Electronic Journal of Differential Equations, Vol. 2006(2006), No. 28, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF SOLUTIONS FOR DISCONTINUOUS HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS IN BANACH ALGEBRAS

BAPURAO C. DHAGE, SOTIRIS K. NTOUYAS

Abstract. In this paper, we prove an existence theorem for hyperbolic differential equations in Banach algebras under Lipschitz and Carathéodory conditions. The existence of extremal solutions is also proved under certain monotonicity conditions.

1. Introduction

Let $\mathbb{R}$ denote the real line. Given two closed and bounded intervals $J_a = [0, a]$ and $J_b = [0, b]$ in $\mathbb{R}$, we consider the second order hyperbolic differential equation (HDE)

$$\begin{gathered} \frac{\partial^2}{\partial x \partial y}\left[\frac{u(x,y)}{f(x,y,u(x,y))}\right] = g(x,y,u(x,y)), \quad (x,y) \in J_a \times J_b, \\ u(x,0) = \phi(x), \quad u(0,y) = \psi(y), \end{gathered} \tag{1.1}$$

where $f : J_a \times J_b \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g : J_a \times J_b \times \mathbb{R} \to \mathbb{R}$, and $\phi : J_a \to \mathbb{R}$, $\psi : J_b \to \mathbb{R}$ are continuous functions with $\phi(0) = \psi(0)$.

By a *solution* of the HDE (1.1) we mean a function $u \in AC(J_a \times J_b, \mathbb{R})$ satisfying

(i) the function $(x, y) \mapsto \big(\frac{u(x,y)}{f(x,y,u(x,y))}\big)$ is absolutely continuous, and
(ii) u satisfies the equations in (1.1),

where $AC(J_a \times J_b, \mathbb{R})$ is the space of absolutely continuous real-valued functions on $J_a \times J_b$.

The existence of solutions and the topological properties of the solutions set of hyperbolic differential equations have received much attention during the last two decades, see for example, De Blasi and Myjak [3] and the references cited therein. Lakshmikantham and Pandit [7, 8] coupled the method of upper and lower solutions with the monotone method to obtain existence of extremal solutions for hyperbolic differential equations. The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of the first and second order partial differential equations. We refer to the books by Carl and Heikkila [2], Heikkila and Lakshmikantham

2000 *Mathematics Subject Classification.* 35L70, 35L15.
Key words and phrases. Hyperbolic differential equation; Banach algebras.

Submitted December 22, 2005. Published March 9, 2006.

[6], Lakshmikantham and Pandit [7], Pandit [8] and the references cited therein. The hyperbolic differential equation (1.1) is new to the literature and the physical situations in which HDE (1.1) occurs are yet to be investigated. Existence results for the hyperbolic differential equations (1.1) are proved in Arara et. al, [1] under Carathéodory conditions via nonlinear alternative of Leray-Schauder type. In this paper, we prove existence of extremal solutions under discontinuous nonlinearity involved in the equations. The rest of the paper is organized as follows. In the following section we present notations, definitions and preliminary results needed in the following sections. In Section 3 we prove the main existence result. Section 4 deals with existence theorems for extremal solutions of the HDE (1.1) under certain Lipschitz and monotonicity conditions. Finally, an example illustrating the abstract results is presented in Section 5.

2. Auxiliary Results

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \to X$ is called $\mathcal{D}$-Lipschitz if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \le \psi(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r$ $(\alpha > 0)$, A is called a Lipschitz with a Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with a contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called a nonlinear contraction on X. Sometimes we call the function ψ a D-function for convenience.

An operator $T : X \to X$ is called *compact* if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. Similarly $T : X \to X$ is called *totally bounded* if T maps a bounded subset of X into the relatively compact subset of X. Finally $T : X \to X$ is called *completely continuous* operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true.

The nonlinear alternative of Schaefer type recently proved by Dhage [4] is embodied in the following theorem.

Theorem 2.1 (Dhage [4])**.** *Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) *A is Lipschitz with a Lipschitz constant α,*
(b) *B is compact and continuous, and*
(c) *$\alpha < 1$, where $M = \|B(X)\| := \sup\{\|Bx\| : x \in X\}$.*

Then either

(i) *the equation $\lambda[Ax\,Bx] = x$ has a solution for $\lambda = 1$, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda[Au\,Bu] = u,\ 0 < \lambda < 1\}$ is unbounded.*

A non-empty closed set K in a Banach algebra X is called a *cone* if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \ge 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X. A cone K is called to be *positive* if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\le$ in X as follows. Let $x, y \in X$. Then $x \le y$ if and only if $y - x \in K$. A cone K is called to be *normal* if the norm $\|\cdot\|$ is monotone increasing on K. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Heikkila and Lakshmikantham [6].

Lemma 2.2 (Dhage [5]). *Let K be a positive cone in a real Banach algebra X and let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \le v_1$ and $u_2 \le v_2$. Then $u_1 u_2 \le v_1 v_2$.*

For any $a, b \in X, a \le b$, the order interval $[a, b]$ is a set in X given by

$$[a, b] = \{x \in X : a \le x \le b\}.$$

We use the following fixed point theorem of Dhage [5] for proving the existence of extremal solutions for the HDE (1.1) under certain monotonicity conditions.

Theorem 2.3 (Dhage [5]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two operators such that*

(a) *A is completely continuous,*
(b) *B is totally bounded,*
(c) *$Ax\,By \in [a, b]$ for all $x, y \in [a, b]$, and*
(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has a least and a greatest positive solution in $[a, b]$.

Theorem 2.4 (Dhage [5]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two operators such that*

(a) *A is Lipschitz with a Lipschitz constant α,*
(b) *B is totally bounded,*
(c) *$Ax\,By \in [a, b]$ for all $x, y \in [a, b]$, and*
(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has least and a greatest positive solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Remark 2.5. Note that hypothesis (c) of Theorems 2.3 and 2.4 holds if the operators A and B are positive monotone increasing and there exist elements a and b in X such that $a \le Aa\,Ba$ and $Ab\,Bb \le b$.

3. Existence Results

Let $B(J_a \times J_b, \mathbb{R})$ denote the space of real-valued bounded functions 0n $J_a \times J_b$ and let $C(J_a \times J_b, \mathbb{R})$ be the Banach space of all continuous functions from $J_a \times J_b$ into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup\{|u(x, y)| : (x, y) \in J_a \times J_b\}. \tag{3.1}$$

Define a multiplication " $\cdot$ " by

$$(u \cdot v)(x, y) = u(x, y)v(x, y)$$

for each $(x, y) \in J_a \times J_b$. Then $C(J_a \times J_b, \mathbb{R})$ is a Banach algebra with above norm and multiplication. Let $L^1(J_a \times J_b, \mathbb{R})$ denotes the Banach space of measurable functions $u : J_a \times J_b \longrightarrow \mathbb{R}$ which are Lebesgue integrable normed by

$$\|u\|_{L^1} = \int_0^a \int_0^b |u(x, y)| dx\, dy.$$

The HDE (1.1) is equivalent to the functional integral equation (in short FIE).

$$u(x, y) = \big[f(x, y, u(x, y))\big]\Big(z_0(x, y) + \int_0^x \int_0^y g(t, s, u(t, s))\, ds\, dt\Big) \tag{3.2}$$

for $(x,y) \in J_a \times J_b$, where $z_0(x,y) = \frac{\psi(y)}{f(0,y,\psi(y))} + \frac{\phi(x)}{f(x,0,\phi(x))} - \frac{\phi(0)}{f(0,0,\phi(0))}$.

Note that if the function f is continuous on $J_a \times J_b \times \mathbb{R}$, then from the continuity of ϕ and ψ it follows that $z_0 \in C(J_a \times J_b, \mathbb{R})$.

We need the following definition in the sequel.

Definition 3.1. *A function $\beta : J_a \times J_b \times \mathbb{R} \to \mathbb{R}$ is called* Carathéodory*'s if*

(i) *the function $(x,y) \to \beta(x,y,z)$ is measurable for each $z \in \mathbb{R}$,*
(ii) *the function $z \to \beta(x,y,z)$ is continuous for almost each $(x,y) \in J_a \times J_b$.*

Further a Carathéodory *function $\beta(x,y,z)$ is called L^1-*Carathéodory *if*

(iii) *for each number $r > 0$, there exists a function $h_r \in L^1(J_a \times J_b, \mathbb{R})$ such that*
$$|\beta(x,y,z)| \le h_r(x,y) \quad \text{a.e. } (x,y) \in J_a \times J_b$$
for all $z \in \mathbb{R}$ with $|z| \le r$.

Finally, a Carathéodory *function $\beta(x,y,z)$ is called L^1_X-*Carathéodory *if*

(iv) *there exists a function $h \in L^1(J_a \times J_b, \mathbb{R})$ such that*
$$|\beta(x,y,z)| \le h(x,y) \quad \text{a.e. } (x,y) \in J_a \times J_b$$
for all $z \in \mathbb{R}$.

The following hypotheses will be used in the sequel.

(A1) The function f is continuous on $J_a \times J_b \times \mathbb{R}$.
(A2) There exists a function $\alpha \in B(J_a \times J_b, \mathbb{R}^+)$ such that
$$|f(x,y,z) - f(x,y,\overline{z})| \le \alpha(x,y)|z - \overline{z}|, \quad \text{a.e. } (x,y) \in J_a \times J_b,$$
for all $z, \overline{z} \in \mathbb{R}$.
(A3) The function g is L^1_X-Carathéodory.

Theorem 3.2. *Assume that hypotheses (A1)-(A4) hold. If*
$$\|\alpha\|_\infty \big[\|z_0\|_\infty + \|h\|_{L^1}\big] < 1,$$
then the hyperbolic equation (1.1) *has a solution on $J_a \times J_b$.*

Proof. Let $X = C(J_a \times J_b, \mathbb{R})$. Define two operators A and B on X by
$$Au(x,y) = f(x,y,u(x,y)), \quad (x,y) \in J_a \times J_b, \tag{3.3}$$
$$Bu(x,y) = z_0(x,y) + \int_0^x \int_0^y g(t,s,u(t,s))\,ds\,dt, \quad (x,y) \in J_a \times J_b. \tag{3.4}$$
Clearly A and B define the operators $A, B : X \to X$. Now solving (1.1) is equivalent to solving FIE (3.1), which is further equivalent to solving the operator equation
$$Au(x,y)\,Bu(x,y) = u(x,y), \quad (x,y) \in J_a \times J_b. \tag{3.5}$$
We show that operators A and B satisfy all the assumptions of Theorem 2.1. First we shall show that A is a Lipschitz. Let $u_1, u_2 \in X$. Then by (A2),
$$\begin{aligned}|Au_1(x,y) - Au_2(x,y)| &= |f(x,y,u_1(x,y)) - f(x,y,u_2(x,y))| \\ &\le \alpha(x,y)|u_1(x,y) - u_2(x,y)| \\ &\le \|\alpha\|_\infty \|u_1 - u_2\|_\infty.\end{aligned}$$
Taking the maximum over (x,y), in the above inequality yields
$$\|Au_1 - Au_2\|_\infty \le \|\alpha\|_\infty \|u_1 - u_2\|_\infty,$$

and so A is a Lipschitz with a Lipschitz constant $\|\alpha\|_\infty$.

Next, we show that B is compact operator on X. Let $\{u_n\}$ be a sequence in X. From $(A3)$ it follows that

$$\|Bu_n\|_\infty \leq \|z_0\|_\infty + \|h\|_{L^1},$$

where h is given in Definition 3.1 (iv). As a result $\{Bu_n : n \in \mathbb{N}\}$ is a uniformly bounded set in X. Let $(x_1, y_1), (x_2, y_2) \in J_a \times J_b$. Then

$$\begin{aligned}
&|Bu_n(x_1, y_1) - Bu_n(x_2, y_2)| \\
&\leq |z_0(x_1, y_1) - z_0(x_2, y_2)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} |g(t, s, u_n(t, s))| ds\, dt \\
&\leq |z_0(x_1, y_1) - z_0(x_2, y_2)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} h(t, s) ds\, dt \\
&\to 0, \quad \text{as } (x_1, y_1) \to (x_2, y_2).
\end{aligned}$$

From this we conclude that $\{Bu_n : \ n \in \mathbb{N}\}$ is an equicontinuous set in X. Hence $B : X \to X$ is compact by Arzelà-Ascoli theorem. Moreover,

$$\begin{aligned}
M &= \|B(X)\| \\
&\leq |z_0(x, y)| + \sup_{(x,y) \in J_a \times J_b} \int_0^x \int_0^y |g(t, s, u(t, s))|\, ds\, dt \\
&\leq \|z_0\|_\infty + \|h\|_{L^1},
\end{aligned}$$

and so,

$$\alpha M \leq \|\alpha\|_\infty (\|z_0\|_\infty + \|h\|_{L^1}) < 1,$$

by assumption. To finish, it remain to show that either the conclusion (i) or the conclusion (ii) of Theorem 2.1 holds. We now will show that the conclusion (ii) is not possible. Let $u \in X$ be any solution to (1.1). Then, for any $\lambda \in (0, 1)$ we have

$$u(x, y) = \lambda [f(x, y, u(x, y))] \Big(z_0(x, y) + \int_0^x \int_0^y g(t, s, u(t, s))\, ds\, dt \Big),$$

for $(x, y) \in J_a \times J_b$. Therefore,

$$\begin{aligned}
|u(x, y)| &\leq \big[f(x, y, u(x, y)) \big] \Big(|z_0(x, y)| + \int_0^x \int_0^y |g(t, s, u(t, s))|\, dt\, ds \Big) \\
&\leq \big[|f(x, y, u(x, y)) - f(x, y, 0)| + |f(x, y, 0)| \big] \times \\
&\quad \times \Big(|z_0(x, y)| + \int_0^x \int_0^y h(s, t)\, dt\, ds \Big) \\
&\leq \big[\|\alpha\|_\infty |u(x, y)| + F \big] \left(|z_0(x, y)| + \|h\|_{L^1} \right) \\
&\leq \big[\|\alpha\|_\infty \|u\|_\infty + F \big] [\|z_0\|_\infty + \|h\|_{L^1}],
\end{aligned}$$

where $F = |f(x, y, 0)|$, and consequently

$$\|u\|_\infty \leq \frac{F(\|z_0\|_\infty + \|h\|_{L^1})}{1 - \|\alpha\|_\infty (\|z_0\|_\infty + \|h\|_{L^1})} := M.$$

Thus the conclusion (ii) of Theorem 2.1 does not hold. Therefore the hyperbolic differential equation (1.1) has a solution on $J_a \times J_b$. This completes the proof. $\square$

4. Existence Results for Extremal Solutions

We equip the space $C(J_a \times J_b, \mathbb{R})$ with the order relation $\leq$ with the help of the cone defined by

$$K = \{u \in C(J_a \times J_b, \mathbb{R}) : \; u(x, y) \geq 0, \quad \forall (x, y) \in J_a \times J_b\}.$$

Thus $u \leq \bar{u}$ if and only if $u(x, y) \leq \bar{u}(x, y)$ for each $(x, y) \in J_a \times J_b$.

It is well-known that the cone K is positive and normal in $C(J_a \times J_b, \mathbb{R})$. If $\underline{u}, \bar{u} \in C(J_a \times J_b, \mathbb{R})$ and $\underline{u} \leq \bar{u}$, we put

$$[\underline{u}, \overline{u}] = \{u \in C(J_a \times J_b, \mathbb{R}) : \underline{u} \leq u \leq \bar{u}\}.$$

Definition 4.1. *A function $\beta : J_a \times J_b \times \mathbb{R} \to \mathbb{R}$ is called* Chandrabhan *if*

- (i) *the function $(x, y) \to \beta(x, y, z)$ is measurable for each $z \in \mathbb{R}$,*
- (ii) *the function $z \to \beta(x, y, z)$ is nondecreasing for almost each $(x, y) \in J_a \times J_b$.*

Further a Chandrabhan *function $\beta(x, y, z)$ is called L^1-*Chandrabhan *if*

- (iii) *for each number $r > 0$, there exists a function $h_r \in L^1(J_a \times J_b, \mathbb{R})$ such that*
$$|\beta(x, y, z)| \leq h_r(x, y) \quad a.e. \; (x, y) \in J_a \times J_b,$$
for all $z \in \mathbb{R}$ with $|z| \leq r$.

Definition 4.2. *A function $\underline{u}(\cdot, \cdot) \in C(J_a \times J_b, \mathbb{R})$ is said to be a lower solution of* (1.1) *if we have*

$$\frac{\partial^2}{\partial x \partial y}\Big[\frac{\underline{u}(x, y)}{f(x, y, \underline{u}(x, y))}\Big] \leq g(x, y, \underline{u}(x, y)), \quad (x, y) \in J_a \times J_b,$$
$$\underline{u}(x, 0) \leq \varphi(x), \quad \underline{u}(0, y) \leq \psi(y),$$

for each $(x, y) \in J_a \times J_b$. Similarly a function $\bar{u}(\cdot, \cdot) \in C(J_a \times J_b, \mathbb{R})$ is said to be an upper solution of (1.1) *if we have*

$$\frac{\partial^2}{\partial x \partial y}\Big[\frac{\bar{u}(x, y)}{f(x, y, \bar{u}(x, y))}\Big] \geq g(x, y, \bar{u}(x, y)), \quad (x, y) \in J_a \times J_b,$$
$$\bar{u}(x, 0) \geq \varphi(x), \quad \bar{u}(0, y) \geq \psi(y),$$

for each $(x, y) \in J_a \times J_b$.

Definition 4.3. *A solution u_M of the problem* (1.1) *is said to be maximal if for any other solution u to the problem* (1.1) *one has $u(x, y) \leq u_M(x, y)$, for all $(x, y) \in J_a \times J_b$. Again a solution u_m of the problem* (1.1) *is said to be minimal if $u_m(x, y) \leq u(x, y)$, for all $(x, y) \in J_a \times J_b$ where u is any solution of the problem* (1.1) *on $J_a \times J_b$.*

The following hypotheses will be used in the sequel.

- (H1) $f : J_a \times J_b \times \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$, $g : J_a \times J_b \times \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(y) \geq 0$ on J_b and $\frac{\phi(x)}{f(x, 0, \phi(x))} \geq \frac{\phi(0)}{f(0, 0, \phi(0))}$ for all $x \in J_a$.
- (H2) The functions $f(x, y, u)$ and $g(x, y, u)$ are nondecreasing in u almost everywhere for $(x, y) \in J_a \times J_b$.
- (H3) The function g is L^1-Chandrabhan.
- (H4) The hyperbolic differential equation (1.1) has a lower solution $\underline{u}$ and an upper solution $\overline{u}$ with $\underline{u} \leq \overline{u}$.

Remark 4.4. Assume that (H2)-(H4) hold. Define a function $h : J_a \times J_b \to \mathbb{R}^+$ by

$$h(x, y) = |g(x, y, \overline{u}(x, y))| = g(x, y, \overline{u}(x, y)) \quad \forall\, (x, y) \in J_a \times J_b.$$

Then h is Lebesgue integrable and

$$|g(x, y, z)| \le h(x, y), \quad \text{a.e. } (x, y) \in J_a \times J_b, \quad \forall z \in [\underline{u}, \overline{u}].$$

Theorem 4.5. *Assume that hypotheses* $(A2), (H1) - (H4)$ *hold. If*

$$\|\alpha\|_\infty \big[\|z_0\|_\infty + \|h\|_{L^1}\big] < 1,$$

then the hyperbolic equation (1.1) *has a minimal and a maximal positive solution on* $J_a \times J_b$.

Proof. Let $X = C(J_a \times J_b, \mathbb{R})$ and consider a closed interval $[\underline{u}, \overline{u}]$ in X which is well defined in view of hypothesis (H4). Define two operators $A, B : [\underline{u}, \overline{u}] \to X$ by (3.3) and (3.4) respectively. Clearly A and B define the operators $A, B : [\underline{u}, \overline{u}] \to K$.

Now solving (1.1) is equivalent to solving (3.2), which is further equivalent to solving the operator equation

$$Au(x, y)\, Bu(x, y) = u(x, y), \quad (x, y) \in J_a \times J_b. \tag{4.1}$$

We show that operators A and B satisfy all the assumptions of Theorem 2.4. As in Theorem 3.2 we can prove that A is Lipschitz with a Lipschitz constant $\|\alpha\|_\infty$ and B is completely continuous operator on $[\underline{u}, \overline{u}]$.

Now the hypothesis (H2) implies that A and B are nondecreasing on $[\underline{u}, \overline{u}]$. To see this, let $u_1, u_2 \in [\underline{u}, \overline{u}]$ be such that $u_1 \le u_2$. Then by (H2),

$$Au_1(x, y) = f(x, y, u_1(x, y)) \le f(x, y, u_2(x, y)) = Au_2(x, y), \quad \forall (x, y) \in J_a \times J_b,$$

and

$$\begin{aligned} Bu_1(x, y) &= z_0(x, y) + \int_0^x \int_0^y g(t, s, u_1(t, s))\, ds\, dt \\ &\le z_0(x, y) + \int_0^x \int_0^y g(t, s, u_2(t, s))\, ds\, dt \\ &= Bu_2(x, y), \quad \forall (x, y) \in J_a \times J_b. \end{aligned}$$

So A and B are nondecreasing operators on $[\underline{u}, \overline{u}]$. Again hypothesis (H4) imply

$$\begin{aligned} \underline{u}(x, y) &= [f(x, y, \underline{u}(x, y))]\Big(z_0(x, y) + \int_0^x \int_0^y g(t, s, \underline{u}(t, s))\ , ds\, dt\Big) \\ &\le [f(x, y, z(x, y))]\Big(z_0(x, y) + \int_0^x \int_0^y g(t, s, z(t, s)) ds dt\Big) \\ &\le [f(x, y, \overline{u}(x, y))]\Big(z_0(x, y) + \int_0^x \int_0^y g(t, s, \overline{u}(t, s))\, ds\, dt\Big) \\ &\le \overline{u}(x, y), \end{aligned}$$

for all $(x, y) \in J_a \times J_b$ and $z \in [\underline{u}, \overline{u}]$. As a result

$$\underline{u}(x, y) \le Az(x, y)Bz(x, y) \le \overline{u}(x, y), \quad \forall (x, y) \in J_a \times J_b \text{ and } z \in [\underline{u}, \overline{u}].$$

Hence $Az\, Bz \in [\underline{u}, \overline{u}]$, for all $z \in [\underline{u}, \overline{u}]$.

Notice for any $u \in [\underline{u}, \overline{u}]$,

$$
\begin{aligned}
M &= \|B([\underline{u}, \overline{u}])\| \\
&\le |z_0(x,y)| + \sup_{(x,y)\in J_a\times J_b} \int_0^x \int_0^y |g(t,s,u(t,s))|\,ds\,dt \\
&\le \|z_0\|_\infty + \|h_r\|_{L^1},
\end{aligned}
$$

and so,

$$
\alpha M \le \|\alpha\|_\infty (\|z_0\|_\infty + \|h_r\|_{L^1}) < 1.
$$

Thus the operators A and B satisfy all the conditions of Theorem 2.4 and so the operator equation (3.4) has a least and a greatest solution in $[\underline{u}, \overline{u}]$. This further implies that the hyperbolic differential equation (1.1) has a minimal and a maximal positive solution on $J_a \times J_b$. This completes the proof. □

Theorem 4.6. *Assume that hypotheses* (A1), (A2), (H1)-(H4) *hold. Then the hyperbolic equation* (1.1) *has a minimal and a maximal positive solution on* $J_a \times J_b$.

Proof. Let $X = C(J_a \times J_b, \mathbb{R})$. Consider the order interval $[\underline{u}, \overline{u}]$ in X and define two operators A and B on $[\underline{u}, \overline{u}]$ by (3.3) and (3.4) respectively. Then HDE (1.1) is transformed into an operator equation $Au(x,y)\,Bu(x,y) = u(x,y)$, $(x,y) \in J_a \times J_b$ in a Banach algebra X. Notice that (H1) implies $A, B : [\underline{u}, \overline{u}] \to K$. Since the cone K in X is normal, $[\underline{u}, \overline{u}]$ is a norm bounded set in X.

Next we show that A is completely continuous on $[\underline{u}, \overline{u}]$. Now the cone K in X is normal, so the order interval $[\underline{u}, \overline{u}]$ is norm-bounded. Hence there exists a constant $r > 0$ such that $\|u\| \le r$ for all $u \in [\underline{u}, \overline{u}]$. As f is continuous on compact set $J_a \times J_b \times [-r, r]$, it attains its maximum, say M. Therefore, for any subset S of $[\underline{u}, \overline{u}]$ we have

$$
\begin{aligned}
\|A(S)\| &= \sup\{\|Au\| : u \in S\} \\
&= \sup\Big\{ \sup_{(x,y)\in J_a\times J_b} |f(x,y,u(x,y))| : u \in S \Big\} \\
&\le \sup\Big\{ \sup_{(x,y)\in J_a\times J_b} |f(x,y,u)| : u \in [-r,r] \Big\} \\
&\le M.
\end{aligned}
$$

This shows that $A(S)$ is a uniformly bounded subset of X.

We note that the function $f(x,y,u)$ is uniformly continuous on $J_a \times J_b \times [-r,r]$. Therefore, for any $(x_1,y_1), (x_2,y_2) \in J_a \times J_b$ we have

$$
|f(x_1,y_1,u) - f(x_2,y_2,u)| \to 0 \quad \text{as } (x_1,y_1) \to (x_2,y_2),
$$

for all $u \in [-r,r]$. Similarly for any $u_1, u_2 \in [-r,r]$

$$
|f(x,y,u_1) - f(x,y,u_2)| \to 0 \quad \text{as } u_1 \to u_2,
$$

for all $(x,y) \in J_a \times J_b$. Hence any $(x_1,y_1), (x_2,y_2) \in J_a \times J_b$ and for any $u \in S$ one has

$$
\begin{aligned}
|Au(x_1,y_1) - Au(x_2,y_2)| &= |f(x_1,y_1,u(x_1,y_1)) - f(x_2,y_2,u(x_2,y_2))| \\
&\le |f(x_1,y_1,u(x_1,y_1)) - f(x_2,y_2,u(x_1,y_1)| \\
&\quad + |f(x_2,y_2,x(x_1,y_1)) - f(x_2,y_2,x(x_2,y_2))| \\
&\to 0 \quad \text{as } (x_1,y_1) \to (x_2,y_2).
\end{aligned}
$$

This shows that $A(S)$ is an equi-continuous set in K. Now an application of Arzelà-Ascoli theorem yields that A is a completely continuous operator on $[\underline{u}, \overline{u}]$.

Next it can be shown as in the proof of Theorem 4.5 that B is a compact operator on $[\underline{u}, \overline{u}]$. Now an application of Theorem 2.3 yields that the hyperbolic differential equation (1.1) has a minimal and maximal positive solution on J. This completes the proof. $\square$

5. An Example

Let $J_a = [0,1] = J_b$ and let $\phi, \psi : [0,1] \to \mathbb{R}$ be two functions defined by

$$\phi(x) = x^2 \quad \text{and} \quad \psi(x) = x. \tag{5.1}$$

Define two functions $f, g : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,y,u) = \begin{cases} 1, & \text{if } u < 0 \\ 1 + \frac{u}{9}, & \text{if } u \geq 0, \end{cases} \tag{5.2}$$

and

$$g(x,y,u) = \begin{cases} 0, & \text{if } u < 0 \\ \frac{[u]}{15+[u]}, & \text{if } u \geq 0 \end{cases} \tag{5.3}$$

for all $x, y \in [0,1]$, where $[u]$ is the greatest integer less than or equal to u.

Now consider the hyperbolic differential equation (1.1) with the functions ϕ, ψ f and g defined by (5.1), (5.2) and (5.3) respectively.

We show that the functions ϕ, ψ, f and g satisfy all the hypotheses of Theorem 4.2. Clearly $\phi(0) = 0 = \psi(0)$. Again, here we have $f : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$, $g : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}_+$, $\psi(x) \geq 0$ on $[0,1]$ and $\frac{\phi(x)}{f(x,0,\phi(x))} \geq \frac{\phi(0)}{f(0,0,\phi(0))}$ for all $x \in [0,1]$.

Also the maps $u \mapsto f(x,y,u)$ and $u \mapsto g(x,y,u)$ are nondecreasing in $\mathbb{R}$ for all $x, y \in [0,1]$. Note that f is continuous and g is L^1-Chandrabhan on $[0,1] \times [0,1] \times \mathbb{R}$. Further, it is easy to verify that identically zero function $\underline{u} \equiv 0$ and the constant function $\overline{u} \equiv 3$ are the lower and upper solutions of the (1.1) respectively. Hence by Theorem 4.2, the HDE (1.1) has a maximal and a minimal positive solution in the order interval $[0,3]$ in the space $C([0,1] \times [0,1], \mathbb{R})$ defined on $[0,1] \times [0,1]$.

Remark 5.1. Note that the function g in the above example is not continuous, but Lebesgue integrable on $[0,1] \times [0,1]$.

References

[1] A. Arara, M. Benchohra and B. C. Dhage, *Existence theorems for a class of hyperbolic differential equations in Banach algebras*, preprint.
[2] S. Carl and S. Heikkila, *Nonlinear Differential Equations in Ordered Spaces*, Chapman & Hall, 2000.
[3] F. De Blasi and J. Myjak, *On the structure of the set of solutions of the Darboux problem for hyperbolic equations*, Proc. Edinburgh Math. Soc. **29** (1986), 7-14.
[4] B. C. Dhage, *A nonlinear alternative in Banach algebras with applications to functional differential equations*, Nonlinear Funct. Anal. & Appl. **8** (2004), 563–575.
[5] B. C. Dhage, *Some algebraic fixed point theorems for multi-valued mappings with applications*, Diss. Math. Differential inclusions, Control & Optim. **26** (2006).
[6] S. Heikkila and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Pure and Applied Mathematics, Marcel Dekker, New York, 1994.

[7] V. Lakshmikantham and S. G. Pandit, *The Method of upper, lower solutions and hyperbolic partial differential equations*, J. Math. Anal. Appl. **105** (1985), 466-477.
[8] S. G. Pandit, *Monotone methods for systems of nonlinear hyperbolic problems in two independent variables*, Nonlinear Anal. **30** (1997), 2735-2742.

Bapurao C. Dhage
Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur Maharashtra, India
E-mail address: bcd20012001@yahoo.co.in

Sotiris K. Ntouyas
Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail address: sntouyas@cc.uoi.gr