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ASYMPTOTICALLY ALMOST PERIODIC AND ALMOST PERIODIC SOLUTIONS FOR A CLASS OF PARTIAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this note, we establish the existence of asymptotically almost periodic and almost periodic solutions for a class of partial integrodifferential equations.

1. Introduction

In this short note, we study the existence of asymptotically almost periodic and almost periodic solutions for a class of abstract partial integrodifferential equations of the form

$$u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + g(t, u(t)), \tag{1.1}$$

$$u(0) = x_0, \tag{1.2}$$

where $A : D(A) \subset X \to X$, $B(t) : D(B(t)) \subset X \to X$, $t \geq 0$, are linear, closed and densely defined operators on a Banach space X; $D(B(t)) \supset D(A)$ for every $t \geq 0$ and $g(\cdot)$ is a continuous function.

Abstract partial integrodifferential equations arise in many areas of applied mathematics and for this reason this type of equation has received much attention in recent years, see for example [8, 10, 13, 14]. The existence and qualitative properties of solutions for different types of abstracts partial integrodifferential systems have been treated in several works, see for instance [11, 12] and the associated references.

The existence of almost periodic and asymptotically almost periodic solutions is one of the most attracting topics in the qualitative theory of differential equations due to their significance in physical sciences. For the cases of ordinary differential equations and abstract partial differential equations, this problem has been treated in several works and respect to this matter we cite [4, 5, 15, 22, 18, 19, 20, 21] and the references therein.

The almost periodicity of solutions of abstract partial integrodifferential equations have been studied in some research works. Prüss studied in [16] conditions

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under which a solution of $u'(t) = Au(t) + \int_0^\infty dB(r)u(t-r) + g(t)$ is periodic or almost periodic if g has the corresponding property. On the other hand, Jakubowski & Ruess investigated in [9] when some asymptotic properties (such as asymptotic almost periodicity and weak almost periodicity in the sense of Eberlein) of the forcing term g are inherited by the solutions of the integrodifferential equation

$$\frac{d}{dt}\Big(\kappa(u(t)-u(0)) + \int_0^t K(t-s)(u(s)-u(0))ds\Big) + Au(t) \ni g(t),$$

where A is a multi-valued m-accretive operator. To the best of our knowledge, the study of the existence of asymptotically almost periodic and almost periodic solutions for semi-linear integrodifferential equation (the case $f(t,u(t))$) is a untreated topic, and this fact is the main motivation of this work.

To obtain our results we will use the theory of resolvent of bounded linear operators. This theory is related to partial integrodifferential equations in the same manner that semigroup theory is related to first order linear partial differential equations. The existence of solutions and wellposedness of (1.1)-(1.2), (equivalently, the existence of a resolvent of bounded linear operators associated to (1.1)-(1.2)), have been considered in many works and under different assumptions on the operators $A, B(t)$. For additional details respect this theory and their applications to partial integrodifferential equations, we suggest the reader the Grimmer works [3, 6, 7].

Next, we review some notations and properties needed to establish our results. In this paper, $(X, \|\cdot\|)$ is a abstract Banach space; $A : D(A) \subset X \to X$ and $B(t) : D(B(t)) \subset X \to X$, $t \geq 0$, are linear, closed and densely defined operator on X with $D(B(t)) \supset D(A)$ for each $t \geq 0$. To obtain our results we will assume that the integrodifferential abstract Cauchy problem

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds, \quad t > 0, \tag{1.3}$$

$$x(0) = x_0 \in X, \tag{1.4}$$

has associated a resolvent operator $(R(t))_{t\geq 0}$ on X.

Definition 1.1. A one parameter family $(R(t))_{t\geq 0}$ of bounded linear operators from X into X is called a strongly continuous resolvent operator for (1.3)-(1.4) if the following conditions are verified.

(i) $R(0) = I_d$ and the function $R(t)x$ is continuous on $[0,\infty)$ for every $x \in X$.
(ii) $R(t)D(A) \subset D(A)$ for all $t \geq 0$ and for $x \in D(A)$, $AR(t)x$ is continuous on $[0,\infty)$ and $R(t)x$ is continuously differentiable on $[0,\infty)$.
(iii) For $x \in D(A)$, the next resolvent equations are verified,

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds, \quad t \geq 0,$$

$$R'(t)x = R(t)Ax + \int_0^t R(t-s)B(s)xds. \quad t \geq 0.$$

In this paper, we always assume that the resolvent operator $(R(t))_{t\geq 0}$ is uniformly exponentially stable and that $\widetilde{M}, \delta$ are positive constants such that $\|R(t)\| \leq \widetilde{M}e^{-\delta t}$ for every $t \geq 0$.

In the sequel, we mention a few results, definitions and notations related to asymptotically almost periodic and almost periodic functions. Next, $(Z, \|\cdot\|_Z)$,

$(W, \|\cdot\|_W)$ are Banach spaces and $C_0([0,\infty);Z)$ is the subspace of $C([0,\infty);Z)$ formed by the functions that vanishes at infinity.

Definition 1.2. A set $P \subset \mathbb{R}$ is said to be relatively dense in $\mathbb{R}$ if there exists a number $l > 0$ such that $[a, a+l] \bigcap P \neq \emptyset$ for every $a \in \mathbb{R}$.

Definition 1.3. A function $F \in C(\mathbb{R}; Z)$ is called almost periodic (a.p.) if for every $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, F, Z)$, such that $\|F(t+\xi) - F(t)\|_Z < \varepsilon$ for every $t \in \mathbb{R}$ and all $\xi \in \mathcal{H}(\varepsilon, F, Z)$.

Definition 1.4. A function $F \in C([0,\infty); Z)$ is called asymptotically almost periodic (a.a.p.) if there exists an almost periodic function $g(\cdot)$ and $w \in C_0([0,\infty); Z)$ such that $F(\cdot) = g(\cdot) + w(\cdot)$.

The next Lemma is a useful characterization of a.a.p function.

Lemma 1.5. [19, Theorem 5.5] *A function $F \in C([0,\infty); Z)$ is asymptotically almost periodic if and only if, for every $\varepsilon > 0$ there exists $L(\varepsilon, F, Z) > 0$ and a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{T}(\varepsilon, F, Z)$, such that*

$$\|F(t+\xi) - F(t)\|_Z < \varepsilon, \quad t \geq L(\varepsilon, F, Z),\ \xi \in \mathcal{T}(\varepsilon, F, Z).$$

In this paper, $AP(Z)$ and $AAP(Z)$ are the spaces

$$AP(Z) = \{F \in C(\mathbb{R}; Z) : F \text{ is a.p.}\},$$
$$AAP(Z) = \{F \in C([0,\infty); Z) : F \text{ is a.a.p.}\},$$

endowed with the norms $|\ \|u\|\big|_Z = \sup_{s\in\mathbb{R}} \|u(s)\|$ and $\|u\|_Z = \sup_{s\geq 0} \|u(s)\|$ respectively. We know from [19] that $AP(Z)$ and $AAP(Z)$ are Banach spaces.

Definition 1.6. Let Ω be an open subset of W.

(a) A function $F \in C(\mathbb{R} \times \Omega; Z)$ is called pointwise almost periodic (p.a.p.) if $F(\cdot, x) \in AP(Z)$ for every $x \in \Omega$.
(b) A function $F \in C([0,\infty) \times \Omega; Z)$ is called pointwise asymptotically almost periodic (p.a.a.p.) if $F(\cdot, x) \in AAP(Z)$ for every $x \in \Omega$.
(c) A function $F \in C(\mathbb{R} \times \Omega; Z)$ is called uniformly almost periodic (u.a.p.), if for every $\varepsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, F, K, Z)$, such that
$$\|F(t+\xi, y) - F(t, y)\|_Z \leq \varepsilon \quad (t, \xi, y) \in \mathbb{R} \times \mathcal{H}(\varepsilon, F, K, Z) \times K.$$
(d) A function $F : C([0,\infty) \times \Omega; Z)$ is called uniformly asymptotically almost periodic (u.a.a.p.), if for every $\varepsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{T}(\varepsilon, F, K, Z)$, and a constant $L(\varepsilon, F, K, Z) > 0$ such that
$$\|F(t+\xi, y) - F(t, y)\|_Z \leq \varepsilon, \quad t \geq L(\varepsilon, F, K, Z),\ (\xi, y) \in \mathcal{T}(\varepsilon, F, K, Z) \times K.$$

The next lemma summarize some properties which are fundamental to obtain our results. This results can be obtained from [18, Theorem 1.2.7].

Lemma 1.7. *Let $\Omega \subset W$ be an open set. Then the following properties hold.*

(a) *If $F \in C(\mathbb{R} \times \Omega; Z)$ is p.a.p. and satisfies a local Lipschitz condition at $x \in \Omega$, uniformly at t, then F is u.a.p.*
(b) *If $F \in C([0,\infty) \times \Omega; Z)$ is p.a.a.p. and satisfies a local Lipschitz condition at $x \in \Omega$, uniformly at t, then F is u.a.a.p.*

(c) *If* $F \in C(\mathbb{R}\times\Omega; Z)$ *is u.a.p. and* $y \in AP(W)$ *is such that* $\overline{\{y(t) : t \in \mathbb{R}\}}^W \subset \Omega$*, then* $F(t, y(t)) \in AP(Z)$.

(d) *If* $F \in C([0,\infty) \times \Omega; Z)$ *is u.a.a.p;* $y \in AAP(W)$ *and* $\overline{\{y(t) : t \geq 0\}}^W \subset \Omega$*, then* $F(t, y(t)) \in AAP(Z)$.

2. Existence Results

In this section we study the existence of asymptotically almost periodic and almost periodic solutions of (1.1). The next result is proved using the ideas and estimates in [19, Example 2.2].

Lemma 2.1. *Let* $v \in AAP(X)$ *and* $u : [0,\infty) \to X$ *be the function defined by*

$$u(t) = \int_0^t R(t-s)v(s)ds, \quad t \geq 0.$$

Then $u \in AAP(X)$.

To prove our existence results we always assume that the next condition holds.

(H1) The function $g : \mathbb{R} \times X \to X$ is continuous and there exists a continuous and nondecreasing function $L_g : [0,\infty) \to [0,\infty)$ such that

$$\|g(t,x_1) - g(t,x_2)\| \leq L_g(r)\|x_1 - x_2\|, \quad t \in \mathbb{R}, x_i \in B_r(0,X).$$

From Grimmer [6], we adopt the following concept of mild solutions of (1.1).

Definition 2.2. A function $u \in AAP(X)$ is a asymptotically almost periodic mild solution of system (1.1) if

$$u(t) = R(t)u(0) + \int_0^t R(t-s)g(s,u(s))ds, \quad t \geq 0.$$

Definition 2.3. A function $u \in AP(X)$ is a almost periodic mild solution of system (1.1) if

$$u(t) = R(t-\sigma)u(\sigma) + \int_\sigma^t R(t-s)g(s,u(s))ds, \quad t,\sigma \in \mathbb{R},\, t \geq \sigma.$$

Remark 2.4. It is easy to see that $u \in AP(X)$ is a almost periodic mild solution of system (1.1) if, and only if,

$$u(t) = \int_{-\infty}^t R(t-s)g(s,u(s))ds, \quad t \in \mathbb{R}.$$

Now, we can to establish our first existence result.

Theorem 2.5. *Assume that* $g(\cdot)$ *is p.a.a.p. If* $L_g(0) = 0$ *and* $g(t,0) = 0$ *for every* $t \in \mathbb{R}$*, then there exists* $\varepsilon > 0$ *such that for each* $x_0 \in B_\varepsilon(0,X)$ *there exists a asymptotically almost periodic mild solution* $u(\cdot, x_0)$ *of (1.1) such that* $u(0,x_0) = x_0$.

Proof. Let $r > 0$ and $0 < \lambda < 1$ be such that $\widetilde{M}\lambda + \frac{\widetilde{M}L_g(r)}{\delta} < 1$. We affirm that the assertion holds for $\varepsilon = \lambda r$. In fact, let $x_0 \in B_\varepsilon(0,X)$. On the space

$$\mathfrak{D} = \{u \in AAP(X) : u(0) = x_0,\, \|u(t)\| \leq r,\, t \geq 0\}$$

endowed with the metric $d(u,v) = \|u-v\|_X$, we define the map $\Gamma : \mathfrak{D} \to C([0,\infty); X)$ by

$$\Gamma u(t) = R(t)x_0 + \int_0^t R(t-s)g(s,u(s))ds, \quad t \geq 0.$$

From the properties of $(R(t))_{t\geq 0}$ and $g(\cdot)$, we infer that $\Gamma u(\cdot)$ is well defined and that $\Gamma u \in C([0,\infty);X)$. Moreover, from Lemmas 1.7 and 2.1 it follows that $\Gamma u \in AAP(X)$.

Now, we prove that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$. From the definition of Γ, for $u \in \mathfrak{D}$ and $t \geq 0$ we get

$$\|\Gamma u(t)\| \leq \widetilde{M}\lambda r + \int_0^t \widetilde{M} e^{-\delta(t-s)} L_g(r) r ds \leq \Big(\widetilde{M}\lambda + \frac{\widetilde{M} L_g(r)}{\delta}\Big) r,$$

which implies that $\Gamma(\mathfrak{D}) \subset \mathfrak{D}$. On the other hand, for $u, v \in \mathfrak{D}$ we see that

$$\begin{aligned}\|\Gamma u(t) - \Gamma v(t)\| &\leq \widetilde{M} \int_0^t L_g(r) e^{-\delta(t-s)} \|u(s) - v(s)\| ds \\ &\leq \frac{\widetilde{M} L_g(r)}{\delta} \|u - v\|_X,\end{aligned}$$

which shows that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$. The assertion of the theorem is now a consequence of the contraction mapping principle. □

The next result is proved using the ideas and estimates in the proof of the previous theorem. The proof will be omit.

Theorem 2.6. *If $g(\cdot)$ is p.a.a.p; $L_g(t) = L_g$ for all $t \geq 0$ and $\frac{\widetilde{M} L_g}{\delta} < 1$, then for every $x_0 \in X$ there exists a unique asymptotically almost periodic mild solution $u(\cdot, x_0)$ of (1.1) such that $u(0, x_0) = x_0$.*

Now we discuss the existence of almost periodic solution for (1.1). In the next results, we will assume that $g(t,x) = p(t,x) + \varphi(t)$, $(t,x) \in \mathbb{R} \times X$, where $\varphi \in AP(X)$ and the following condition.

(H2) The function $p : \mathbb{R} \times X \to X$ is continuous and there exists a continuous and nondecreasing function $L_p : \mathbb{R} \to [0,\infty)$ such that
$$\|p(t,x_1) - p(t,x_2)\| \leq L_p(r)\|x_1 - x_2\|, \quad t \in \mathbb{R}, x_i \in B_r(0,X).$$

Theorem 2.7. *Assume that $p(\cdot)$ is p.a.p. If $L_p(0) = 0$ and $p(t,0) = 0$ for every $t \in \mathbb{R}$, then there exists $\eta > 0$ such that for every $\varphi \in B_\eta(0, AP(X))$ there exists a unique almost periodic mild solution of (1.1).*

Proof. Let $1 > r > 0$ and $\eta > 0$ be such that $\frac{\widetilde{M}}{\delta}(L_p(r) r + \eta) < r$. On the space $B_r = \{u \in AP(X) :| \|u\||_X \leq r\}$ we define the operator $\Gamma : B_r \to C_b(\mathbb{R}; X)$ by

$$\Gamma u(t) = \int_{-\infty}^t R(t-s) g(s, u(s)) ds, \quad t \in \mathbb{R}.$$

From the assumption, it is easy to see that $\Gamma u(\cdot)$ is continuous and from Lemma 1.7 we infer that $v(t) = g(t, u(t)) \in AP(X)$. Consequently, for $t \in \mathbb{R}$ and $\xi \in \mathcal{H}(\varepsilon, v, X)$ we get

$$\begin{aligned}\|\Gamma u(t+\xi) - \Gamma u(t)\| &\leq \int_{-\infty}^t \widetilde{M} e^{-\delta(t-s)} \|g(s+\xi, u(s+\xi)) - g(s, u(s))\| ds \\ &\leq \int_{-\infty}^t \widetilde{M} e^{-\delta(t-s)} \varepsilon\, ds \leq \frac{\widetilde{M}\varepsilon}{\delta},\end{aligned}$$

which implies that $\Gamma u \in AP(X)$. Moreover, for $u \in B_r$ we see that

$$\begin{aligned}\|\Gamma u(t)\| &\leq \int_{-\infty}^{t} \widetilde{M} e^{-\delta(t-s)} L_g(r) \|u(s)\| ds + \int_{-\infty}^{t} \widetilde{M} e^{-\delta(t-s)} \mid \|\varphi\| | ds \\ &\leq \frac{\widetilde{M}}{\delta}(L_g(r) r + \eta)\end{aligned}$$

which shows that $\Gamma(B_r) \subset B_r$. On the other hand, for $u, v \in B_r$ we find that

$$\begin{aligned}\|\Gamma u(t) - \Gamma v(t)\| &\leq \int_{-\infty}^{t} \widetilde{M} e^{-\delta(t-s)} L_g(r) \|u(s) - v(s)\| ds \\ &\leq \frac{\widetilde{M} L_g(r)}{\delta} \mid \|u - v\| |_X,\end{aligned}$$

which allow us conclude that Γ is a contraction from B_r into B_r.

The existence of an almost periodic mild solution for (1.1) is now a consequence of the contraction on mapping principle. This completes the proof. □

In a similar manner we can prove the next result.

Theorem 2.8. *Assume condition (H2) holds and that $p(\cdot)$ is p.a.p. If $L_g(t) = L_g$ for all $t \geq 0$ and $\frac{\widetilde{M} L_g}{\delta} < 1$, then there exists a unique almost periodic mild solution of* (1.1).

3. Example

In this section we apply our abstract results to establish the existence of almost periodic and asymptotically almost periodic solutions for the partial integrodifferential differential

$$\begin{aligned}C\theta''(t) + \beta(0)\theta'(t) &= \alpha(0)\Delta\theta(t) - \int_0^t \beta'(t-s)\theta'(s) ds \\ &\quad + \int_0^t \alpha'(t-s)\Delta\theta(s) ds + a_1(t) a_2(\theta(t)),\end{aligned} \tag{3.1}$$

which arise in the study of heat conduction in materials with fading memory, see [14, 7, 2].

In the sequel, $X = H_0^1(\Omega) \times L^2(\Omega)$ where $\Omega \subset \mathbb{R}^3$ is a open set with smooth boundary of class C^∞; $\alpha(\cdot), \beta(\cdot)$ are $\mathbb{R}$-valued functions of class C^2 on $[0, \infty)$ with $\alpha(0) > 0$, $\beta(0) > 0$ and $A : D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \to X$ is the operator defined by

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ \alpha(0)\Delta - \beta(0) y \end{bmatrix}$$

where Δ is the Laplacian on Ω with boundary condition $\theta\big|_{\partial\Omega} = 0$. We know from Chen [1], that A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t \geq 0}$ on X and that there are positive constant $\widetilde{M}, \gamma$ such that $\|T(t)\| \leq \widetilde{M} e^{-\gamma t}$ for all $t \geq 0$.

Let $B(t) = AF(t)$ where $F(t) : X \to X$, $t \geq 0$, is defined by

$$F = (F_{ij}) = \begin{bmatrix} 0 & 0 \\ -\beta'(t) + \beta(0)\frac{\alpha'(t)}{\alpha(0)} & \frac{\alpha'(t)}{\alpha(0)} \end{bmatrix}.$$

Assume functions $\alpha^{(i)}(\cdot)$, $\beta^{(i)}(\cdot)$, $i=1,2$, be bounded, uniformly continuous and that

$$\max\{\|F_{22}(t)\|,\|F_{21}(t)\|\} \le \frac{\gamma e^{-\gamma t}}{2M}, \quad t\ge 0,$$
$$\max\{\|F'_{22}(t)\|,\|F'_{21}(t)\|\} \le \frac{\gamma^2 e^{-\gamma t}}{4M^2}, \quad t\ge 0.$$

Under these conditions, the abstract integrodifferential system

$$x'(t) = Ax(t) + \int_0^t AF(t-s)x(s)ds, \tag{3.2}$$

has associated a resolvent of operator $(R(t))_{t\ge 0}$ on X such that $\|R(t)\| \le \widetilde{M}e^{\frac{-\gamma t}{2}}$ for $t\ge 0$, see Grimmer [7, p. 343] for details.

Consider the integrodifferential system

$$\begin{aligned}\begin{bmatrix}\theta'(t)\\ \eta'(t)\end{bmatrix} &= \begin{bmatrix}0 & I\\ \alpha(0)\Delta & -\beta(0)I\end{bmatrix}\begin{bmatrix}\theta(t)\\ \eta(t)\end{bmatrix} + \int_0^t \begin{bmatrix}0 & I\\ \alpha'(t-s)\Delta & -\beta'(t-s)I\end{bmatrix}\begin{bmatrix}\theta(s)\\ \eta(s)\end{bmatrix}\\ &\quad + \begin{bmatrix}0\\ a_1(t)a_2(\theta(t))\end{bmatrix}\end{aligned} \tag{3.3}$$

where the functions $a_i:\mathbb{R}\to\mathbb{R}$, $i=1,2$, are continuous. If $g(\cdot)$ is the function defined by

$$g(t,\begin{bmatrix}x\\ y\end{bmatrix})(\xi) = a_1(t)\begin{bmatrix}0\\ a_2(x(\xi))\end{bmatrix}$$

the system (3.3) can be transformed into the abstract integrodifferential equation (1.1).

The next result follows from Theorems 2.5 and 2.7. We will omit the proof.

Theorem 3.1. *Assume that the previous conditions are satisfied and that there exists a constant $L>0$ such that*

$$|a_2(t)-a_2(s)| \le L|t-s|, \quad t,s\in\mathbb{R}.$$

If $a_1(\cdot)$ is asymptotically almost periodic (resp. almost periodic) and $\frac{2\|a_1\|_{\mathbb{R}}ML}{\gamma} < 1$, then there exist a asymptotically almost periodic mild solution (resp. a almost periodic mild solution) of (3.3).

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