Electronic Journal of Differential Equations, Vol. 2006(2006), No. 36, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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POSITIVE SOLUTIONS FOR BOUNDARY-VALUE PROBLEMS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider the existence and multiplicity of positive solutions for the nonlinear fractional differential equation boundary-value problem

$$\begin{gathered}\mathbf{D}_{0+}^{\alpha} u(t) = f(t, u(t)), \quad 0 < t < 1 \\ u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0\end{gathered}$$

where $1 < \alpha \leq 2$ is a real number, and $\mathbf{D}_{0+}^{\alpha}$ is the Caputo's fractional derivative, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. By means of a fixed-point theorem on cones, some existence and multiplicity results of positive solutions are obtained.

1. Introduction

Fractional calculus has played a significant role in engineering, science, economy, and other fields. Many papers and books on fractional calculus, fractional differential equations have appeared recently, (see [6, 7, 8, 9]). As cited in [1] "There have appeared lots of works, in which fractional derivatives are used for a better description considered material properties, mathematical modelling base on enhanced rheological models naturally leads to differential equations of fractional order-and to the necessity of the formulation of initial conditions to such equations. Applied problems require definitions of fractional derivatives allowing the utilization of physically in interpretable initial conditions, which contain $f(a)$, $f'(a)$, etc". In fact, there has the same requirements for boundary conditions. Caputo's fractional derivative exactly satisfies these demands. Here, we consider the existence and multiplicity of positive solutions of nonlinear fractional differential equation boundary-value problem involving Caputo's derivative.

$$\begin{gathered}\mathbf{D}_{0+}^{\alpha} u(t) = f(t, u(t)), \quad 0 < t < 1 \\ u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0\end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$ is a real number and $\mathbf{D}_{0+}^{\alpha}$ is the Caputo's fractional derivative, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. As far as we known, there has few papers which deal with the boundary-value problem for nonlinear fractional differential equation.

2000 *Mathematics Subject Classification.* 34B15.
Key words and phrases. Caputo's fractional derivative; fractional differential equation; boundary-value problem; positive solution; fractional Green's function; fixed-point theorem.

Submitted November 30, 2005. Published March 21, 2006.

In [7], the authors consider the existence and multiplicity of positive solutions of nonlinear fractional differential equation boundary-value problem

$$\begin{gathered} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1 \\ u(0) = u(1) = 0 \end{gathered} \tag{1.2}$$

where $1 < \alpha \leq 2$ is a real number. D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, and $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous. Due to the reasons cited above, when conditions of (1.2) are not zero boundary value, the Riemann-Liouville fractional derivative D_{0+}^{α} is not suitable. Therefore, in the sense of practicable demand, we investigate boundary-value problem (1.1) involving the Caputo's fractional derivative.

In this paper, analogy with boundary-value problem for differential equations of integer order, we firstly derive the corresponding Green' function-named by fractional Green' function. Consequently problem (1.1) is reduced to a equivalent Fredholm integral equation of the second kind. Finally, using some fixed-point theorems, the existence and multiplicity of positive solutions are obtained.

2. Preliminaries

For completeness, in this section, we will demonstrate and study the definitions and some fundamental facts of Caputo's derivatives of fractional order which can been founded in [5].

Definition. [5, (2.138)] Caputo's derivative for a function $f : [0, \infty) \to R$ can been written as

$$\mathbf{D}_{0+}^{s} f(x) = \frac{1}{\Gamma(n-s)} \int_0^x \frac{f^n(t)dt}{(x-t)^{s+1-n}}, \quad n = [s] + 1 \tag{2.1}$$

where $[s]$ denotes the integer part of real number s.

Remark 2.1. Under natural conditions on the function $f(x)$, for $s \to n$ Caputo's derivative becomes a conventional n-th derivative of the function $f(x)$. See [5, 79]

Definition. [6, Definition 2.1] The integral

$$I_{0+}^{s} f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}} dt, \quad x > 0$$

where $s > 0$, is called Riemann-Liouville fractional integral of order s.

Definition. [6, page 36-37] For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{s} f(x) = \frac{1}{\Gamma(n-s)} (\frac{d}{dx})^n \int_0^x \frac{f(t)}{(x-t)^{s-n+1}} dt$$

where $n = [s] + 1, [s]$ denotes the integer part of number s, is called the Riemann-Liouville fractional derivative of order s.

As examples, for $\mu > -1$, we have

$$\mathbf{D}_{0+}^{\alpha} x^{\mu} = \mu(\mu-1)\dots(\mu-n+1)\frac{\Gamma(1+\mu-n)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}$$

$$D_{0+}^{\alpha} x^{\mu} = \frac{\Gamma(1+\mu-n)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}$$

where $n = [\alpha] + 1$.

From the definition of Caputo's derivative and Remark 2.1, we can obtain the statement.

Lemma 2.2. *Let $\alpha > 0$, then the differential equation*

$$\mathbf{D}_{0+}^{\alpha} u(t) = 0$$

has solutions $u(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_n t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n$, $n = [\alpha] + 1$.

From the lemma above, we deduce the following statement.

Lemma 2.3. *Let $\alpha > 0$, then*

$$I_{0+}^{\alpha} \mathbf{D}_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_n t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n$, $n = [\alpha] + 1$.

The following theorems will play major role in our next analysis.

Lemma 2.4 ([3])**.** *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that, either*

(1) $\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_2$, *or*
(2) $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \le \|w\|$ $w \in P \cap \partial\Omega_2$

Then S has a fixed point in $P \cap \overline{\Omega}_2 \backslash \Omega_1$.

Definition A map δ is said to be a nonnegative continuous concave functional on K if $\delta : K \to [0, +\infty)$ is continuous and

$$\delta(tx + (1-t)y) \ge t\delta(x) + (1-t)\delta(y)$$

for all $x, y \in K$ and $0 \le t \le 1$. And let

$$K(\delta, a, b) = \{u \in K | a \le \delta(u), \|u\| \le b\}$$

Lemma 2.5 ([4])**.** *Let K be a cone and $K_c = \{y \in K | \|y\| \le c\}$, and $A : \overline{K_c} \to \overline{K_c}$ be completely continuous and α be a nonnegative continuous concave function on K such that $\alpha(y) \le \|y\|$, for all $y \in \overline{K_c}$. Suppose there exist $0 < a < b < d \le c$ such that*

(C1) $\{y \in K(\alpha, b, d) | \alpha(y) > b\} \ne \emptyset$ *and* $\alpha(Ay) > b$, *for all* $y \in K\{\alpha, b, d\}$,
(C2) $\|Ay\| < a$, *for* $\|y\| \le a$, *and*
(C3) $\alpha(Ay) > b$, *for* $y \in K\{\alpha, b, c\}$ *with* $\|Ay\| > d$.

Then A has at least three fixed points y_1, y_2, y_3 satisfying

$$\|y_1\| < a, \quad b < \alpha(y_2), \quad \textit{and} \quad \|y_3\| > a \quad \textit{with} \quad \alpha(y_3) < b$$

3. Main Results

In this section, we consider the existence and multiplicity of positive solutions of problem (1.1) by means of the Lemmas 2.4 and 2.5. First of all, we find the Green's function for boundary-value problem (1.1).

Lemma 3.1. *Let $h(t) \in C[0, 1]$ be a given function, then the boundary-value problem*

$$\begin{gathered} \mathbf{D}_{0+}^{\alpha} u(t) = h(t), \quad 0 < t < 1 \\ u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0 \end{gathered} \tag{3.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)ds \tag{3.2}$$

where

$$G(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1}(1-t)+(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & s \le t \\ \frac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & t \le s \end{cases} \tag{3.3}$$

Here $G(t,s)$ is called the Green's function of boundary-value problem (3.1)*.*

Proof. By the Lemma 2.3, we can reduce the equation of problem (3.1) to an equivalent integral equation

$$u(t) = I_{0+}^{\alpha}h(t) - c_1 - c_2 t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - c_1 - c_2 t$$

for some constants $c_1, c_2 \in \mathbb{R}$. On the other hand, by relations $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$ and $I_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\alpha+\beta} u(t)$, for $\alpha, \beta > 0$, $u \in L(0,1)$ (see [6]), we have

$$u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}h(s)ds - c_2$$

As boundary conditions for problem (3.1), we have

$$\begin{gathered} -c_1 - c_2 = 0 \\ -c_1 - 2c_2 = -I_{0+}^{\alpha}h(1) - I_{0+}^{\alpha-1}h(1); \end{gathered}$$

that is,

$$\begin{gathered} c_1 = -I_{0+}^{\alpha}h(1) - I_{0+}^{\alpha-1}h(1) \\ c_2 = I_{0+}^{\alpha}h(1) + I_{0+}^{\alpha-1}h(1) \end{gathered}$$

Therefore, the unique solution of (3.1) is

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)ds - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds \\ &\quad - \frac{t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)ds \\ &= \int_0^t \Big(\frac{(1-s)^{\alpha-1}(1-t)+(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}\Big)h(s)ds \\ &\quad + \int_t^1 \Big(\frac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}\Big)h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds \end{aligned}$$

which completes the proof. □

Lemma 3.2. *Let $h(t) \in C[0,1]$ be a given function, then function $G(t,s)$ defined by* (3.3) *has the following properties:*

(R1) *$G(t,s) \in C([0,1] \times [0,1))$, and $G(t,s) > 0$ for $t, s \in (0,1)$;*

(R2) *There exists a positive function* $\gamma \in C(0,1)$ *such that*

$$\begin{gathered}\min_{1/4\le t\le 3/4} G(t,s) \ge \gamma(s)M(s), \quad s\in(0,1)\\ \max_{0\le t\le 1} G(t,s)\le M(s),\end{gathered} \tag{3.4}$$

where

$$M(s)=\frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, \quad s\in[0,1) \tag{3.5}$$

Proof. From the expression of $G(t,s)$, it is obvious that $G(t,s)\in C([0,1]\times[0,1))$ and $G(t,s)\ge 0$ for $s,t\in(0,1)$. Next, we will prove (R2). From the definition of $G(t,s)$, we can known that, for given $s\in(0,1)$, $G(t,s)$ is decreasing with respect to t for $t\le s$, we let

$$\begin{gathered}g_1(t,s)=\frac{(1-t)(1-s)^{\alpha-1}+(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, \quad s\le t\\ g_2(t,s)=\frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, \quad t\le s\end{gathered}$$

That is, $g_1(t,s)$ is a continuous function for $\frac{1}{4}\le t\le\frac{3}{4}$, and $g_2(t,s)$ is decreasing with respect to t. Hence, we have

$$\begin{aligned}g_1(t,s)&\ge\frac{(1-s)^{\alpha}}{4\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, \quad \text{for } 1/4\le t\le 3/4\\ \max_{0\le t\le 1} g_1(t,s)&\le\frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\ \min_{1/4\le t\le 3/4} g_2(t,s)&=g_2(\frac{3}{4},s)=\frac{(1-s)^{\alpha-1}}{4\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}\\ \max_{0\le t\le 1} g_2(t,s)&=g_2(0,s)=\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\ &<\frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\end{aligned}$$

Thus, we have

$$\min_{1/4\le t\le 3/4} G(t,s)\ge m(s)=\frac{(1-s)^{\alpha-1}}{4\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, \quad s\in[0,1) \tag{3.6}$$

$$\max_{0\le t\le 1} G(t,s)\le M(s)=\frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, \quad s\in[0,1) \tag{3.7}$$

Let

$$\gamma(s)=m(s)/M(s)=\frac{1}{4}\frac{(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}}{2(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}}, \quad s\in(0,1) \tag{3.8}$$

It is obviously that $\gamma(s)\in C((0,1),(0,+\infty))$. The proof is completed. □

Remark 3.3. From the definition of function $\gamma(s)$, we see that $\gamma(s)\ge\frac{1}{8}$.

Let $E=C[0,1]$ be endowed with the ordering $u\le v$ if $u(t)\le v(t)$ for all $t\in[0,1]$, and the maximum norm, $\|u\|=\max_{0\le t\le 1}|u(t)|$, Define the cone $K\subset E$ by

$$K=\{u\in E|u(t)\ge 0, \min_{1/4\le t\le 3/4}\ge\frac{1}{8}\|u\|\}$$

and the nonnegative continuous concave functional φ on the cone K by

$$\varphi(u) = \min_{1/4 \le t \le 3/4} |u(t)|$$

Lemma 3.4. *Assume that $f(t,u)$ is continuous on $[0,1]\times[0,\infty)$. A function $u \in K$ is a solution of boundary-value problem* (1.1) *if and only if it is a solution of the integral equation* (3.2).

Proof. Let $u \in K$ be a solution of boundary-value problem (1.1). Applying the operator I_{0+}^{α} to both sides of equation of problem (1.1), we have

$$u(t) = c_1 + c_2 t + I_{0+}^{\alpha} f(t, u(t))$$

for some $c_1, c_2 \in \mathbb{R}$. By the same methods as obtaining the Green's function of problem (1.1) (Lemma 3.1), by boundary value conditions of problem (1.1), we can calculate out constants c_1 and c_2, so

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) ds$$

From Lemma 3.2 and Remark 3.3, we obtain that $\int_0^1 G(t,s) f(s,u(s)) ds \in K$. Hence, u is also a solution of integral equation (3.2).

Let $u \in K$ be a solution of integral equation (3.2). If we denote the right-hand side of integral equation (3.2) by $w(t)$, then, applying Caputo's fractional operator to both sides of integral equation (3.2), by the Definition of function $G(t,s)$, since

$$\begin{aligned}
w(t) &= \int_0^1 G(t,s) f(s,u(s)) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,u(s)) ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s)) ds - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,u(s)) ds \\
&\quad - \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s)) ds\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
w'(t) &= \frac{d}{dt} I_{0+}^{\alpha} f(t,u(t)) - I_{0+}^{\alpha} f(1,u(1)) - I_{0+}^{\alpha-1} f(1,u(1)) \\
&= D_{0+}^{1} I_{0+}^{\alpha} f(t,u(t)) - I_{0+}^{\alpha} f(1,u(1)) - I_{0+}^{\alpha-1} f(1,u(1)) \\
&= D_{0+}^{1} I_{0+}^{1} I_{0+}^{\alpha-1} f(t,u(t)) - I_{0+}^{\alpha} f(1,u(1)) - I_{0+}^{\alpha-1} f(1,u(1)) \\
&= I_{0+}^{\alpha-1} f(t,u(t)) - I_{0+}^{\alpha} f(1,u(1)) - I_{0+}^{\alpha-1} f(1,u(1))
\end{aligned}$$

and

$$\begin{gathered}
w''(t) = D_{0+}^{1} I_{0+}^{\alpha-1} f(t,u(t)) = D_{0+}^{2-\alpha} f(t,u(t)) \\
\mathbf{D}_{0+}^{\alpha} w(t) = I_{0+}^{2-\alpha} w''(t) = I_{0+}^{2-\alpha} D_{0+}^{2-\alpha} f(t,u(t)) = f(t,u(t))
\end{gathered}$$

here, use the relation $I_{0+}^{s} I_{0+}^{t} g(t) = I_{0+}^{s+t} g(t)$, $D_{0+}^{s} I_{0+}^{s} g(t) = g(t)$, $s > 0$, $t > 0$, $g \in L(0,1)$ and $I_{0+}^{s} D_{0+}^{s} g(t) = g(t)$, $s > 0$, $g \in C[0,1]$ (see [6]), where D_{0+}^{s} is

Riemann-Liouville fractional derivative. That is, $\mathbf{D}_{0+}^{\alpha}u(t) = f(t, u(t))$. On the other hand, one has

$$\begin{aligned} u(0) &= \int_0^1 \Big(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big) f(s,u(s))ds \\ u'(0) &= -\int_0^1 \Big(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big) f(s,u(s))ds \\ u(1) &= \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \\ u'(1) &= -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds\,. \end{aligned}$$

We obtain

$$u(0) + u'(0) = 0, u(1) + u'(1) = 0$$

which implies that $u \in K$ is a solution of problem (1.1). □

Lemma 3.5. *Assume that $f(t, u)$ is continuous on $[0, 1] \times [0, \infty)$, and define the operator $T : K \to E$ by*

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))ds$$

Then $T : K \to K$ is completely continuous.

Proof. Firstly, we prove that $T : K \to K$. In view of the expression of $G(t, s)$, it is clear that, $Tu(t) \geq 0$, $t \in [0, 1]$, $Tu(t)$ is continuous for $u \in K$. And that, for $u \in K$, by means of the Lemma 3.1 and Remark 3.3, we have

$$\min_{1/4 \leq t \leq 3/4} Tu(t) = \min_{1/4 \leq t \leq 3/4} \int_0^1 G(t,s) f(s,u(s))ds \geq \frac{1}{8} \int_0^1 M(s) f(s,u(s))ds$$

On the other hand,

$$\|Tu\| = \max_{0 \leq t \leq 1} |Tu(t)| \leq \int_0^1 M(s) f(s,u(s))ds\,.$$

Thus, we obtain

$$\min_{1/4 \leq t \leq 3/4} Tu(t) \geq \frac{1}{8} \|Tu\|$$

which implies $T : K \to K$.

Let $P \subset K$ be bounded, i.e. there exists a positive constant $L > 0$ such that $\|u\| \leq L$, for all $u \in P$. Let $M = \max_{0 \leq t \leq 1, 0 \leq u \leq L} |f(t, u)| + 1$, then for $u \in P$, from the Lemma 3.1, one has

$$|Tu(t)| \leq \int_0^1 |G(t,s) f(t,u(s))| ds \leq M \int_0^1 M(s) ds$$

Hence, $T(P)$ is bounded. For all $\varepsilon > 0$, each $u \in P$, $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, let

$$\eta = \min\{\frac{1}{2}, \frac{\Gamma(\alpha)\varepsilon}{12M}, \frac{\Gamma(1+\alpha)\varepsilon}{8M}\}$$

we will prove that $|Tu(t_2) - Tu(t_1)| < \varepsilon$, when $t_2 - t_1 < \eta$. One has

$$\begin{aligned}
&|Tu(t_2) - Tu(t_1)| \\
&= |\int_0^1 G(t_2,s)f(s,u(s))ds - \int_0^1 G(t_1,s)f(s,u(s))ds| \\
&\leq \int_0^{t_1} |(G(t_2,s) - G(t_1,s))f(s,u(s))|ds + \int_{t_2}^1 |(G(t_2,s) - G(t_1,s))f(s,u(s))|ds \\
&\quad + \int_{t_1}^{t_2} |(G(t_2,s) - G(t_1,s))f(s,u(s))|ds \\
&\leq M(\int_0^{t_1} |(G(t_2,s) - G(t_1,s))|ds + \int_{t_2}^1 |(G(t_2,s) - G(t_1,s))|ds \\
&\quad + \int_{t_1}^{t_2} |(G(t_2,s) - G(t_1,s))|ds) \\
&= M(\int_0^{t_1} (\frac{(t_2-t_1)(1-s)^{\alpha-1} + ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})}{\Gamma(\alpha)} \\
&\quad + \frac{(t_2-t_1)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)})ds \\
&\quad + \int_{t_2}^1 (\frac{(t_2-t_1)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t_2-t_1)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)})ds \\
&\quad + \int_{t_1}^{t_2} (\frac{(t_2-t_1)(1-s)^{\alpha-1} + (t_2-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t_2-t_1)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)})ds) \\
&\leq M(\int_0^{t_1} (\frac{\eta + ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})}{\Gamma(\alpha)} + \frac{\eta(1-s)^{\alpha-2}}{\Gamma(\alpha-1)})ds \\
&\quad + \int_{t_2}^1 (\frac{\eta}{\Gamma(\alpha)} + \frac{\eta(1-s)^{\alpha-2}}{\Gamma(\alpha-1)})ds + \int_{t_1}^{t_2} (\frac{\eta + (t_2-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(1-s)^{\alpha-2}}{\Gamma(\alpha-1)})ds) \\
&\leq M(\frac{2\eta}{\Gamma(\alpha)} + \frac{t_2^\alpha - t_1^\alpha}{\Gamma(1+\alpha)} + \frac{2\eta}{\Gamma(\alpha)} + \frac{2\eta}{\Gamma(\alpha)} + \frac{2\eta^\alpha}{\Gamma(1+\alpha)}) \\
&= M(\frac{6\eta}{\Gamma(\alpha)} + \frac{2\eta^\alpha + (t_2^\alpha - t_1^\alpha)}{\Gamma(1+\alpha)}) \\
&< M(\frac{6\eta}{\Gamma(\alpha)} + \frac{2\eta + (t_2^\alpha - t_1^\alpha)}{\Gamma(1+\alpha)})
\end{aligned}$$

In order to estimate $t_2^\alpha - t_1^\alpha$, we can apply a method used in [7]; that is, for $\eta \leq t_1 < t_2 \leq 1$, by means of mean value theorem of differentiation, we have

$$t_2^\alpha - t_1^\alpha \leq \alpha(t_2 - t_1) < \alpha\eta \leq 2\eta$$

for $0 \leq t_1 < \eta, t_2 < 2\eta$, we have

$$t_2^\alpha - t_1^\alpha \leq t_2^\alpha < (2\eta)^\alpha \leq 2\eta\,.$$

while for $0 \leq t_1 < t_2 \leq \eta$, there has

$$t_2^\alpha - t_1^\alpha \leq t_2^\alpha < \eta^\alpha < 2\eta\,.$$

Thus, we obtain

$$|Tu(t_2) - Tu(t_1)| < \frac{6M\eta}{\Gamma(\alpha)} + \frac{4M\eta}{\Gamma(1+\alpha)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

By means of the Arzela-Ascoli theorem, $T: K \to K$ is completely continuous. $\square$

Theorem 3.6. *Assume that $f(t,u)$ is continuous on $[0,1]\times[0,\infty)$, and satisfies one of the following conditions*

(H1) *There exist $0<\mu_1,\nu_1\le 1$ such that*
$$\lim_{u\to\infty}\frac{f(t,u(t))}{u^{\mu_1}}=0,\quad \lim_{u\to 0}\frac{f(t,u(t))}{u^{\nu_1}}=\infty$$
for all $t\in[0,1]$

(H1') *There exist $\mu_2,\nu_2\ge 1$ such that*
$$\lim_{u\to\infty}\frac{f(t,u(t))}{u^{\mu_2}}=\infty,\quad \lim_{u\to 0}\frac{f(t,u(t))}{u^{\nu_2}}=0$$
for all $t\in[0,1]$.

Then problem (1.1) *has one positive solution.*

Proof. By the Lemma 3.4, we know that we only need to consider existence of fixed point of operator T in K. It follows from the Lemma 3.5 that $T:K\to K$ is a completely continuous operator. Assume that (H1) holds, then there exist $N_1>0, N_2>0$, such that for all $0<\varepsilon<(2\int_0^1 M(s)ds)^{-1}$ and $\rho>\frac{64}{\int_{1/4}^{3/4}M(s)ds}>0$. Then
$$\begin{gathered}
f(t,u(t))\le \varepsilon u^{\mu_1},\quad \text{for } t\in[0,1], u\ge N_1\\
f(t,u(t))>\rho u^{\nu_1},\quad \text{for } t\in[0,1], 0\le u\le N_2
\end{gathered}$$
So we have
$$f(t,u(t))\le \varepsilon u^{\mu_1}+c,\quad \text{for } t\in[0,1], u\in[0,+\infty)$$
where
$$c=\max_{0\le t\le 1, 0\le u\le N_1}|f(t,u(t))|+1$$
Let
$$\Omega_1=\{u\in K; \|u\|<R_1\}$$
where $R_1>\{1, 2c\int_0^1 M(s)ds\}$. For $u\in\partial\Omega_1$, from the Lemma 3.2, we have
$$\begin{aligned}
|Tu(t)| &= \int_0^1 G(t,s)f(s,u(s))ds\\
&\le \int_0^1 M(s)(\varepsilon|u|^{\mu_1}+c)ds\\
&\le \varepsilon R_1^{\mu_1}\int_0^1 M(s)+c\int_0^1 M(s)ds\\
&\le \frac{R_1}{2}+\frac{R_1}{2}=R_1\,,
\end{aligned}$$
$\|Tu\|\le R_1=\|u\|$. Let
$$\Omega_2=\{u\in K; \|u\|<R_2\}$$

where $0 < R_2 < \{1, N_2\}$, then for $u \in \partial\Omega_2$, we obtain

$$\begin{aligned}
|Tu(t)| &= |\int_0^1 G(t,s)f(s,u(s))ds| \\
&\geq \int_{1/4}^{3/4} G(t,s)f(s,u(s))ds \\
&> \frac{\rho}{8}\int_{1/4}^{3/4} M(s)u(s)^{\nu_1}ds \\
&\geq \frac{\rho}{64}\int_{1/4}^{3/4} M(s)\|u\|^{\nu_1}ds \\
&= \frac{\rho}{64}\int_{1/4}^{3/4} M(s)R_2R_2^{\nu_1-1}ds \\
&\geq \frac{\rho}{64}\int_{1/4}^{3/4} M(s)R_2ds \\
&> R_2 = \|u\|
\end{aligned}$$

so $\|Tu\| \geq R_2 = \|u\|$. Then Lemma 2.4 implies that operator T has one fixed point $u^*(t) \in \overline{\Omega_1}\backslash\Omega_2$. Then $u^*(t)$ is one positive solution of problem (1.1).

For condition (H1'), we can obtain the result in a similarly way. Now, we give a briefly description. Assume that (H1') holds, thus, there exist $M_1 > 0, M_2 > 0$, such that for all $0 < \varepsilon < (\int_0^1 M(s)ds)^{-1}$ and $\lambda > (\frac{\int_{1/4}^{3/4} M(s)ds}{64})^{-1} > 0$, we have have

$$\begin{gathered}
f(t,u(t)) > \lambda u^{\mu_2}, \quad \text{for } t \in [0,1], u \geq M_1 \\
f(t,u(t)) \leq \varepsilon u^{\nu_2}, \quad \text{for } t \in [0,1], 0 \leq u \leq M_2
\end{gathered}$$

Let

$$\Omega_1 = \{u \in K; \|u\| < R_1\}, \quad \Omega_2 = \{u \in K; \|u\| < R_2\}$$

where $R_1 > \{1, 8M_1\}$, $0 < R_2 < \{1, M_2\}$. Then for $u \in \partial\Omega_1$, for $\frac{1}{4} \leq t \leq \frac{3}{4}$, one has $u(t) \geq \min_{1/4 \leq t \leq 3/4} u(t) \geq \frac{1}{8}\|u\| = \frac{R_1}{8} > M_1$. thus, from the Lemma 3.2, we have

$$\begin{aligned}
|Tu(t)| &\geq \int_{1/4}^{3/4} G(t,s)f(s,u(s))ds \\
&\geq \frac{\lambda}{64}\int_{1/4}^{3/4} M(s)\|u\|^{\mu_2}ds \\
&\geq \frac{\lambda}{64}\int_{1/4}^{3/4} M(s)\|u\|ds \\
&> R_1 = \|u\|
\end{aligned}$$

for $u \in \partial\Omega_2$, we obtain

$$|Tu(t)| \leq \int_0^1 M(s)\varepsilon\|u\|^{\nu_2}ds \leq \varepsilon R_2\int_0^1 M(s)ds \leq R_2$$

Thence, the Lemma 2.4 implies that operator T has one fixed point $u^*(t) \in \overline{\Omega_1}\backslash\Omega_2$, then $u^*(t)$ is a positive solution of problem (1.1). $\square$

Let

$$M = (\int_0^1 M(s)ds)^{-1}, \quad N = (\int_{1/4}^{3/4} \gamma(s)M(s)ds)^{-1}$$

Theorem 3.7. *Assume that $f(t,u)$ is continuous on $[0,1]\times[0,\infty)$, and there exist constants $0<b<c$ such that:*

- (H2) *There exists $r \ge c$ such that $f(t,u(t)) < Mu$ for all $(t,u) \in [0,1]\times[0,r]$;*
- (H3) *$f(t,u) \ge Nb$, for all $(t,u) \in [\frac14,\frac34]\times[b,c]$.*

Then problem (1.1) has at least three positive solutions u_1,u_2,u_3 with

$$\|u_1\| < a, \quad b < \min_{1/4\le t\le 3/4} |u_2(t)|$$
$$a < \|u_3\|, \quad \min_{1/4\le t\le 3/4} |u_3(t)| < b$$

Proof. We will apply the Lemma 2.5 to prove this result. Next, we show that all conditions of the Lemma 2.5 are satisfied. By the Lemma 3.4, we know that we only need to consider existence of fixed point of operator T in K. It follows from the Lemma 3.5 that $T: K \to K$ is a completely continuous operator.
By (H2), there exist r, such that

$$f(t,u(t)) < Mu, \quad \text{for } t\in[0,1], 0\le u\le r$$

Let $0<a<b<c\le r$, then if $u \in \overline{K}_c$ ($\overline{K}_c = \{u\in K | \|u\| \le c\}$,$K_c = \{u\in K|\|u\| < c\}$), we can obtain

$$\|Tu\| = \max_{0\le t\le 1} |\int_0^1 G(t,s)f(s,u(s))ds| < M\int_0^1 M(s)\|u\|ds = \|u\| \le c$$

Hence, combining with the Lemma 3.5, we know that $T:\overline{K}_c \to \overline{K}_c$ is completely continuous. In the same way, let $0<a<c$, then if $u\in\overline{K}_a$, we can also obtain that $\|Tu\| < a$ which satisfies the condition (C2) of the Lemma 2.5. Now, we check condition (C1) of the Lemma 2.5. Let $u(t) = \frac{b+c}{2}$, $0\le t\le 1$. It is obvious that $u(t) = \frac{b+c}{2} \in K(\delta,b,c)$, $\delta(u) = \frac{b+c}{2} > b$, thus, $\{u\in K(\delta,b,c)|\delta(u) > b\} \neq \emptyset$. Thence, if $u\in K(\delta,b,c)$, then $b\le u(t)\le c$ for $1/4\le t\le 3/4$, by assumption (H_3), we have $f(t,u)\ge Nb$, for $\frac14 \le t\le \frac34$, so, by the Lemma 3.2, there has

$$\delta(Tu) = \min_{1/4\le t\le 3/4} |Tu(t)| > \int_{1/4}^{3/4} \gamma(s)M(s)Nbds = b$$

By Lemma 2.5, problem (1.1) has at least three positive solutions u_1,u_2,u_3 with the required conditions; which completes the proof. □

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Addendum posted on November 9, 2009.

The definition $n = [\alpha] + 1$ in Lemmas 2.2 and 2.3 is incomplete. It should be

$$n = \begin{cases} [\alpha] + 1 & \text{if } n \notin \{0, 1, 2, \dots\} \\ \alpha & \text{if } n \in \{0, 1, 2, \dots\} . \end{cases}$$

The author wants to thank Yige Zhao, Shurong Sun, Zhenlai Han, and Meng Zhang (at the University of Jinan) for pointing out this misprint.

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