_{12}\epsilon,\quad \forall i,j\in\{1,\dots,n\}.\tag{4.10}$$

If $\epsilon<\frac{\delta_0}{2\widetilde{C}}$, then (4.10) and (G4) yields the estimate

$$\delta_0/2\leq -\widetilde{C}\epsilon+\delta_0\leq -\widetilde{C}\epsilon+\det\frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)\leq \det\partial_\theta\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta),\tag{4.11}$$

with $\widetilde{C}>0$ If ϵ is such that (4.6) and (4.11) hold, then the mapping given in (4.7) is a global diffeomorphism of $\mathbb{R}^{3n}$. Hence there exists a mapping

$$\theta:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\ni(x,\widetilde{x},\xi)\to\theta(x,\widetilde{x},\xi)\in\mathbb{R}^n$$

such that

$$\begin{gathered}\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta(x,\widetilde{x},\xi))=\xi\\ \theta(x,\widetilde{x},\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta))=x\\ \partial^\alpha\theta(x,\widetilde{x},\xi)=\mathcal{O}(1),\quad\forall\alpha\in\mathbb{N}^{3n}\backslash\{0\}\end{gathered}\tag{4.12}$$

If we change the variable ξ by $\theta(x,\widetilde{x},\xi)$ in $K_{1,\epsilon}(x,\widetilde{x})$, we obtain

$$K_{1,\epsilon}(x,\widetilde{x})=\int_{\mathbb{R}^n}e^{i\langle x-\tilde{x},\xi\rangle}b_{1,\epsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))\big|\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)\big|\widehat{d\xi}.\tag{4.13}$$

From (4.12) we have, for $k=0,1$, that $b_{1,\epsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))|\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)|$ belongs to $\Gamma_k^{2m}(\mathbb{R}^{3n})$ if $a\in\Gamma_k^m(\mathbb{R}^{2n})$.

Applying the stationary phase theorem (c.f. [12]) to 4.13, we obtain the expression of the symbol of the pseudodifferential operator FF^*,

$$\sigma(FF^*)=b_{1,\epsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))\big|\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)\big|_{|\widetilde{x}=x}+R(x,\xi)$$

where $R(x,\xi)$ belongs to $\Gamma_k^{2m-2}(\mathbb{R}^{2n})$ if $a\in\Gamma_k^m(\mathbb{R}^{2n})$, $k=0,1$.

For $\tilde{x}=x$, we have $b_{1,\epsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))=|a(x,\theta(x,x,\xi))|^2$ where $\theta(x,x,\xi)$ is the inverse of the mapping $\theta\to\partial_x S(x,\theta)=\xi$. Thus

$$\sigma(FF^*)(x,\partial_x S(x,\theta))\equiv|a(x,\theta)|^2\big|\det\frac{\partial^2 S}{\partial\theta\partial x}(x,\theta)\big|^{-1}.$$

From (4.2) and (4.3), we obtain the expression of F^*F: $\forall v \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}(\mathcal{F}(F^*F)\mathcal{F}^{-1})v(\theta) &= \int_{\mathbb{R}^n} e^{-iS(x,\theta)}\overline{a}(x,\theta)(F(\mathcal{F}^{-1}v))(x)dx \\ &= \int_{\mathbb{R}^n} e^{-iS(x,\theta)}\overline{a}(x,\theta)\Big(\int_{\mathbb{R}^n} e^{iS(x,\widetilde{\theta})}a(x,\widetilde{\theta})(\mathcal{F}(\mathcal{F}^{-1}v))(\widetilde{\theta})\widehat{d\widetilde{\theta}}\Big)dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(S(x,\theta)-S(x,\tilde{\theta}))}\,\overline{a}(x,\theta)\,a(x,\widetilde{\theta})v(\tilde{\theta})\widehat{d\tilde{\theta}}dx.\end{aligned}$$

Hence the distribution kernel of the integral operator $\mathcal{F}(F^*F)\mathcal{F}^{-1}$ is

$$\widetilde{K}(\theta,\widetilde{\theta}) = \int_{\mathbb{R}^n} e^{-i(S(x,\theta)-S(x,\tilde{\theta}))}\overline{a}(x,\theta)a(x,\tilde{\theta})\widehat{dx}.$$

We remark that we can deduce $\widetilde{K}(\theta,\widetilde{\theta})$ from $K(x,\widetilde{x})$ by replacing x by θ. On the other hand, all assumptions used here are symmetrical on x and θ; therefore, $\mathcal{F}(F^*F)\mathcal{F}^{-1}$ is a nice pseudodifferential operator with symbol

$$\sigma(\mathcal{F}(F^*F)\mathcal{F}^{-1})(\theta,-\partial_\theta S(x,\theta)) \equiv |a(x,\theta)|^2\Big|\det\frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)\Big|^{-1}.$$

Thus the symbol of F^*F is given by (c.f. [9])

$$\sigma(F^*F)(\partial_\theta S(x,\theta),\theta) \equiv |a(x,\theta)|^2\Big|\det\frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)\Big|^{-1}.$$

□

Corollary 4.2. *Let F be the integral operator with the distribution kernel*

$$K(x,y) = \int_{\mathbb{R}^n} e^{i(S(x,\theta)-y\theta)}a(x,\theta)\widehat{d\theta}$$

where $a \in \Gamma_0^m(\mathbb{R}^{2n}_{x,\theta})$ and S satisfies (G1), (G4) and (G5). Then, we have:

1. *For any m such that $m \le 0$, F can be extended as a bounded linear mapping on $L^2(\mathbb{R}^n)$*
2. *For any m such that $m < 0$, F can be extended as a compact operator on $L^2(\mathbb{R}^n)$.*

Proof. It follows from Theorem 4.1 that F^*F is a pseudodifferential operator with symbol in $\Gamma_0^{2m}(\mathbb{R}^{2n})$.
(1) If $m \le 0$, the weight $\lambda^{2m}(x,\theta)$ is bounded, so we can apply the Caldéron-Vaillancourt theorem (see [2, 12, 13]) for F^*F and obtain the existence of a positive constant $\gamma(n)$ and a integer $k(n)$ such that

$$\|(F^*F)\,u\|_{L^2(\mathbb{R}^n)} \le \gamma(n)\,Q_{k(n)}(\sigma(FF^*))\|u\|_{L^2(\mathbb{R}^n)}, \quad \forall u \in \mathcal{S}(\mathbb{R}^n)$$

where

$$Q_{k(n)}(\sigma(FF^*)) = \sum_{|\alpha|+|\beta|\le k(n)}\sup_{(x,\theta)\in\mathbb{R}^{2n}}\big|\partial_x^\alpha\partial_\theta^\beta\sigma(FF^*)(\partial_\theta S(x,\theta),\theta)\big|$$

Hence, for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\|Fu\|_{L^2(\mathbb{R}^n)} \le \|F^*F\|^{1/2}_{\mathcal{L}(L^2(\mathbb{R}^n))}\|u\|_{L^2(\mathbb{R}^n)} \le (\gamma(n)\,Q_{k(n)}(\sigma(FF^*)))^{1/2}\|u\|_{L^2(\mathbb{R}^n)}.$$

Thus F is also a bounded linear operator on $L^2(\mathbb{R}^n)$.
(2) If $m < 0$, $\lim_{|x|+|\theta|\to+\infty}\lambda^m(x,\theta) = 0$, and the compactness theorem (see [12, 13]) show that the operator F^*F can be extended as a compact operator on $L^2(\mathbb{R}^n)$.

Thus, the Fourier integral operator F is compact on $L^2(\mathbb{R}^n)$. Indeed, let $(\varphi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$, then

$$\|F^*F - \sum_{j=1}^{n}\langle\varphi_j, .\rangle F^*F\varphi_j\| \to 0 \quad \text{as } n \to +\infty.$$

Since F is bounded, for all $\psi \in L^2(\mathbb{R}^n)$,

$$\big\|F\psi - \sum_{j=1}^{n}\langle\varphi_j, \psi\rangle F\varphi_j\big\|^2 \le \big\|F^*F\psi - \sum_{j=1}^{n}\langle\varphi_j, \psi\rangle F^*F\varphi_j\big\|\big\|\psi - \sum_{j=1}^{n}\langle\varphi_j, \psi\rangle\varphi_j\big\|,$$

it follows that

$$\|F - \sum_{j=1}^{n}\langle\varphi_j, .\rangle F\varphi_j\| \to 0 \quad \text{as } n \to +\infty$$

□

Example 4.3. We consider the function given by

$$S(x,\theta) = \sum_{|\alpha|+|\beta|=2,\, \alpha,\beta\in\mathbb{N}^n} C_{\alpha,\beta}x^\alpha\theta^\beta, \quad \text{for } (x,\theta) \in \mathbb{R}^{2n}$$

where $C_{\alpha,\beta}$ are real constants. This function satisfies (G1), (G4) and (G5).

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