Electronic Journal of Differential Equations, Vol. 2006(2006), No. 46, pp. 1–30.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ANISOTROPIC NONLINEAR ELLIPTIC SYSTEMS WITH MEASURE DATA AND ANISOTROPIC HARMONIC MAPS INTO SPHERES

MOSTAFA BENDAHMANE, KENNETH H. KARLSEN

Abstract. We prove existence results for distributional solutions of anisotropic nonlinear elliptic systems with a measure valued right-hand side. The functional setting involves anisotropic Sobolev spaces as well as weak Lebesgue (Marcinkiewicz) spaces. In a special case we also prove maximal regularity and uniqueness results. Some of the obtained results are applied, along with an anisotropic variant of the div-curl lemma in the Hardy one space, to prove that the space of anisotropic harmonic maps into spheres is compact in the weak topology of the relevant anisotropic Sobolev space.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$ ($N \geq 2$) with Lipchitz boundary $\partial\Omega$. Our aim is to prove the existence of at least one distributional solution $u = (u_1, \dots, u_m)^\top$ ($m \geq 1$) to the anisotropic nonlinear elliptic system

$$\begin{aligned} -\sum_{l=1}^{N} \frac{\partial}{\partial x_l} \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) &= \mu, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where the right-hand side $\mu = (\mu_1, \dots, \mu_m)^\top$ is a given vector-valued Radon measure on Ω of finite mass.

We assume that the vector fields $\sigma_l : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$, $l = 1, \dots, N$, satisfy the following conditions concerning continuity, coercivity, growth, and strict monotonicity:

$$\begin{aligned} &\sigma_l(x,\xi) \text{ is measurable in } x \in \Omega \text{ for every } \xi \in \mathbb{R}^m \text{ and} \\ &\sigma_l(x,\xi) \text{ is continuous in } \xi \in \mathbb{R}^m \text{ for a.e. } x \in \Omega; \\ &\sigma_l(x,\xi) \cdot \xi \geq c_1 \big|\xi\big|^{p_l} - c_2, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^m; \\ &\big|\sigma_l(x,\xi)\big| \leq c_1' \,|\xi|^{p_l - 1} + c_2', \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^m; \end{aligned} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35A05, 35J70, 58E20.
Key words and phrases. Elliptic system; anisotropic; nonlinear; measure data; weak solution; existence; harmonic map.

Submitted February 7, 2005. Published April 6, 2006.

and for all $x \in \Omega$, and all $\xi, \xi' \in \mathbb{R}^m$,

$$(\sigma_l(x,\xi) - \sigma_l(x,\xi')) \cdot (\xi - \xi') \geq \begin{cases} c_3 \big|\xi - \xi'\big|^{p_l}, & \text{if } p_l \geq 2, \\ c_4 \frac{\left|\xi - \xi'\right|^2}{\left(\left|\xi\right| + \left|\xi'\right|\right)^{2-p_l}}, & \text{if } 1 < p_l < 2, \end{cases} \tag{1.3}$$

for some positive constants $c_1, c_2, c_1', c_2', c_3, c_4$.

We assume that the exponents $p_1, \dots, p_N > 1$ satisfy

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_l < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \overline{p} < N. \quad l = 1, \dots, N, \tag{1.4}$$

where $\overline{p}$ denotes the harmonic mean of $p_1, \dots, p_N$, i.e.,

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{l=1}^{N} \frac{1}{p_l}. \tag{1.5}$$

The relevance of (1.4) is discussed in Remark 1. Here it suffices to say that the lower bound implies that solutions belong at least to $W^{1,1}$, so that we can understand the partial derivatives in (1.1) in the distributional sense.

Fundamentally different from the scalar case ($m = 1$), it is well-known [27, 28, 18, 17, 19, 6] that an additional structure condition is needed to have existence of solutions to elliptic systems with L^1 or measure data. Here we shall mainly use the following anisotropic version of the so-called (right-)angle condition (but see Section 5 for a different condition):

$$\begin{gathered} \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^m, \text{ and } \forall a \in \mathbb{R}^m \text{ with } \left|a\right| \leq 1, \\ \sigma_l(x,\xi) \cdot \left[(I - a \otimes a)\, \xi\right] \geq 0, \quad l = 1, \dots, N, \end{gathered} \tag{1.6}$$

where $(I - a \otimes a)$ is the rank $m-1$ orthogonal projector onto the space orthogonal to the unit vector $a \in \mathbb{R}^m$. If $\sigma_{i,l}$, $i = 1, \dots, m$, denotes the components of the vector σ_l, then the angle condition can be stated more explicitly as

$$\sum_{i,j=1}^{m} \sigma_{i,l}(x,\xi)\xi_j(\delta_{i,j} - a_i a_j) \geq 0.$$

Clearly, condition (1.6) is void in the scalar case.

A prototype example that is covered by our assumptions is the anisotropic p-harmonic, or $(p_1, \dots, p_N)$-harmonic, system

$$-\sum_{l=1}^{N} \frac{\partial}{\partial x_l}\Big(\big|\frac{\partial u}{\partial x_l}\big|^{p_l-2}\frac{\partial u}{\partial x_l}\Big) = \mu. \tag{1.7}$$

We prove herein the existence of a solution to (1.1). The proof is based on the usual strategy of deriving a priori estimates for a sequence of suitable approximate solutions $(u_\varepsilon)_{0<\varepsilon\leq 1}$ (for which existence is straightforward to prove) and then to pass to the limit as $\varepsilon \to 0$. Introduce the numbers

$$q = \frac{N(\overline{p}-1)}{N-1}, \quad q^\star = \frac{Nq}{N-q} = \frac{N(\overline{p}-1)}{N-\overline{p}}. \tag{1.8}$$

We derive a priori estimates for u_ε and the partial derivatives $\frac{\partial u_\varepsilon}{\partial x_l}$ in the weak Lebesgue spaces $\mathcal{M}^{q^\star}$ and $\mathcal{M}^{p_l q/\overline{p}}$, respectively (see Section 2 for the definition of weak Lebesgue spaces). To prove the weak Lebesgue space estimates we employ an anisotropic Sobolev inequality [45]. Having derived the weak priori estimates, we then prove a.e. convergence of the partial derivatives $\frac{\partial u_\varepsilon}{\partial x_l}$, which can be turned into strong L^1 convergence thanks to the $\mathcal{M}^{p_l q/\overline{p}}$ estimates and, by (1.4), $p_l q/\overline{p} > 1$. Equipped with this convergence we pass to the limit in the strong L^1 sense in the nonlinear vector fields $\sigma_l(x, \frac{\partial u_\varepsilon}{\partial x_l})$, and finally conclude that the approximate solutions u_ε converge to a solution of (1.1).

Our existence result and the method of proof rely heavily on previous work by Dolzmann, Hungerbühler, and Müller [18] (see also [17, 19, 22, 47, 15]) dealing with the isotropic p-harmonic system

$$-\operatorname{div}\left(\left|Du\right|^{p-2} Du\right) = \mu. \tag{1.9}$$

Under the assumption $2-\frac{1}{N} < p < N$, the work [18] proves existence and regularity results for distributional solutions of the p-harmonic system. These solutions satisfy $u \in \mathcal{M}^{q^\star}$ and $Du \in \mathcal{M}^q$, where q, $q^\star$ are defined as in (1.8) but with $\overline{p}$ replaced by p. The lower bound on the exponent p is known to be optimal (also in the scalar case). Regarding the anisotropic system (1.1), note that (1.4) implies $2 - \frac{1}{N} < \overline{p} < N$.

Even when $p_l \equiv p$ for all l, so that (1.4) implies $2 - \frac{1}{N} < p < N$ and our results yield the existence of a solution u to (1.7) such that $u \in \mathcal{M}^{q^\star}$, $\frac{\partial u}{\partial x_l} \in \mathcal{M}^q$ for all l, (1.7) does not coincide with (1.9).

While (1.9) can be viewed as the Euler-Lagrange system of the classical energy functional

$$I[w] := \int_\Omega \frac{1}{p}\left|Du\right|^p dx \tag{1.10}$$

on the Sobolev space $W_0^{1,p}$, (1.7) can be viewed as the Euler-Lagrange system of the anisotropic energy functional

$$I[w] := \int_\Omega \sum_{l=1}^N \frac{1}{p_l}\left|\frac{\partial w}{\partial x_l}\right|^{p_l} dx \tag{1.11}$$

on the anisotropic Sobolev space $W_0^{1,(p_1,\dots,p_N)}$. This illustrates a key difference between (1.7) and (1.9), even when $p_l = p$ for all l.

We recall that in the scalar case ($m = 1$), existence and regularity results for distributional solutions with L^1 or measure data have been obtained in [9, 29, 30, 3] for a class of anisotropic elliptic and parabolic equations. For an anisotropic parabolic reaction-diffusion-advection system similar results have been established in [4]. These works can be viewed as extensions of parts of the well known theory developed for distributional solutions of isotropic elliptic and parabolic equations with measure data, see, e.g., [8, 11, 10] and the references cited therein.

When $p \in \left(1, 2 - \frac{1}{N}\right]$ one cannot expect solutions to belong to $W^{1,1}$, and hence the notions of weak derivatives and distributional solutions break down. This problem is dealt with in the literature on scalar equations using the notion of entropy/renormalized solutions, see, e.g., [5, 7, 11, 16, 32, 35]. For isotropic elliptic systems (such as (1.9)) Dolzmann, Hungerbühler, and Müller [17] introduced a notion of solution based on replacing the weak derivative Du by the approximate derivative $\operatorname{ap} Du$. Moreover, existence results for such solutions were proved.

In our anisotropic setting (1.1), we cannot expect solutions to belong to $W^{1,1}$ as long as $1 < p_l \le \frac{\overline{p}(N-1)}{N(\overline{p}-1)}$, which implies $\overline{p} \in (1, 2 - \frac{1}{N}]$. Although we are not going to pursue this here, let us mention that it seems likely that one can adapt the notion of solution as well as the arguments used in [17], together with the ideas used in the present paper, to analyze (1.1) also in the range $1 < p_l \le \frac{\overline{p}(N-1)}{N(\overline{p}-1)}$.

In [19], Dolzmann, Hungerbühler, and Müller proved maximal regularity and uniqueness of solutions to isotropic N-Laplace type systems. We apply the machinery developed in [19] to prove similar results for anisotropic N-Laplace type systems. A typical example of such a system is (1.7) with $p_l = N$ for all l (which does not coincide with (1.9) with $p = N$).

One of our motivations for studying (1.1) comes from applications to $(p_1, \dots, p_N)$-harmonic maps from Ω into the sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ ($m \ge 2$), sometimes simply called anisotropic harmonic maps.

Let $b : \overline{\Omega} \to \mathbb{S}^{m-1}$ be a smooth function, and consider the anisotropic Dirichlet energy (1.11) with w belonging to the admissibility class

$$\mathcal{A} = \{w \in W^{1,(p_1,\dots,p_N)}(\Omega; \mathbb{S}^{m-1}) : w = b \text{ on } \partial\Omega \text{ in the trace sense}\}. \tag{1.12}$$

The corresponding Euler-Lagrange system is the anisotropic elliptic system

$$-\sum_{l=1}^{N} \frac{\partial}{\partial x_l}\Big(\big|\frac{\partial u}{\partial x_l}\big|^{p_l-2}\frac{\partial u}{\partial x_l}\Big) = \sum_{l=1}^{N}\big|\frac{\partial u}{\partial x_l}\big|^{p_l} u, \tag{1.13}$$

together with the constraint $|u| = 1$ a.e. in Ω. A vector-valued map u of class $W^{1,(p_1,\dots,p_N)}(\Omega; \mathbb{S}^{m-1})$ is called $(p_1, \dots, p_N)$-harmonic if it satisfies (1.13) in the distributional sense. Note that the critical growth right-hand side of (1.13) belongs to $L^1(\Omega; \mathbb{R}^m)$.

Although anisotropic harmonic maps have been very little studied in the literature, harmonic maps (between general manifolds) have been intensively studied over the years in terms of their compactness, existence, uniqueness, and regularity properties. For an excellent introduction to the theory of harmonic maps, we refer to the recent book by Hélein [25].

In the final section of this paper we study the question of compactness of sequences of $(p_1, \dots, p_N)$-harmonic maps with respect to the weak topology of $W^{1,(p_1,\dots,p_N)}$, at least when $1 < \overline{p} < N$. If $(u_\varepsilon)_{0<\varepsilon\le 1}$ is a sequence of such maps that converges weakly to a limit map u as $\varepsilon \to 0$, is it then true that u is $(p_1, \dots, p_N)$-harmonic? This is a highly nontrivial question since the system (1.13) has a nonlinearity of critical growth. Questions like this have been studied by Chen [13], Shatah [38], Evans [21, 20], and Hélein [25] for harmonic maps, which are special cases ($p = 2$) of p-harmonic maps (see also earlier work by Schoen and Uhlenbeck [36] on minimizing maps). A p-harmonic map u from Ω into $\mathbb{S}^{m-1}$ is a distributional solution of

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = |Du|^p u.$$

Compactness properties of p-harmonic maps (between general manifolds) have been studied by Toro and Wang [44] (see also Hardt and Lin [24] and Luckhaus [33] for earlier work on minimizing maps). Inspired by Toro and Wang, we prove that limits of weakly converging sequences of $(p_1, \dots, p_N)$-harmonic maps are again $(p_1, \dots, p_N)$-harmonic. This is done under the assumption that the anisotropy $(p_1, \dots, p_N) > 1$ satisfies

$$\overline{p} < N, \quad \overline{p}^\star > p_{\max}. \tag{1.14}$$

The important condition is the last one, which requires that the anisotropy is not too much spread out. The proof relies on some compactness arguments used for

(1.1) and the important fact that the right-hand side of (1.13) belongs to the local Hardy one space $\mathcal{H}^1_{\mathrm{loc}}(\Omega)$. To deduce this compensated integrability property we rely on an anisotropic variant of the Hardy space version of the div-curl lemma due to Coifman, Lions, Meyer, and Semmes [14], which we prove under assumption (1.14).

The remaining part of this paper is organized as follows: Section 2 is devoted to mathematical preliminaries, including, among other things, a brief discussion of anisotropic Sobolev and weak Lebesgue spaces. We also prove a weak Lebesgue space estimate that will be used later to obtain a priori estimates for our approximate solutions. The main existence result is stated and proved in Section 3. In Sections 4 and 5 we discuss some extensions. In Section 6 we prove maximal regularity and uniqueness results for (1.1) when $p_l = N$ for all l. Finally, in Section 7 we study compactness properties of anisotropic harmonic maps into spheres.

2. Mathematical preliminaries

In this section real-valued functions on Ω are denoted by $g = g(x)$. Let $1 \le p_1, \dots, p_N < \infty$ be N real numbers. Denote by $\overline{p}$ the harmonic mean of these numbers, i.e., $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{l=1}^N \frac{1}{p_l}$, and set $p_{\max} = \max(p_1, \dots, p_N)$, $p_{\min} = \min(p_1, \dots, p_N)$. We always have $p_{\min} \le \overline{p} \le N p_{\min}$. The Sobolev conjugate of $\overline{p}$ is denoted by $\overline{p}^\star$, i.e., $\overline{p}^\star = \frac{N\overline{p}}{N-\overline{p}}$.

2.1. **Anisotropic Sobolev spaces.**

Anisotropic Sobolev spaces were introduced and studied by Nikol'skiĭ [34], Slobodeckiĭ [39], Troisi [45], and later by Trudinger [46] in the framework of Orlicz spaces.

Herein we need the anisotropic Sobolev space

$$W_0^{1,(p_1,\dots,p_N)}(\Omega) = \Big\{ g \in W_0^{1,1}(\Omega) \,:\, \frac{\partial g}{\partial x_l} \in L^{p_l}(\Omega),\, l = 1, \dots, N \Big\}.$$

This is a Banach space under the norm

$$\|g\|_{W_0^{1,(p_1,\dots,p_N)}(\Omega)} = \|g\|_{L^1(\Omega)} + \sum_{l=1}^N \Big\| \frac{\partial g}{\partial x_l} \Big\|_{L^{p_l}(\Omega)}.$$

We use standard notation for the vector- and matrix-valued versions of the space/norm introduced above. For example, the $\mathbb{R}^m$-valued version of $W_0^{1,(p_1,\dots,p_N)}(\Omega)$ is denoted by $W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m)$.

We need the anisotropic Sobolev embedding theorem.

Theorem 2.1 (Troisi [45])**.** *Suppose $g \in W_0^{1,(p_1,\dots,p_N)}(\Omega)$, and let*

$$\begin{cases} q = \overline{p}^\star, & \text{if } \overline{p}^\star < N, \\ q \in [1,\infty), & \text{if } \overline{p}^\star \ge N. \end{cases}$$

Then there exists a constant C, depending on N, $p_1, \dots, p_N$ if $\overline{p} < N$ and also on q and $|\Omega|$ if $\overline{p} \ge N$, such that

$$\|g\|_{L^q(\Omega)} \le C \prod_{l=1}^N \Big\| \frac{\partial g}{\partial x_l} \Big\|_{L^{p_l}(\Omega)}^{1/N}. \tag{2.1}$$

We can replace the geometric mean on the right-hand side of (2.1) by an arithmetic mean. Indeed, the inequality between geometric and arithmetic means implies

$$\|g\|_{L^q(\Omega)} \le \frac{C}{N}\sum_{l=1}^{N}\Big\|\frac{\partial g}{\partial x_l}\Big\|_{L^{p_l}(\Omega)},$$

and thus there is in particular, when $\overline{p} < N$, a continuous embedding of the space $W_0^{1,(p_1,\dots,p_N)}(\Omega)$ into $L^q(\Omega)$ for all $q \in [1, \overline{p}^\star]$.

The exponent $\overline{p}^\star$, which is suggested by the usual scaling argument, is critical if the numbers $p_1, \dots, p_N$ are close enough to ensure $\overline{p}^\star \ge p_{\max}$. It may happen that $\overline{p}^\star < p_{\max}$ if the anisotropy is too much spread out, in which case the true critical exponent is $p_{\max}$ rather than $\overline{p}^\star$. However, this latter case is excluded by our assumptions, see (3.3) below.

2.2. **Weak Lebesgue spaces and a technical lemma.** In this paper we will use the weak Lebesgue (Marcinkiewicz) spaces $\mathcal{M}^q(\Omega)$ $(1 < q < \infty)$, which belong to the scale of Lorentz spaces. They contain the measurable functions $g : \Omega \to \mathbb{R}$ for which the distribution function

$$\lambda_g(\gamma) = \big|\{x \in \Omega : |g(x)| > \gamma\}\big|, \quad \gamma \ge 0,$$

satisfies an estimate of the form

$$\lambda_g(\gamma) \le C\gamma^{-q}, \quad \text{for some finite constant } C.$$

The space $\mathcal{M}^q(\Omega)$ is a Banach space under the norm

$$\|g\|^*_{\mathcal{M}^q(\Omega)} = \sup_{t>0} t^{1/q}\Big(\frac{1}{t}\int_0^t g^*(s)\,ds\Big),$$

where g^* denotes the nonincreasing rearrangement of f:

$$g^*(t) = \inf\{\gamma > 0 : \lambda_g(\gamma) \le t\}.$$

We will in what follows use the pseudo norm

$$\|g\|_{\mathcal{M}^q(\Omega)} = \inf\{C : \lambda_g(\gamma) \le C\gamma^{-q}, \forall \gamma > 0\},$$

which is equivalent to the norm $\|g\|^*_{\mathcal{M}^q(\Omega)}$.

It is clear that $L^q(\Omega) \subset \mathcal{M}^q(\Omega)$, and this inclusion is strict as the function $g(x) = |x|^{-N/q}$ belongs to $\mathcal{M}^q(\Omega)$ but not $L^q(\Omega)$.

A useful property of weak Lebesgue spaces is the following version of Hölder's inequality: Let $E \subset \Omega$, $g \in \mathcal{M}^q(\Omega)$, $r < q$, then

$$\|g\|_{\mathcal{M}^r(E)} \le \big(\frac{q}{q-r}\big)^{1/r}|E|^{\frac{1}{r}-\frac{1}{p}}\|g\|_{\mathcal{M}^q(E)}.$$

It is then immediate that $\mathcal{M}^q(\Omega) \subset \mathcal{M}^r(\Omega)$ if $r < q$. Similarly to the anisotropic Sobolev spaces, we use standard notation for the vector/matrix-valued versions of the weak Lebesgue spaces.

We now prove an "anisotropic version" of a weak Lebesgue space estimate that goes back to Talenti [43] and Benilan *et al.* [5] for isotropic elliptic equations, and Dolzmann, Hungerbühler, and Müller [18, 17] for isotropic elliptic systems.

Lemma 2.2. *Let g be a nonnegative function in $W_0^{1,(p_1,\dots,p_N)}(\Omega)$. Suppose $\overline{p} < N$, and that there exists a constant c such that*
$$\sum_{l=1}^{N} \int_{\{g\le\gamma\}} \big|\frac{\partial g}{\partial x_l}\big|^{p_l}\,dx \le c(\gamma+1), \quad \forall \gamma > 0. \tag{2.2}$$
Then there exists a constant C, depending on c, such that
$$\|g\|_{\mathcal{M}^{\frac{N(\overline{p}-1)}{N-\overline{p}}}(\Omega)} \le C.$$

Proof. For any $\gamma > 0$, the standard scalar truncation function T_γ on $[0,\infty)$ (at height γ) is defined as
$$T_\gamma(r) := \begin{cases} r, & \text{if } r \le \gamma,\\ \gamma, & \text{if } r > \gamma. \end{cases}$$
Then, by (2.2), for $\gamma \ge 1$
$$\int_\Omega \big|\frac{\partial T_\gamma(g)}{\partial x_l}\big|^{p_l}\,dx = \int_{\{g\le\gamma\}} \big|\frac{\partial g}{\partial x_l}\big|^{p_l}\,dx \le C\gamma, \quad l = 1,\dots,N,$$
so that the anisotropic Sobolev inequality (2.1) gives
$$\begin{aligned}\int_\Omega \big|T_\gamma(g)\big|^{\overline{p}^\star}\,dx &\le C_1\Big[\prod_{l=1}^{N}\Big(\int_\Omega \big|\frac{\partial T_\gamma(g)}{\partial x_l}\big|^{p_l}\,dx\Big)^{\frac{1}{p_l N}}\Big]^{\overline{p}^\star}\\ &\le C_2\Big[\prod_{l=1}^{N}\gamma^{\frac{1}{p_l N}}\Big]^{\overline{p}^\star} = C_2\gamma^{\frac{\overline{p}^\star}{\overline{p}}}.\end{aligned}$$
Hence, for $\gamma \ge 1$,
$$\lambda_g(\gamma) \le \gamma^{-\overline{p}^\star}\int_\Omega \big|T_\gamma(g)\big|^{\overline{p}^\star}\,dx \le C_2\gamma^{-\overline{p}^\star+\frac{\overline{p}^\star}{\overline{p}}} = C_2\gamma^{-\frac{N(\overline{p}-1)}{N-\overline{p}}}.$$
For $\gamma < 1$, we have trivially that $\lambda_g(\gamma) \le |\Omega| \le |\Omega|\gamma^{-\frac{N(\overline{p}-1)}{N-\overline{p}}}$. This shows that $g \in \mathcal{M}^{\frac{N(\overline{p}-1)}{N-\overline{p}}}(\Omega)$. ☐

2.3. **Truncation function.** For any $\gamma > 0$, define the spherial (radially symmetric) truncation function $T_\gamma : \mathbb{R}^m \to \mathbb{R}^m$ by
$$T_\gamma(r) := \begin{cases} r, & \text{if } |r| \le \gamma,\\ \frac{r}{|r|}\gamma, & \text{if } |r| > \gamma. \end{cases} \tag{2.3}$$
This function will be used repeatedly to derive a priori estimates for our approximate solutions. Observe that
$$DT_\gamma(r) = \begin{cases} I, & \text{if } |r| < \gamma,\\ \frac{\gamma}{|r|}\big(I - \frac{r\otimes r}{|r|^2}\big), & \text{if } |r| > \gamma. \end{cases}$$
In particular, (1.6) implies for all $\xi, r \in \mathbb{R}^m$ the crucial property
$$\sigma_l(x,\xi)\cdot DT_\gamma(r)\xi \ge \sigma_l(x,\xi)\cdot\xi\,\chi_{|r|<\gamma}, \quad l = 1,\dots,N. \tag{2.4}$$

We refer to Landes [28] for a discussion of T_γ and other test functions for elliptic systems, which indeed is a delicate issue.

3. Existence of a solution

3.1. Statement of main theorem.

Definition 3.1. A distributional solution of (1.1) is a vector-valued function $u : \Omega \to \mathbb{R}^m$ satisfying

$$u \in W_0^{1,1}(\Omega;\mathbb{R}^m), \quad \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \in L^1(\Omega;\mathbb{R}^m), \quad l = 1,\dots,N, \tag{3.1}$$

and for all $\varphi \in C_c^\infty(\Omega;\mathbb{R}^m)$,

$$\int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \cdot \frac{\partial \varphi}{\partial x_l}\, dx = \int_\Omega \varphi\, d\mu.$$

Theorem 3.1. *Suppose* (1.2)-(1.6) *hold. Let $\mu = (\mu_1,\dots,\mu_m)^\top$ be a Radon measure on Ω of finite mass. Then there exists at least one distributional solution $u = (u_1,\dots,u_m)^\top$ of* (1.1)*. Moreover,*

$$u \in \mathcal{M}^{q^\star}(\Omega;\mathbb{R}^m), \quad \frac{\partial u}{\partial x_l} \in \mathcal{M}^{p_l q/\overline{p}}(\Omega;\mathbb{R}^m), \quad l = 1,\dots,N, \tag{3.2}$$

where the exponents q and $q^\star$ are defined in (1.8).

This theorem will be an immediate consequence of the results proved in the subsections that follow.

Remark 1. The fact that $\overline{p} > 2 - \frac{1}{N}$ (which is a consequence of the lower bound in (1.4)) yields $\overline{p} > \frac{2N}{N+1} > 1$ (since $N \ge 2$). This in turn implies $\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < \frac{\overline{p}(N-1)}{N-\overline{p}}$ and also $q^\star > 1$. Moreover, the lower bound in (1.4) is equivalent to $p_l q/\overline{p} > 1$ for all l. The upper bound in (1.4) is equivalent to $p_l q/\overline{p} > p_l - 1$ for all l, which is needed for proving strong convergence of the nonlinear vector fields $\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big)$, $l = 1,\dots,N$. The upper bound is also equivalent to having $q^\star > p_l q/\overline{p}$ for all l.

We do not know if the upper condition in (1.4) is optimal for having existence of a solution to (1.1), but note that it is equivalent to having

$$\overline{p}^\star > p_{\max} + \frac{\overline{p}}{N - \overline{p}}, \quad p_{\max} = \max(p_1,\dots,p_N). \tag{3.3}$$

Roughly speaking, this condition requires that the anisotropy $(p_1,\dots,p_N)$ is not too much spread out. The case $\overline{p}^\star < p_{\max}$ (i.e., when the anisotropy is highly spread out) seems difficult to handle since the anisotropic Sobolev inequality does not imply $f \in L^{p_{\max}}$ when $f \in W_0^{1,(p_1,\dots,p_N)}$. On the other hand, one may wonder if it is possible to prove existence under the less restrictive condition $\overline{p}^\star \ge p_{\max}$ (but we do not know how to do it). In the scalar case [4, 30, 29, 3], conditions similar to (1.4) have also been imposed in order to have existence of a solution. We recall that there are well known examples of minimizers of anisotropic integral functionals that are unbounded when the anisotropy is too spread out [23], see also [2, 12, 41, 42] for regularity results for minimizers of anisotropic integral functionals that hold under the assumption that the anisotropy is not too spread out.

3.2. Approximate solutions.

To prove existence of a solution to (1.1) we introduce approximating problems for which existence is easy to prove. To this end, let $(f_\varepsilon)_{0<\varepsilon\le 1} \subset C_c^\infty(\Omega;\mathbb{R}^m)$ be a sequence defined by $f_\varepsilon = \mu \star \omega_\varepsilon$, where $\omega_\varepsilon(x) = \frac{1}{\varepsilon^N}\omega_0\big(\frac{x}{\varepsilon}\big) \ge 0$ and ω_0 is a nonnegative function in $C_c^\infty(B(0,1))$ with

$\int \omega_0\,dx = 1$. It is always understood that ε takes values in a sequence in $(0,\infty)$ tending to zero. Clearly,

$$\begin{gathered} |f_\varepsilon| \le C(\varepsilon) \quad \text{and} \quad \int_\Omega |f_\varepsilon|\,dx \le |\mu|, \\ f_\varepsilon \overset{\star}{\rightharpoonup} \mu \text{ in the sense of measures as } \varepsilon \to 0. \end{gathered} \tag{3.4}$$

For $u, v \in W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m)$, we denote by $\mathbf{A}$ the operator

$$\mathbf{A} : u \mapsto \Big(v \mapsto \int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \cdot \frac{\partial v}{\partial x_l}\,dx \Big).$$

Clearly, $\mathbf{A}$ is well-defined and monotone. We recall that monotone means

$$\langle \mathbf{A}(u) - \mathbf{A}(v), u - v \rangle \ge 0$$

for all $u, v \in W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m)$. Here $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m)$ and $\sum_{l=1}^N W^{-1,p_l'}(\Omega;\mathbb{R}^m)$ ($p_l' = \frac{p_l}{p_l-1}$). It is not difficult to deduce from the coercivity condition in (1.2) that $\mathbf{A}$ is coercive. The growth condition of our operator $\mathbf{A}$ implies that $\mathbf{A}$ is hemicontinuous, i.e., the mapping $\lambda \to \langle \mathbf{A}(u+\lambda v), w \rangle$ is continuous on the real axis for $u, v, w \in W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m)$. On the other hand, by (1.2),

$$\big|\langle \mathbf{A}u, v \rangle\big| \le c \sum_{l=1}^N \Big(\int_\Omega \Big(\big|\frac{\partial u}{\partial x_l}\big|^{p_l-1} + 1 \Big)^{\frac{p_l}{p_l-1}}\,dx \Big)^{\frac{p_l-1}{p_l}} \Big(\int_\Omega \big|\frac{\partial v}{\partial x_l}\big|^{p_l}\,dx \Big)^{1/p_l},$$

which implies the boundedness of $\mathbf{A}$. Then, using a standard theorem for monotone operators (see, e.g., [31, Theorem 2.1/Chapter 2]), it follows that $\mathbf{A}$ is bijective, and hence there exists a sequence of functions

$$(u_\varepsilon)_{0<\varepsilon\le 1} \subset W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m),$$

each of them satisfying the weak formulation

$$\int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \cdot \frac{\partial \varphi}{\partial x_l}\,dx = \int_\Omega f_\varepsilon \cdot \varphi\,dx, \quad \forall \varphi \in W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m). \tag{3.5}$$

Now the proof of Theorem 3.1 consists of two main steps. First, we prove ε-uniform a priori estimates in weak Lebesgue spaces for u_ε and $\frac{\partial u_\varepsilon}{\partial x_l}$. Second, we pass to the limit in (3.5) as $\varepsilon \to 0$.

3.3. **A priori estimates.**

Lemma 3.2. *There exists a constant c, not depending on ε, such that*

$$\sum_{l=1}^N \int_{\{|u_\varepsilon| \le \gamma\}} \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l}\,dx \le c(\gamma + 1), \quad \forall \gamma > 0. \tag{3.6}$$

Proof. Inserting $\varphi = T_\gamma(u_\varepsilon)$ into (3.5) gives

$$\int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \cdot DT_\gamma(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_l}\,dx = \int_\Omega f_\varepsilon \cdot T_\gamma(u_\varepsilon)\,dx.$$

Using (2.4) and the coercivity condition in (1.2), we obtain (3.6). $\square$

Lemma 3.3. *There exists a constant C, not depending on ε, such that*

$$\|u_\varepsilon\|_{\mathcal{M}^{q^\star}(\Omega;\mathbb{R}^m)} \le C \tag{3.7}$$

and

$$\Big\|\frac{\partial u_\varepsilon}{\partial x_l}\Big\|_{\mathcal{M}^{p_l q/\overline{p}}(\Omega;\mathbb{R}^m)} \le C, \quad l = 1,\dots,N. \tag{3.8}$$

where the exponents q and $q^\star$ are defined in (1.8).

Proof. Let $a = \frac{N(\overline{p}-1)}{N-\overline{p}}$. By Lemma 3.2 and $\big|\frac{\partial}{\partial x_l}|u_\varepsilon|\big| \le \big|\frac{\partial}{\partial x_l}u_\varepsilon\big|$,

$$\sum_{l=1}^N \int_{\{|u_\varepsilon|\le\gamma\}} \Big|\frac{\partial |u_\varepsilon|}{\partial x_l}\Big|^{p_l}\,dx \le c(\gamma+1).$$

Applying Lemma 2.2 to $|u_\varepsilon|$ gives $\big\|\,|u_\varepsilon|\,\big\|_{\mathcal{M}^a(\Omega)} \le C$, which also proves (3.7). By (3.6) and (3.7), we have for any $\alpha,\gamma \ge 1$

$$\begin{aligned}
\lambda_{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|}(\alpha) &\le \Big|\{x\in\Omega : \Big|\frac{\partial u_\varepsilon}{\partial x_l}\Big| > \alpha,\ |u_\varepsilon|\le\gamma\}\Big| \\
&\quad + \Big|\{x\in\Omega : \Big|\frac{\partial u_\varepsilon}{\partial x_l}\Big| > \alpha,\ |u_\varepsilon| > \gamma\}\Big| \\
&\le \frac{1}{\alpha^{p_l}} \int_{\{|u_\varepsilon|\le\gamma\}} \Big|\frac{\partial u_\varepsilon}{\partial x_l}\Big|^{p_l}\,dx + \lambda_{|u_\varepsilon|}(\gamma) \\
&\le C\Big(\frac{\gamma}{\alpha^{p_l}} + \gamma^{-a}\Big).
\end{aligned}$$

Optimizing with respect to γ gives $\gamma = \frac{1}{a}\alpha^{\frac{p_l}{a+1}}$, which in turn yields the bound $\lambda_{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|}(\alpha) \le C\alpha^{-\frac{ap_l}{a+1}}$. With the choice $a = q^\star$, see (1.8),

$$\lambda_{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|}(\alpha) \le C\alpha^{-\frac{p_l}{\overline{p}}\frac{N(\overline{p}-1)}{N-1}}, \quad \alpha \ge 1.$$

For $\alpha < 1$, $\lambda_{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|}(\alpha) \le |\Omega| \le |\Omega|\alpha^{-\frac{p_l}{\overline{p}}\frac{N(\overline{p}-1)}{N-1}}$. This proves (3.8). □

3.4. **Strong L^1 convergence of nonlinear vector fields.** In view of Lemma 3.3, u_ε is uniformly bounded in $L^{s_0}(\Omega;\mathbb{R}^m)$ for some $s_0 < q^\star$ with $s_0 > p_l q/\overline{p}$ for all l, and $\frac{\partial u_\varepsilon}{\partial x_l}$ is uniformly bounded in $L^{s_l}(\Omega;\mathbb{R}^m)$ for some $s_l > 1$ with $p_l - 1 < s_l < p_l q/\overline{p}$, $l = 1,\dots,N$. From this we get that u_ε is uniformly bounded in the isotropic Sobolev space

$$W_0^{1,s_{\min}}(\Omega;\mathbb{R}^m), \quad s_{\min} = \min(s_1,\dots,s_N).$$

Consequently, we can assume without loss of generality that as $\varepsilon \to 0$

$$\begin{aligned}
&u_\varepsilon \to u \quad \text{a.e. in } \Omega \text{ and in } L^{s_{\min}}(\Omega;\mathbb{R}^m), \\
&u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,s_{\min}}(\Omega;\mathbb{R}^m), \\
&\Big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial u}{\partial x_l}\Big| \rightharpoonup h_l \quad \text{in } L^{s_l}(\Omega), \quad l = 1,\dots,N, \\
&\sigma_l\Big(x, \frac{\partial u_\varepsilon}{\partial x_l}\Big) \rightharpoonup \beta_l \quad \text{in } L^{s_l'}(\Omega;\mathbb{R}^m), \quad l = 1,\dots,N, \\
&f_\varepsilon \overset{\star}{\rightharpoonup} \mu \quad \text{in the sense of measures on } \Omega.
\end{aligned} \tag{3.9}$$

Of course, the convergences provided by (3.9) are not strong enough if we want to pass to the limit $\varepsilon \to 0$ in the nonlinear system (3.5), and the proof of Theorem 3.1 will be completed by Lemma 3.4 below. To prove this lemma we follow closely the argument used in [18] for the isotropic p-harmonic system (1.9) (see also [20]), which is based on using a regularized test function and a localization procedure to handle the problem that u does not in general belong to the anisotropic Sobolev space $W_0^{1,(p_1,\dots,p_N)}$.

Lemma 3.4. *For $l = 1, \dots, N$, as $\varepsilon \to 0$ we have*

$$\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \to \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \quad \text{a.e. in } \Omega \text{ and in } L^1(\Omega;\mathbb{R}^m). \tag{3.10}$$

Proof. The main part of the proof consists in showing that

$$h_l(x) = 0 \quad \text{for a.e. } x \in \Omega, \quad l = 1, \dots, N, \tag{3.11}$$

where h_l is defined in (3.9). Suppose for the moment the validity of (3.11), and fix any one of the directions $l = 1, \dots, N$. Then, by Vitali's theorem,

$$\frac{\partial u_\varepsilon}{\partial x_l} \to \frac{\partial u}{\partial x_l} \quad \text{in } L^1(\Omega;\mathbb{R}^m),$$

and, after extracting a subsequence if necessary, $\frac{\partial u_\varepsilon}{\partial x_l} \to \frac{\partial u}{\partial x_l}$ a.e. in Ω. From this we also have $\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \to \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big)$ a.e. in Ω. As $\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big)$ is uniformly bounded in $L^{s_l'}(\Omega;\mathbb{R}^m)$, Vitali's theorem gives

$$\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \to \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \quad \text{in } L^{t_l}(\Omega;\mathbb{R}^m),$$

for any $1 \le t_l < s_l'$, which proves (3.10).

We now set out to prove (3.11). Choose a nonnegative function $\alpha \in C^\infty([0,\infty) \cap L^\infty([0,\infty))$ such that $\alpha(t) = t$ for $t \in [0,\delta]$ for some $\delta > 0$, $\alpha' \ge 0$, and $\alpha'(t)t \le \alpha(t)$ for all $t \ge 0$ (see [17] for an explicit example of such a function). Then define the function $\psi : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\psi(r) = \frac{r}{|r|}\alpha(|r|),$$

and note that $\psi(r) = r$ when $|r| \le \delta$. We also need two scalar functions η, ϕ of the following type:

$$\begin{gathered} \eta \in C_c^\infty(\mathbb{R}^m), \quad 0 \le \eta \le 1, \quad \operatorname{supp}(\eta) \subset [0,\delta), \\ \phi \in C_c^\infty(\mathbb{R}^n), \quad 0 \le \phi \le 1, \quad \int \phi \, dx = 1. \end{gathered}$$

In what follows, let us fix any one of the directions $l = 1, \dots, N$. Denoting by v a comparison function in $C^1(\Omega;\mathbb{R}^m)$ (to be chosen later), we proceed by using the triangle and Hölder inequalities:

$$
\begin{aligned}
&\int_\Omega \sum_{l=1}^N \big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial u}{\partial x_l}\big|\eta(u_\varepsilon - v)\phi\,dx \\
&\le \int_\Omega \sum_{l=1}^N \big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\big|\eta(u_\varepsilon - v)\phi\,dx + \int_\Omega \sum_{l=1}^N \big|\frac{\partial v}{\partial x_l} - \frac{\partial u}{\partial x_l}\big|\eta(u_\varepsilon - v)\phi\,dx \\
&\le \sum_{l=1}^N \Big(\int_\Omega \big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\big|^{p_l}\eta(u_\varepsilon - v)\phi\,dx\Big)^{\frac{1}{p_l}} \Big(\int_\Omega \eta(u_\varepsilon - v)\phi\,dx\Big)^{\frac{p_l-1}{p_l}} \\
&\quad + \int_\Omega \sum_{l=1}^N \big|\frac{\partial v}{\partial x_l} - \frac{\partial u}{\partial x_l}\big|\eta(u_\varepsilon - v)\phi\,dx.
\end{aligned}
$$

Equipped with this and (3.9), using in particular that $u_\varepsilon \to u$ a.e. and the fact that η, ψ, $D\psi$ are continuous and bounded functions, we deduce

$$
\begin{aligned}
&\int_\Omega \sum_{l=1}^N h_l(x)\eta(u-v)\phi\,dx \\
&\le \sum_{l=1}^N L_l^{\frac{1}{p_l}} \Big(\int_\Omega \eta(u-v)\phi\,dx\Big)^{\frac{p_l-1}{p_l}} + \int_\Omega \sum_{l=1}^N \big|\frac{\partial v}{\partial x_l} - \frac{\partial u}{\partial x_l}\big|\eta(u-v)\phi\,dx,
\end{aligned} \tag{3.12}
$$

where

$$
L_l = L_l(\eta,\phi,\psi) := \limsup_{\varepsilon\to 0} \int_\Omega \big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\big|^{p_l}\eta(u_\varepsilon - v)\phi\,dx.
$$

We must analyze L_l, and start with the case $p_l \ge 2$. By (1.3),

$$
\begin{aligned}
&\int_\Omega c_3\big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\big|^{p_l}\eta(u_\varepsilon - v)\phi\,dx \\
&\le \int_\Omega \sum_{l=1}^N \Big(\sigma_l\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big) - \sigma_l\big(x,\frac{\partial v}{\partial x_l}\big)\Big)\cdot\Big(\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\Big)\eta(u_\varepsilon - v)\phi\,dx \\
&= \int_\Omega \sum_{l=1}^N \Big(\sigma_l\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big) - \sigma_l\big(x,\frac{\partial v}{\partial x_l}\big)\Big)\cdot\frac{\partial\psi(u_\varepsilon - v)}{\partial x_l}\eta(u_\varepsilon - v)\phi\,dx \\
&= \int_\Omega \sum_{l=1}^N \sigma_l\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big)\cdot\frac{\partial\psi(u_\varepsilon - v)}{\partial x_l}\phi\,dx \\
&\quad - \int_\Omega \sum_{l=1}^N \sigma_l\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big)\cdot\frac{\partial\psi(u_\varepsilon - v)}{\partial x_l}(1-\eta(u_\varepsilon - v))\phi\,dx \\
&\quad - \int_\Omega \sum_{l=1}^N \sigma_l\Big(x,\frac{\partial v}{\partial x_l}\big)\cdot\big(\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\Big)\eta(u_\varepsilon - v)\phi\,dx \\
&=: E_1 + E_2 + E_3,
\end{aligned} \tag{3.13}
$$

In the case $p_l < 2$, we employ (1.3) instead as follows:

$$\begin{aligned}
&\int_\Omega \big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\big|^{p_l} \eta(u_\varepsilon - v)\phi\, dx \\
&\le \Big(\int_\Omega \frac{\big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\big|^2}{\big(\big|\frac{\partial u_\varepsilon}{\partial x_l}\big| + \big|\frac{\partial v}{\partial x_l}\big|\big)^{2-p_l}} \eta(u_\varepsilon - v)\phi\, dx \Big)^{\frac{p_l}{2}} \\
&\quad \times \Big(\int_\Omega \big(\big|\frac{\partial u_\varepsilon}{\partial x_l}\big| + \big|\frac{\partial v}{\partial x_l}\big|\big)^{p_l} \eta(u_\varepsilon - v)\phi\, dx \Big)^{\frac{2-p_l}{2}} \\
&\le c_4^{-p_l/2} \left(E_1 + E_2 + E_3\right)^{\frac{p_l}{2}} \left(\int_\Omega \big(\big|\frac{\partial u_\varepsilon}{\partial x_l}\big| + \big|\frac{\partial v}{\partial x_l}\big|\big)^{p_l} \eta(u_\varepsilon - v)\phi\, dx \right)^{\frac{2-p_l}{2}} .
\end{aligned} \tag{3.14}$$

Thanks to (1.1),

$$E_1 = \int_\Omega f_\varepsilon \cdot \psi(u_\varepsilon - v)\phi\, dx - \int_\Omega \sum_{l=1}^N \sigma_l\Big(x, \frac{\partial u_\varepsilon}{\partial x_l}\Big) \cdot \psi(u_\varepsilon - v) \frac{\partial \phi}{\partial x_l}\, dx.$$

Since

$$D\psi(r) = \alpha'(|r|)\frac{r \otimes r}{|r|^2} + \frac{\alpha(|r|)}{|r|}\big(I - \frac{r \otimes r}{|r|^2}\big),$$

there holds

$$\sigma_l(x,\xi) \cdot D\psi(r)\xi \ge 0, \quad \forall \xi, r \in \mathbb{R}^m.$$

This follows from (1.6), since

$$\sigma_l(x,\xi) \cdot D\psi(r)\xi = \frac{\alpha\big|r\big|}{\big|r\big|}\, \sigma_l(x,\xi) \cdot \Big(I - \Big[\big(1 - \frac{\alpha'(\big|r\big|)\big|r\big|}{\alpha(\big|r\big|)}\big)\, \frac{r \otimes r}{\big|r\big|^2}\Big]\Big)\xi,$$

where the term inside the square brackets can be written as $a \otimes a$ for some $a \in \mathbb{R}^m$ with $\big|a\big| \le 1$ (recall that $\alpha'(t)t \le \alpha(t)$). Hence

$$E_2 \le \int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \cdot D\psi(u_\varepsilon - v)\frac{\partial v}{\partial x_l}(1 - \eta(u_\varepsilon - v))\phi\, dx. \tag{3.15}$$

Since $u_\varepsilon \to u$ a.e. and η, ψ, $D\psi$ are continuous and bounded functions, we deduce from (3.13) that

$$\begin{aligned}
L_l \le \sup|\psi| \int_\Omega \phi\, d\mu &- \int_\Omega \sum_{l=1}^N \beta_l \cdot \psi(u - v)\frac{\partial \phi}{\partial x_l}\, dx \\
&+ \int_\Omega \sum_{l=1}^N \beta_l \cdot D\psi(u - v)\frac{\partial v}{\partial x_l}\left(1 - \eta(u - v)\right)\phi\, dx \\
&- \int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial v}{\partial x_l}\big) \cdot \big(\frac{\partial u}{\partial x_l} - \frac{\partial v}{\partial x_l}\big)\eta(u - v)\phi\, dx.
\end{aligned} \tag{3.16}$$

At this stage we specify the functions v, η, ψ, ϕ. Fix any point $x = a \in \Omega$ that is simultaneously a Lebesgue point of $\frac{\partial u}{\partial x_l}, h_l, \beta_l$, $l = 1, \dots, N$, and the measure μ. Choose v as the first order Taylor polynomial of u around $x = a$:

$$v(x) = u(a) + Du(a)(x - a),$$

and replace ϕ, η, ψ in the above calculations by the following functions:

$$\eta_\rho(r) = \tilde{\eta}\big(\frac{r}{\rho}\big), \quad \tilde{\eta} \in C_c^\infty(B(0,1)), \quad \tilde{\eta}|_{B(0,\frac{1}{2})} \equiv 1,$$
$$\phi_\rho(x) = \frac{1}{\rho^n}\tilde{\phi}\big(\frac{x-a}{\rho}\big), \quad \tilde{\phi} \in C_c^\infty(B(0,1)), \quad \int \tilde{\phi} = 1,$$

and $\psi_\rho(r) = \rho\psi\big(\frac{r}{\rho}\big)$. Denote by $L_l(\rho)$ the corresponding L_l, that is, $L_l(\rho) := L_l(\eta_\rho, \phi_\rho, \psi_\rho)$. We deduce $\limsup_{\rho\to 0} L_l(\rho) = 0$, since as $\rho \to 0$,

$$\frac{1}{\big|B(a,\rho)\big|}\int_{B(a,\rho)} \big|\frac{u-v}{\rho}\big|\,dx \to 0,$$
$$\frac{1}{\big|B(a,\rho)\big|}\int_{B(a,\rho)} \sum_{l=1}^N \big|\frac{\partial u}{\partial x_l} - \frac{\partial v}{\partial x_l}\big|\,dx \to 0,$$
$$\frac{1}{\big|B(a,\rho)\big|}\int_{B(a,\rho)} \sum_{l=1}^N \big|\beta_l(x) - \beta_l(a)\big|\,dx \to 0,$$

where the second and third terms in (3.16) tend to zero as we have

$$\psi_\rho(u-v)\frac{\partial\phi}{\partial x_l} = \mathcal{O}\big(\frac{u-v}{\rho}\big), \quad 1 - \eta_\rho(u-v) = \mathcal{O}\big(\frac{u-v}{\rho}\big).$$

The first term tends to zero since

$$\limsup_{\rho\to 0} \mu(B(a,\rho))/\rho^n < \infty,$$

and thus $\sup|\psi_\rho| \int_\Omega \phi_\rho\,d\mu \le C\rho\mu(B(a,\rho))/\rho^n$. In the case $p_l < 2$, we also use that the term $(\cdots)^{\frac{2-p_l}{2}}$ in (3.14) stays finite in the above localization procedure (since $N \ge 2$). Since

$$\frac{1}{\big|B(a,\rho)\big|}\int_{B(a,\rho)} \sum_{l=1}^N \big|h_l(x) - h_l(a)\big|\,dx \to 0 \quad \text{as } \rho \to 0,$$

it follows, via (3.12), that $h(a) = 0$. This completes the proof of (3.11), and hence the lemma. □

4. An extension

In this section we show that the results obtained for (1.1) can be extended to more general anisotropic elliptic systems of the form

$$\begin{gathered} -\sum_{l=1}^N \frac{\partial}{\partial x_l}\sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) + g(x,u) = \mu, \quad \text{in } \Omega, \\ u = 0, \quad \text{in } \partial\Omega, \end{gathered} \tag{4.1}$$

where the vector fields $\sigma_1, \dots, \sigma_N$ are as before. We assume that the nonlinearity $g(x,r) : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is measurable in $x \in \Omega$ for all $r \in \mathbb{R}^m$, continuous in r for a.e. $x \in \Omega$, and satisfies the following conditions:

$$g(x,r)\cdot(r - r') \ge 0, \quad \forall r, r' \in \mathbb{R}^m \text{ with } |r'| \le |r|, \tag{4.2}$$
$$\sup\big\{\big|g(x,r)\big| : \big|r\big| \le \tau\big\} \in L^1(\Omega;\mathbb{R}^m), \quad \forall r \in \mathbb{R}^m \text{ and } \forall \tau \in \mathbb{R}. \tag{4.3}$$

Condition (4.2), often called the angle condition, is also assumed in the recent work [6]. A prototype example of (4.1) is provided by the equation

$$-\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\big(\big|\frac{\partial u}{\partial x_l}\big|^{p_l-2}\frac{\partial u}{\partial x_l}\big)+|u|^{\theta-1}u=\mu\,,$$

for some $\theta>1$. We look for distributional solutions to (4.1) in the following sense:

Definition 4.1. A distributional solution of (4.1) is a function $u:\Omega\to\mathbb{R}^m$ such that (3.1) and $g(x,u)\in L^1(\Omega;\mathbb{R}^m)$ hold, and $\forall\varphi\in C_c^\infty(\Omega;\mathbb{R}^m)$

$$\int_\Omega\sum_{l=1}^{N}\sigma_l\big(x,\frac{\partial u}{\partial x_l}\big)\cdot\frac{\partial\varphi}{\partial x_l}\,dx+\int_\Omega g(x,u)\varphi\,dx=\int_\Omega\varphi\,d\mu.$$

Our main results are collected in the following theorem.

Theorem 4.1. *Let $\mu=(\mu_1,\dots,\mu_m)^\top$ be a vector-valued Radon measure on Ω of finite mass. Then, under the assumptions stated above and in Section 1,* (4.1) *has at least one distributional solution u. Moreover, u has regularity as stated in* (3.2)*.*

Proof. Let f_ε be as in Section 3. Then, by classical arguments, there exists a sequence of approximate solutions $(u_\varepsilon)_{0<\varepsilon\le1}$ satisfying the weak formulation

$$\int_\Omega\sum_{l=1}^{N}\sigma_l\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big)\cdot\frac{\partial\varphi}{\partial x_l}\,dx+\int_\Omega g(x,u_\varepsilon)\cdot\varphi\,dx=\int_\Omega f_\varepsilon\cdot\varphi\,dx,\tag{4.4}$$

for all $\varphi\in W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m)$. Substituting $\varphi=T_\gamma(u_\varepsilon)$ in (4.4), we get

$$\int_\Omega\sum_{l=1}^{N}\sigma\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big)\cdot\frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l}\,dx+\int_\Omega g(x,u_\varepsilon)\cdot T_\gamma(u_\varepsilon)\,dx=\int_\Omega f_\varepsilon T_\gamma(u_\varepsilon)\,dx.\tag{4.5}$$

By (4.2), $\int_{\{|u_\varepsilon|\le\gamma\}}g(x,u_\varepsilon)\cdot T_\gamma(u_\varepsilon)\,dx\ge0$, and thus we deduce

$$c_1\sum_{l=1}^{N}\int_{\{|u|\le\gamma\}}\big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l}\,dx+\gamma\int_{\{|u_\varepsilon|>\gamma\}}\big|g(x,u_\varepsilon)\big|\,dx\le C.\tag{4.6}$$

We obtain from (4.6) and Lemma 3.3 the weak Lebesgue space estimates

$$\big\|u_\varepsilon\big\|_{\mathcal{M}^{q^\star}(\Omega;\mathbb{R}^m)}\le C,\quad\big\|\frac{\partial u_\varepsilon}{\partial x_l}\big\|_{\mathcal{M}^{p_lq/\overline{p}}(\Omega;\mathbb{R}^m)}\le C,\quad l=1,\dots,N,$$

where the exponents q and $q^\star$ are defined in (1.8), and C is a constant independent of ε. Consequently, we can assume without loss of generality that the convergence in (3.9) hold for our sequence $(u_\varepsilon)_{0<\varepsilon\le1}$.

Taking $\gamma=1$ in (4.6) and using (4.3), we deduce

$$\int_\Omega\big|g(x,u_\varepsilon)\big|\,dx\le C,\tag{4.7}$$

where C is a constant independent of ε. We also have $g(x,u_\varepsilon)\to g(x,u)$ a.e. in Ω. In view of Vitali's theorem, to show that $g(x,u_\varepsilon)$ converges strongly in $L^1(\Omega)$ it remains to prove that $g(x,u_\varepsilon)$ is equi-integrable. To this end, let B be a measurable set in Ω. As usual, we split the integral into two parts

$$\int_B\big|g(x,u_\varepsilon)\big|\,dx=\int_{B\cap\{|u_\varepsilon|\le\gamma\}}\big|g(x,u_\varepsilon)\big|\,dx+\int_{B\cap\{|u_\varepsilon|>\gamma\}}\big|g(x,u_\varepsilon)\big|\,dx.$$

Let us call the first and second integrals on the right-hand side for I_1 and I_2, respectively. In view of (4.3), $\lim_{|B|\to 0} I_1 = 0$. Let $0 < M < \gamma$, and observe that

$$\big|T_\gamma(u_\varepsilon)\big| \le \big|T_\gamma(u_\varepsilon)\big|\mathbf{1}_{\{|u_\varepsilon|\le M\}} + \big|T_\gamma(u_\varepsilon)\big|\mathbf{1}_{\{|u_\varepsilon|> M\}} \le M + \gamma\mathbf{1}_{\{|u_\varepsilon|> M\}},$$

Using this decomposition in (4.5) yields

$$\gamma\int_{\{|u_\varepsilon|>\gamma\}}\big|g(x,u_\varepsilon)\big|\,dx \le M\int_\Omega\big|f_\varepsilon\big|\,dx + \gamma\int_{\{|u_\varepsilon|>M\}}\big|f_\varepsilon\big|\,dx.$$

From this inequality we obtain

$$\lim_{\gamma\to\infty}\Big(\sup_{0<\varepsilon\le 1}\int_{\{|u_\varepsilon|>\gamma\}}\big|g(x,u_\varepsilon)\big|\,dx\Big) = o\big(\frac{1}{M}\big),$$

and, by sending $M\to\infty$, we conclude the equi-integrability of $g(x,u_\varepsilon)$.

The proof of Lemma 3.4 remains more or less unchanged, except that the term E_1 rewrites in our problem (4.1) as

$$\begin{aligned} E_1 &= \int_\Omega f_\varepsilon\psi(u_\varepsilon - v)\phi\,dx - \int_\Omega g(x,u_\varepsilon)\psi(u_\varepsilon - v)\phi\,dx \\ &\quad - \int_\Omega\sum_{l=1}^N \sigma_l\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big)\psi(u_\varepsilon - v)\frac{\partial\phi}{\partial x_l}\,dx, \end{aligned} \tag{4.8}$$

and estimate (3.16) rewrites as

$$\begin{aligned} L_l &\le \sup\big|\psi\big|\Big(\int_\Omega\phi\,d\mu + \int_\Omega\big|g(x,u)\big|\phi\,dx\Big) \\ &\quad - \int_\Omega\sum_{l=1}^N\beta_l\cdot\psi(u-v)\frac{\partial\phi}{\partial x_l}\,dx \\ &\quad + \int_\Omega\sum_{l=1}^N\beta_l\cdot D\psi(u-v)\frac{\partial v}{\partial x_l}\,(1-\eta(u-v))\,\phi\,dx \\ &\quad - \int_\Omega\sum_{l=1}^N\sigma_l\big(x,\frac{\partial v}{\partial x_l}\big)\cdot\big(\frac{\partial u}{\partial x_l} - \frac{\partial v}{\partial x_l}\big)\eta(u-v)\phi\,dx. \end{aligned} \tag{4.9}$$

Letting $x = a$ be a Lebesgue point simultaneously of μ, $g(x,u)$, h, u, Du, and $\beta = (\beta_1,\dots\beta_N)$, we can proceed as in the proof of Lemma 3.4. $\square$

5. A different structure condition

Zhou [47] proved that the results of Dolzmann, Hungerbühler, and Müller [18, 17, 19] continue to hold under the so-called (isotropic) sign condition. Moreover, he gave an example of an isotropic elliptic system that satisfies the sign condition but not the the angle condition.

In this section we return to problem (1.1) under assumptions (1.2)-(1.6), but we want to replace the anisotropic angle condition (1.6) by the following anisotropic sign condition:

$$\sigma_{i,l}(x,\xi)\xi_i \ge 0, \quad \forall(x,\xi)\in\Omega\times\mathbb{R}^N, \tag{5.1}$$

for $i = 1,\dots,m$, $l = 1,\dots,N$. Here $\sigma_{i,l}$ and ξ_i are the ith components of vectors σ_l and ξ, respectively. When $m = 2$, (1.6) implies (5.1). To see this, recall that

$(I - a \otimes a)$ projects orthogonally onto the space orthogonal to a, and then choose $a = (1,0)^\top$, $a = (0,1)^\top$ in (1.6).

It is easy to give an example of an elliptic system which satisfies (1.2), (1.3), and (5.1), but does not satisfy (1.6). For example, take $m = 2$, $N = 2$, and

$$\sigma_l(x,\xi) = \big|\xi\big|^{p_l-2}(\alpha\xi_1,\xi_2)^\top, \quad l = 1,2, \quad \xi = (\xi_1,\xi_2)^\top,$$

where $0 < \alpha \le 0.2$. It is clear that assumptions (1.2), (1.3), and the anisotropic sign condition (5.1) hold.

Let us verify that the anisotropic angle condition (1.6) does not hold. To this end, take $a = \big(\alpha^{1/2}, (1-\alpha)^{1/2}\big)^\top$ and $\xi = (1,1)^\top$. Then $|a| = 1$ and

$$(I - a \otimes a)\,\xi = \Big(1 - \alpha - \alpha^{1/2}(1-\alpha)^{1/2}, \alpha - \alpha^{1/2}(1-\alpha)^{1/2}\Big)^\top,$$

so that

$$\begin{aligned}
&\sigma_l(x,\xi)\cdot[(I - a\otimes a)\,\xi]\\
&= 2^{\frac{p_l-2}{2}}\Big[\alpha\Big(1-\alpha-\alpha^{1/2}\,(1-\alpha)^{1/2}\Big) + \alpha - \alpha^{1/2}\,(1-\alpha)^{1/2}\Big]\\
&< 2^{\frac{p_l-2}{2}}\Big[2\alpha - \alpha^{1/2}(1-\alpha)^{1/2}\Big] \le 0, \quad l = 1,2,
\end{aligned}$$

which implies that (1.6) does not hold.

The purpose of this section is to prove the following theorem.

Theorem 5.1. *Theorem 3.1 continues to hold when the anisotropic angle condition* (1.6) *is replaced by the anisotropic sign condition* (5.1).

Proof. Compared to the proof of Theorem 3.1, the main new idea is to use, instead of (2.3), the following cubic truncation function

$$\Theta_\gamma(r) = \Big(\max(-\gamma,\min(\gamma,r_1)),\dots,\max(-\gamma,\min(\gamma,r_N))\Big)^\top,$$

where $r = (r_1,\dots,r_N)^\top \in \mathbb{R}^N$. Substituting $\varphi = \Theta_\gamma(u_\varepsilon)$ in (3.5) yields

$$\sum_{i=1}^m \int_{\{|u_{\varepsilon,i}|\le\gamma\}} \sum_{l=1}^N \sigma_{i,l}\Big(x, \frac{\partial u_\varepsilon}{\partial x_l}\Big)\frac{\partial u_{\varepsilon,i}}{\partial x_l}\,dx \le C. \tag{5.2}$$

Using assumptions (1.2), we deduce from (5.2) that

$$\begin{aligned}
&\int_{\{|u_\varepsilon|\le\gamma\}} \sum_{l=1}^N \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l}\,dx\\
&\le \frac{1}{c_1}\int_{\{\max(|u_{\varepsilon,1}|,\dots,|u_{\varepsilon,N}|)\le\gamma\}} \sum_{l=1}^N \sigma_l\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big)\cdot\frac{\partial u_\varepsilon}{\partial x_l}\,dx + \frac{c_2}{c_1}|\Omega| \qquad (5.3)\\
&\le \frac{1}{c_1}\sum_{i=1}^m \int_{\{|u_{\varepsilon,i}|\le\gamma\}} \sum_{l=1}^N \sigma_{i,l}\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big)\frac{\partial u_{\varepsilon,i}}{\partial x_l}\,dx + \frac{c_2}{c_1}|\Omega| \le C.
\end{aligned}$$

Making similar changes due to the new truncation function in the rest of the proof of Theorem 3.1, we conclude eventually that Theorem 5.1 holds. $\square$

6. Maximal regularity and a uniqueness result

We collect our results in Theorem 6.1 (existence/regularity of solutions) and Theorem 6.2 (uniqueness of solutions) below.

Before stating the theorems, let us introduce some notation. First of all, we say that a set $E \subset \mathbb{R}^N$ is of type A if there exists a constant K such that for all $x \in \overline{E}$ and for all $0 < \rho < \operatorname{diam}(E)$ there holds $\left|Q(x,\rho) \cap E\right| \geq K\rho^N$, where $Q(x,\rho)$ denotes the cube $\left\{y \in \mathbb{R}^N : |x_l - y_l| < \frac{\rho}{2}, l = 1, \dots, N\right\}$.

In what follows we regard all relevant functions as defined in $\mathbb{R}^N$ by setting them to zero outside Ω. A function g belongs to the space $\mathrm{BMO}(\mathbb{R}^N)$ of functions of bounded mean oscillation if $g \in L^N(\mathbb{R}^N)$ and

$$|g|_{\mathrm{BMO}(\mathbb{R}^N)} = \Big(\sup_{y\in\mathbb{R}^N}\sup_{\rho>0}\frac{1}{\rho^N}\int_{Q(y,\rho)}\left|g-(g)_{y,\rho}\right|^N dx\Big)^{1/N} < \infty,$$

where $(g)_{y,\rho}$ denotes the mean value of g on the cube $Q(y,\rho)$. The space $\mathrm{BMO}(\mathbb{R}^N)$ is a Banach space under the norm

$$\|g\|_{\mathrm{BMO}(\mathbb{R}^N)} = \|g\|_{L^N(\mathbb{R}^N)} + |g|_{\mathrm{BMO}(\mathbb{R}^N)}.$$

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set such that $\Omega^c = \mathbb{R}^N \setminus \Omega$ is a domain of type A. Suppose* (1.2)-(1.6) *hold and $p_l = N$ for all $l = 1, \dots, N$. Let $\mu = (\mu_1, \dots, \mu_m)^\top$ be a Radon measure on Ω of finite mass. Then problem* (1.1) *has a solution $u \in W_0^{1,(s_1,\dots,s_N)}(\Omega;\mathbb{R}^m) \cap \mathrm{BMO}(\Omega;\mathbb{R}^m)$ for any set of exponents $1 \leq s_1, \dots, s_N < N$, and the following a priori estimate holds:*

$$\|u\|_{\mathrm{BMO}(\Omega;\mathbb{R}^m)} \leq C_1\Big(\|\mu\|_{\mathcal{M}^{\frac{1}{N-1}}(\Omega;\mathbb{R}^m)} + C_2\Big). \tag{6.1}$$

Moreover, Du belongs to the weak Lebesgue space $\mathcal{M}^N(\Omega;\mathbb{R}^{m\times N})$ and

$$\|Du\|_{\mathcal{M}^N(\Omega;\mathbb{R}^{m\times N})} \leq C_3\left(\|\mu\|_{\mathcal{M}^{\frac{1}{N-1}}} + C_4\right). \tag{6.2}$$

The constants C_i, $i = 1,2,3,4$, depend only on c_1, c_2, c_1', c_2', N, $|\Omega|$, and the constant K in the definition of property A.

Theorem 6.2. *Suppose* (1.2), (1.3), *and* (1.6) *hold and $p_l = N$ for all $l = 1, \dots, N$. Let $\mu = (\mu_1, \dots, \mu_m)^\top$ be a Radon measure on Ω of finite mass. Let u, v be two solutions of* (1.1) *such that u, $v \in W_0^{1,1}(\Omega;\mathbb{R}^m)$, $u - v \in W_0^{1,1}(\Omega;\mathbb{R}^m)$, and $Du, Dv \in \mathcal{M}^N(\Omega;\mathbb{R}^{m\times N})$. Then $u = v$ a.e. in Ω.*

Let us now embark on the proofs of Theorems 6.1 and 6.2.

Proof of Theorem 6.1. The following lemma contains so-called Caccioppoli estimates, which are at the heart of the matter of the regularity theory developed in [19].

Lemma 6.3. *Let $u \in W_0^{1,N}(\Omega;R^m)$ be a solution of* (1.1) *with $p_l = N$ for all $l = 1, \dots, N$, $\mu = f$, and $f \in L^1(\Omega;\mathbb{R}^m)$. Fix two positive numbers ρ, R such that $0 < \rho < R$. Then there exist constants C_1, C_2 such that for all cubes $Q(y,R) \subset \Omega$,* *for all $\beta \in \mathbb{R}^m$, and for all $\gamma > 0$*

$$\begin{aligned} &\int_{\{|u-\beta|\leq\gamma\}\cap Q(y,\rho)} \sum_{l=1}^N \big|\frac{\partial u}{\partial x_l}\big|^N\,dx \\ &\leq \frac{C_1}{(R-\rho)^N}\int_{Q(y,R)\setminus Q(y,\rho)} \big|u-\beta\big|^N\,dx + C_2\Big(\gamma\int_{Q(y,R)}\big|f\big|\,dx + R^N\Big), \end{aligned} \tag{6.3}$$

and for all cubes $Q(y,R) \subset \mathbb{R}^N$ and for all $\gamma > 0$

$$\begin{aligned} &\int_{\{|u|<\alpha\}\cap Q(y,\rho)} \sum_{l=1}^N \big|\frac{\partial u}{\partial x_l}\big|^N dx \\ &\leq \frac{C_1}{(R-\rho)^N}\int_{Q(y,R)\setminus Q(y,\rho)} \big|u\big|^N dx + C_2\Big(\gamma\int_{Q(y,R)}\big|f\big|\,dx + R^N\Big). \end{aligned} \tag{6.4}$$

Proof. Following [19], let $\chi \in C_c^\infty(Q(y,R))$ be a cut-off function satisfying

$$\begin{gathered} \chi(x) = 1 \text{ if } x \in Q(y,\rho),\ 0 \leq \chi \leq 1, \text{ and} \\ \big|\frac{\partial\chi}{\partial x_l}\big| \leq C/(R-\rho), \quad l = 1,\dots,N, \end{gathered}$$

for some finite constant C. Let $\alpha_\gamma : \mathbb{R} \to \mathbb{R}$ be any smooth function with the following properties:

$$\begin{gathered} \alpha_\gamma(s) = s \text{ if } s \in [0,\gamma],\ 0 \leq \alpha_\gamma \leq N\gamma,\ \alpha_\gamma' \leq 1, \\ 0 < c\big(\frac{\alpha_\gamma(s)}{s}\big)^{N/(N-1)} \leq \alpha_\gamma'(s) \leq \frac{\alpha_\gamma(s)}{s} \leq 1 \quad \text{on } (0,\infty), \end{gathered} \tag{6.5}$$

where $c > 0$ is a constant. An example of such a function can be found in [17]. Now we define the cut-off function $\psi_\gamma : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\psi_\gamma(r) = \frac{r}{|r|}\,\alpha_\gamma(|r|).$$

A calculation reveals that

$$\begin{aligned} D\psi_\gamma(r) &= \alpha_\gamma'(|r|)\frac{r\otimes r}{|r|^2} + \frac{\alpha_\gamma(|r|)}{|r|}\big(I - \frac{r\otimes r}{|r|^2}\big) \\ &= \alpha'(|r|)I + \big[\frac{\alpha_\gamma(|r|)}{|r|} - \alpha'(|r|)\big]\big(I - \frac{r\otimes r}{|r|^2}\big). \end{aligned}$$

Hence, by (1.6) and (6.5), there holds

$$\sigma_l(x,\xi)\cdot D\psi_\gamma(r)\xi \geq \sigma_l(x,\xi)\cdot\xi\alpha_\gamma'(|r|), \quad \forall\xi, r \in \mathbb{R}^m,\ l = 1,\dots,N,$$

and, by (1.2),

$$\begin{aligned} &\sum_{l=1}^N \sigma_l\big(x,\frac{\partial u}{\partial x_l}\big)\cdot\frac{\partial}{\partial x_l}\psi_\gamma(u) \\ &\geq \alpha_\gamma'(|u|)\sum_{l=1}^N \sigma_l\big(x,\frac{\partial u}{\partial x_l}\big)\cdot\frac{\partial u}{\partial x_l} \geq \alpha_\gamma'(|u|)\Big(c_1\sum_{l=1}^N\big|\frac{\partial u}{\partial x_l}\big|^N - c_2N\Big). \end{aligned} \tag{6.6}$$

Using $\chi^N\psi_\gamma(u-\beta)$ as a test function in the weak formulation of (1.1) yields

$$\begin{aligned}
&\int_\Omega \chi^N \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big)\cdot \frac{\partial}{\partial x_l}\psi_\gamma(u-\beta)\,dx \\
&= -\int_\Omega N\chi^{N-1}\psi_\gamma(u-\beta)\,dx \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big)\cdot\frac{\partial \chi}{\partial x_l}\,dx \\
&+ \int_\Omega \chi^N f\psi_\gamma(u-\beta)\,dx.
\end{aligned} \tag{6.7}$$

Using (6.5), (6.6), (1.2), and Hölder's inequality, we deduce from (6.7)

$$\begin{aligned}
&c_1\int_\Omega \chi^N \sum_{l=1}^N \big|\frac{\partial u}{\partial x_l}\big|^N \alpha'_\gamma(\big|u-\beta\big|)\,dx \\
&\le \frac{C}{(R-\rho)}\Big(\int_\Omega \chi^N\alpha'_\gamma(\big|u-\beta\big|)\Big(c'_1\sum_{l=1}^N\big|\frac{\partial u}{\partial x_l}\big|^{N-1} + Nc'_2\Big)^{\frac{N}{N-1}}\,dx\Big)^{\frac{N-1}{N}} \\
&\quad\times\Big(\int_{Q(y,R)\setminus Q(y,\rho)}\big|u-\beta\big|^N dx\Big)^{1/N} + \tilde{C}\Big(\gamma\int_{Q(y,R)}\big|f\big|\,dx + R^N\Big).
\end{aligned} \tag{6.8}$$

An application of Young's inequality yields

$$\begin{aligned}
&\int_\Omega \chi^N\alpha'_\gamma(\big|u-\beta\big|)\sum_{l=1}^N\big|\frac{\partial u}{\partial x_l}\big|^N\,dx \\
&\le \frac{C_1}{(R-\rho)^N}\int_{Q(y,R)\setminus Q(y,\rho)}\big|u-\beta\big|^N\,dx + C_2\Big(\gamma\int_{Q(y,\rho)}\big|f\big|\,dx + R^N\Big),
\end{aligned}$$

for some constants C_1, C_2. Now (6.3) follows from the definition of α_γ. Using $\chi^N\psi_\gamma(u)$ as a test function in the weak formulation of (1.1) and proceeding as in the proof of (6.3), we deduce easily (6.4). □

We quote the following key lemma from [19].

Lemma 6.4 (Dolzmann, Hungerbühler, and Müller [19])**.** *Suppose u belongs to $W_0^{1,N}(\Omega;R^m)$ and there exists $f\in L^1(\Omega;\mathbb{R}^m)$ such that the Caccioppoli estimates (6.3), (6.4) hold. Then $u\in \mathrm{BMO}(\Omega;\mathbb{R}^m)$, $Du\in\mathcal{M}^N(\Omega;\mathbb{R}^{m\times N})$, and*

$$\big|u\big|_{\mathrm{BMO}(\Omega;\mathbb{R}^m)} + \big\|Du\big\|_{\mathcal{M}^N(\Omega;\mathbb{R}^{m\times N})} \le C\big(\big\|f\big\|_{L^1(\Omega;\mathbb{R}^m)}^{1/(N-1)} + 1\big),$$

where $C>0$ is a constant depending only on N, $\big|\Omega\big|$, and the constant K in the definition of property A.

Concluding the proof of Theorem 6.1. It is possible to construct a sequence of approximate solutions $u_\varepsilon\in W_0^{1,N}(\Omega;\mathbb{R}^m)$ satisfying (3.5), with $f_\varepsilon\in L^1(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ satisfying (3.4). In view of Lemmas 6.3 and 6.4, the proof of Theorem 6.1 is obtained by routine arguments.

Proof of Theorem 6.2. The main obstacle that one encounters when attempting to prove uniqueness is that if u, v are two solutions of (1.1), then $w = u - v$ is not in $L^\infty(\Omega;\mathbb{R}^m)$ and therefore cannot be used as a test function in the weak formulation. To handle this problem, we implement the technique developed in Dolzmann, Hungerbühler, and Müller [19], which in turn was motivated by earlier work by Acerbi and Fusco [1]. The idea is to approximate the function w by a Lipschitz function w_λ that coincides with w on a large set. Moreover, precise estimates of the measure of the set where these two functions do not coincide can be provided if w has "maximal regularity".

We start by recalling the key approximation lemma.

Lemma 6.5 (Dolzmann, Hungerbühler, and Müller [19], see also [1]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set such that Ω^c is a domain of type A and fix $1 < p < \infty$. Let $w \in W_0^{1,1}(\Omega;\mathbb{R}^m)$ be such that $Dw \in \mathcal{M}^p(\Omega;\mathbb{R}^{m\times N})$. Then there exists for each $\lambda > 0$ a function $w_\lambda \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $\|w_\lambda\|_{W^{1,\infty}(\Omega;\mathbb{R}^m)} \le C_1\lambda$ and*

$$\big|\{x\in\Omega : w(x)\neq w_\lambda(x)\}\big| \le C_2\lambda^{-p}\|Dw\|_{\mathcal{M}^p(\Omega;\mathbb{R}^{m\times N})}.$$

The constants C_1 and C_2 depend only on $|\Omega|$ and N. If $w\in W^{1,p}(\Omega;\mathbb{R}^m)$, then

$$\big|\{x\in\Omega : w(x)\neq w_\lambda(x)\}\big| = o(\lambda^{-p}).$$

Let $A_\lambda := \{x\in\Omega : w(x)\neq w_\lambda(x)\}$. To prove Theorem 6.2, observe that $w := u - v \in W_0^{1,1}(\Omega;\mathbb{R}^m)$ and introduce according to Lemma 6.5 the function w_λ. Since u and v are solutions, we have

$$\sum_{l=1}^N \frac{\partial}{\partial x_l}\Big(\sigma_l\big(x,\frac{\partial u}{\partial x_l}\big) - \sigma_l\big(x,\frac{\partial v}{\partial x_l}\big)\Big) = 0 \quad \text{in } \mathcal{D}'(\Omega;\mathbb{R}^m). \tag{6.9}$$

Using w_λ as a test function in (6.9) yields

$$\sum_{l=1}^N \int_\Omega \Big(\sigma_l\big(x,\frac{\partial u}{\partial x_l}\big) - \sigma_l\big(x,\frac{\partial v}{\partial x_l}\big)\Big)\cdot\frac{\partial w_\lambda}{\partial x_l}\,dx = 0. \tag{6.10}$$

Since $\frac{\partial w_\lambda}{\partial x_l} = \frac{\partial u}{\partial x_l} - \frac{\partial v}{\partial x_l}$ a.e. on $\Omega\setminus A_\lambda$, we deduce from (6.10) and (1.2), with $p_l = N$ for all $l = 1,\dots,N$,

$$\begin{aligned}
&c_1\sum_{l=1}^N\int_{\Omega\setminus A_\lambda}\big|\frac{\partial u}{\partial x_l} - \frac{\partial v}{\partial x_l}\big|^N\,dx\\
&\le C\lambda\sum_{l=1}^N\int_{A_\lambda}\Big(\big|\frac{\partial u}{\partial x_l}\big|^{N-1} + \big|\frac{\partial v}{\partial x_l}\big|^{N-1} + 1\Big)\,dx\\
&\le C\lambda\sum_{l=1}^N |A_\lambda|^{1/N}\Big(\|\big|\frac{\partial u}{\partial x_l}\big|\|^{N-1}_{\mathcal{M}^N(\Omega;\mathbb{R}^m)} + \|\big|\frac{\partial v}{\partial x_l}\big|\|^{N-1}_{\mathcal{M}^N(\Omega;\mathbb{R}^m)}\Big) + C\lambda N|A_\lambda|\\
&\le \tilde{C},
\end{aligned} \tag{6.11}$$

where the last bound is a consequence of Lemma 6.5. Consequently, sending $\lambda\to\infty$, we have $Dw = D(u-v)\in L^N(\Omega;\mathbb{R}^{m\times N})$. We can therefore use the last part of Lemma 6.5 when sending $\lambda\to\infty$ in (6.11). The result is that $Dw = 0$, which concludes the proof Theorem 6.2.

7. Anisotropic harmonic maps into spheres

Let Ω be a bounded smooth open connected subset of $\mathbb{R}^N$ ($N \ge 2$) and $1 \le p_1,\dots,p_N < \infty$. In this section we need to use the anisotropic Sobolev space

$W^{1,(p_1,\dots,p_N)}(\Omega)$, which is defined by

$$W^{1,(p_1,\dots,p_N)}(\Omega) = \{g \in W^{1,1}(\Omega) : \frac{\partial g}{\partial x_l} \in L^{p_l}(\Omega),\, l = 1,\dots,N\},$$

$$\|g\|_{W^{1,(p_1,\dots,p_N)}(\Omega)} = \sum_{l=1}^{N} \Big(\|g\|_{L^{p_l}(\Omega)} + \|\frac{\partial g}{\partial x_l}\|_{L^{p_l}(\Omega)}\Big).$$

Let u satisfy $I[u] = \min_{w\in\mathcal{A}} I[w]$, where the anisotropic energy functional I and the set of admissible functions $\mathcal{A}$ are defined in (1.11) and (1.12), respectively. Pick any $\phi \in W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$. Since $|u| = 1$ a.e. in Ω, $w(\tau) = (u+\tau\phi)/|u+\tau\phi| \in \mathcal{A}$ for small enough τ's. Hence $J(\tau) = I[w(\tau)]$ has a minimum at $\tau = 0$ and $J'(0) = 0$. A calculation of $J'(0)$ then shows that u solves the Euler-Lagrange system (1.13) in the weak sense, which motivates the next definition.

Definition 7.1. A vector-valued function

$$u = (u_1,\dots,u_m)^\top \in W^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{S}^{m-1})$$

is called a $(p_1,\dots,p_N)$-harmonic map from Ω into $\mathbb{S}^{m-1}$ provided

$$\int_\Omega \sum_{l=1}^{N} \big|\frac{\partial u}{\partial x_l}\big|^{p_l-2} \frac{\partial u}{\partial x_l}\cdot\frac{\partial \phi}{\partial x_l}\,dx = \int_\Omega \sum_{l=1}^{N} \big|\frac{\partial u}{\partial x_l}\big|^{p_l} u\cdot\phi\,dx, \tag{7.1}$$

for all $\phi \in W_0^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$. We also use the term "anisotropic harmonic" for such a map.

Since we have not been able to find the proof of the following anisotropic Sobolev-Poincaré inequality in the literature, we have chosen to include a proof of it by the usual "contradiction method", relying on the following anisotropic Sobolev inequality [45, 2]: Let Q be a cube with faces parallel to the coordinate planes. Suppose $g \in W^{1,(p_1,\dots,p_N)}(Q)$ and $\overline{p} < N$. Then

$$\|g\|_{L^{\overline{p}^\star}(Q)} \le C \prod_{l=1}^{N} \Big(\|\frac{\partial g}{\partial x_l}\|_{L^{p_l}(Q)} + \|g\|_{L^{p_l}(Q)}\Big)^{1/N}, \tag{7.2}$$

and the inequality between geometric and arithmetic means implies that the right-hand side can be bounded by $\frac{C}{N}\sum_{l=1}^{N}\Big(\|\frac{\partial g}{\partial x_l}\|_{L^{p_l}(Q)} + \|g\|_{L^{p_l}(Q)}\Big)$. Hence, the space $W^{1,(p_1,\dots,p_N)}(Q)$ is continuously embedded into $L^{\overline{p}^\star}(Q)$.

Lemma 7.1. *Let $Q(x_0,\rho) = \{x \in \mathbb{R}^N : |x_l - x_{0,l}| < \frac{\rho}{2},\, l = 1,\dots,N\}$, where $x_0 \in \mathbb{R}^N$, $\rho > 0$. Suppose $g \in W^{1,(p_1,\dots,p_N)}(Q(x_0,\rho))$. Suppose the anisotropy $(p_1,\dots,p_N)$ is such that (1.14) holds. Then for each $1 \le p < \overline{p}^\star$*

$$\Big(\frac{1}{\rho^N}\int_{Q(x_0,\rho)} |g - (g)_{x_0,\rho}|^p\,dx\Big)^{1/p} \le C\rho \sum_{l=1}^{N} \Big(\frac{1}{\rho^N}\int_{Q(x_0,\rho)} \big|\frac{\partial g}{\partial x_l}\big|^{p_l}\Big)^{1/p_l}, \tag{7.3}$$

for some constant $C = C(N,p_1,\dots,p_N,p)$. Here $(g)_{x_0,\rho}$ denotes the average value of g over the cube $Q(x_0,\rho)$.

Proof. We divide the proof into two steps.

Step 1 ($x_0 = 0$, $\rho = 1$). We argue by contradiction. Suppose the assertion is not true. Then for each $n = 1,2,\dots$, there would exist a function $g_n \in W^{1,(p_1,\dots,p_N)}(\Omega)$

such that

$$\sum_{l=1}^{N}\big\|\frac{\partial g_n}{\partial x_l}\big\|_{L^{p_l}(Q(0,1))}<\frac{1}{n}\big\|g_n-(g_n)_{0,1}\big\|_{L^{p}(Q(0,1))}, \tag{7.4}$$

where, by the anisotropic Sobolev inequality (7.2), the right-hand side is bounded by a constant (independent of n) times $1/n$. Define

$$h_n=\frac{g_n-(g_n)_{0,1}}{\big\|g_n-(g_n)_{0,1}\big\|_{L^p(Q(0,1))}}.$$

Then $(h_n)_{0,1}=0$ and $\big\|h_n\big\|_{L^p(Q(0,1))}=1$. By (7.4), we have, passing if necessary to a subsequence, that $h_n\to h$ a.e. in $Q(0,1)$ and also in $L^p(Q(0,1))$, where h is some limit function. It follows that

$$(h)_{0,1}=0,\quad \big\|h\big\|_{L^p(Q(0,1))}=1. \tag{7.5}$$

On the other hand, it follows from (7.4) that $\frac{\partial h}{\partial x_l}=0$ for all $i=1,\dots,N$, and hence h is constant, which contradicts (7.5).

Step 2 (the general case). Let $g:Q(x_0,\rho)\to\mathbb{R}$, and scale this function to the unit cube by setting $h(x)=g(x_0+\rho x)$ for $x\in Q(0,1)$. By Step 1,

$$\Big(\int_{Q(0,1)}\big|h\big|^p\,dx\Big)^{\frac{1}{p}}\le C\sum_{l=1}^{N}\Big(\int_{Q(0,1)}\big|\frac{\partial h}{\partial x_l}\big|^{p_i}\,dx\Big)^{\frac{1}{p_l}}.$$

Changing variables in this inequality yields (7.3). □

Before we continue, we need to introduce some additional notations and function spaces. A function $g\in L^1(\mathbb{R}^N)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^N)$ if the grand maximal function $g^\star:=\sup_{\rho>0}\big|g\star\omega_\rho\big|$ belongs to $L^1(\mathbb{R}^N)$, where $\omega_\rho(x)=\rho^{-N}\omega_1(x/\rho)$, $\omega_1\in C_c^\infty(B(0,1))$, $\int\omega_1=1$. The definition does not depend on the choice of ω_1. The Hardy space is a Banach space under the norm $\big\|g\big\|_{\mathcal{H}^1(\mathbb{R}^N)}=\big\|g\big\|_{L^1(\mathbb{R}^N)}+\big\|g^\star\big\|_{L^1(\mathbb{R}^N)}$. If $g\in\mathcal{H}^1(\mathbb{R}^N)$, then necessarily $\int g=0$. The dual space of $\mathcal{H}^1(\mathbb{R}^N)$ is the space $\mathrm{BMO}(\mathbb{R}^N)$ of functions of bounded mean oscillations. Here a function $h\in L^1_{\rm loc}(\mathbb{R}^N)$ belongs to $BMO(\mathbb{R}^N)$ if $\big|h\big|_{\mathrm{BMO}(\mathbb{R}^N)}=\sup_{x,\rho}\frac{1}{\rho^N}\int_{Q(x,r)}\big|h(y)-(h)_{x,\rho}\big|\,dy$ is finite. The space $VMO(\mathbb{R}^N)$ of functions of vanishing mean oscillations, which is defined as the closure of $C_0(\mathbb{R}^N)$ in $BMO(\mathbb{R}^N)$, is the predual of $\mathcal{H}^1(\mathbb{R}^N)$. We shall need the local Hardy space $\mathcal{H}^1_{\rm loc}(\Omega)$. Let K be any compact subset of Ω and set $\epsilon_K=\mathrm{dist}(K,\mathbb{R}^N\setminus\Omega)$. Then $g\in\mathcal{H}^1_{\rm loc}(\Omega)$ if for any compact subset $K\subset\Omega$ there holds $\sup_{0<\rho<\epsilon_K}\big|g\star\omega_\rho\big|\in L^1(K)$. We refer to Stein [40] for more information about the spaces just introduced.

Coifman, Lions, Meyer, and Semmes [14] proved that if two vector fields B and E in conjugate Lebesgue spaces $L^p(\mathbb{R}^N;\mathbb{R}^N)$ and $L^{p'}(\mathbb{R}^N;\mathbb{R}^N)$ satisfy $\operatorname{curl}B=0$ and $\operatorname{div}E=0$ in the sense of distributions, then their scalar product $B\cdot E$, which a priori only belongs to $L^1(\mathbb{R}^N)$ by Hölder's inequality, belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^N)$, which is a strict subspace of $L^1(\mathbb{R}^N)$. Thus the nonlinear quantity $B\cdot E$ possesses a compensated integrability property. We shall require an anisotropic version of (a special case) of this result. The proof follows closely that in [14], with some minor modifications to account for the anisotropy of the involved vector fields.

Theorem 7.2. *Let $1<p_l<\infty$ and $1/p_l+1/p_l'=1$, $l=1,\dots,N$. Suppose $B=D\pi$ for some function $\pi\in W^{1,(p_1,\dots,p_N)}(\mathbb{R}^N)$ and $E=(E_1,\dots,E_N)^\top$, $E_l\in$*

$L^{p_l'}(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, $\operatorname{div} E=0$. *Suppose the anisotropy* $(p_1,\dots,p_N)$ *is such that* (1.14) *holds. Then* $B\cdot E$ *belongs to* $\mathcal{H}^1(\mathbb{R}^N)$ *and*

$$\|B\cdot E\|_{\mathcal{H}^1(\mathbb{R}^N)}\le C\Big(\sum_{l=1}^N\big\|\frac{\partial\pi}{\partial x_l}\big\|_{L^{p_l}(\mathbb{R}^N)}\Big)\Big(\sum_{l=1}^N\|E_l\|_{L^{p_l'}(\mathbb{R}^N)}\Big),$$

where the universal constant C *depends on* $N,p_1,\dots,p_N$. *If the domain of definition* $\mathbb{R}^N$ *for* B *and* E *are replaced by* Ω, *then the theorem remains true with* $\mathcal{H}^1(\mathbb{R}^N)$ *replaced by* $\mathcal{H}^1_{\rm loc}(\Omega)$.

Remark 2. Theorem 7.2 shows that the product $B\cdot E$ has a compensated integrability property as long as the anisotropy $(p_1,\dots,p_N)$ is not too much spread out, which is reflected in the condition $\overline{p}^\star>p_{\max}$.

Proof. It is clear that $D\pi\cdot E\in L^1(\mathbb{R}^N)$ and that $D\pi\cdot E=\sum_{l=1}^N\frac{\partial}{\partial x_l}(\pi E_l)$ in the sense of distributions. For any $x\in\mathbb{R}^N$ and any $\rho>0$, we need to estimate the convolution product $(D\pi\cdot E)\star\omega_\rho(x)$:

$$\begin{aligned}\big|(D\pi\cdot E)\star\omega_\rho(x)\big|&=\Big|\int_{\mathbb{R}^N}(D\pi\cdot E)(y)\omega_\rho(x-y)\,dy\Big|\\&=\Big|\int_{\mathbb{R}^N}\sum_{l=1}^N(\pi E_l)(y)\frac{\partial}{\partial y_l}\omega_\rho(x-y)\,dy\Big|\\&=\Big|\int_{\mathbb{R}^N}\sum_{l=1}^N(\pi(y)-(\pi)_{x,\rho})E_l(y)\frac{\partial}{\partial y_l}\omega_\rho(y-x)\,dy\Big|\\&\le C\frac{1}{\rho^{N+1}}\int_{Q(x,\rho)}\sum_{l=1}^N\big|\pi(y)-(\pi)_{x,\rho}\big|\,\big|E_l(y)\big|\,dy.\end{aligned}$$

Next we choose $(q_1,\dots,q_N)$ such that $q_l<p_l$ for all $l=1,\dots,N$ and $\overline{q}^\star>p_{\max}>q_{\max}$. We can do this since $\overline{p}^\star>p_{\max}$. To be specific, choose $q_l=\theta p_l$, $l=1,\dots,N$, for some $\theta\in\big(\frac{\overline{p}^\star}{\overline{p}^\star+N},1\big)$ to be specified later. One can check that

$$\theta\overline{p}^\star=\frac{N\theta}{N\theta-(1-\theta)\overline{q}^\star}\overline{q}^\star=:e(\theta)\overline{q}^\star.$$

Since $0<\overline{q}^\star<\overline{p}^\star$ and $\theta>\frac{\overline{p}^\star}{\overline{p}^\star+N}$, there holds $1<e(\theta)<\frac{N\theta}{N\theta-(1-\theta)\overline{p}^\star}<\infty$. Moreover, $e(\theta)\downarrow 1$ as we let $\theta\uparrow 1$. Using $\overline{p}^\star>p_{\max}$ to write $\overline{p}^\star=p_{\max}+\kappa$ for some $\kappa>0$, we obtain

$$\overline{q}^\star=\frac{\theta}{e(\theta)}\overline{p}^\star=p_{\max}+\Delta(\theta),\quad \Delta(\theta):=\big(\frac{\theta}{e(\theta)}-1\big)p_{\max}+\frac{\theta}{e(\theta)}\kappa.$$

Clearly, by choosing θ close enough to 1, we can ensure $\Delta(\theta)>0$. Hence, for such a choice of θ, we have $\overline{q}^\star>p_{\max}>q_{\max}$. Having chosen the q_l's, we choose $(s_1,\dots,s_N)$ such that $p_l<s_l<\overline{q}^\star$ for all l. Indeed, we can take $\frac{1}{s_l}=\frac{1}{q_l}-\delta_l$, with $\delta_l\in(0,\frac{1}{q_l}-\frac{1}{\overline{q}^\star})$.

We now continue using Hölder's inequality to obtain

$$\begin{aligned}&\big|(D\pi\cdot E)\star\omega_\rho(x)\big|\\&\le C\sum_{l=1}^N\frac{1}{\rho}\Big(\frac{1}{\rho^N}\int_{Q(x,\rho)}\big|\pi(y)-(\pi)_{x,\rho}\big|^{s_l}\Big)^{1/s_l}\Big(\frac{1}{\rho^N}\int_{Q(x,\rho)}\big|E_l(y)\big|^{s_l'}\Big)^{1/s_l'}.\end{aligned}$$

Since $s_l < \overline{q}^\star$ and $\pi \in W_0^{1,(q_1,\dots,q_N)}(Q(x,\rho))$, we can use the anisotropic Sobolev-Poincaré inequality (see Lemma 7.1) to obtain

$$\begin{aligned}
&\big|(D\pi \cdot E) \star \omega_\rho(x)\big| \\
&\le C \sum_{l=1}^N \sum_{j=1}^N \Big(\frac{1}{\rho^N}\int_{Q(x,\rho)} \big|\frac{\partial \pi(y)}{\partial y_j}\big|^{q_j}\Big)^{1/q_j} \Big(\frac{1}{\rho^N}\int_{Q(x,\rho)} \big|E_l(y)\big|^{s_l'}\Big)^{1/s_l'}.
\end{aligned}$$

We need the Hardy-Littlewood maximal function

$$M[g](x) = \sup_{\rho>0} \frac{1}{\rho^N}\int_{Q(x,\rho)} |g(y)|\,dy,$$

which is bounded on $L^p(\mathbb{R}^N)$, that is,

$$\big\|M[g]\big\|_{L^p(\mathbb{R}^N)} \le C(p)\big\|g\big\|_{L^p(\mathbb{R}^N)},$$

for $1 < p < \infty$. Using the maximal function we find that

$$\sup_{\rho>0}\big|(D\pi \cdot E) \star \omega_\rho(x)\big| \le C \sum_{l=1}^N \sum_{j=1}^N \Big(M\big[\big|\frac{\partial \pi}{\partial y_j}\big|^{q_j}\big](x)\Big)^{1/q_j} \Big(M\big[\big|E_l\big|^{s_l'}\big](x)\Big)^{1/s_l'}.$$

Integrating over $x \in \mathbb{R}^N$, using Hölder's inequality, and finally using the boundedness of the maximal function (recall that $p_j > q_j$ and $p_l' > s_l'$), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \sup_{\rho>0}\big|(D\pi \cdot E) \star \omega_\rho(x)\big|\,dx \\
&\le C \sum_{l=1}^N \sum_{j=1}^N \Big(\int_{\mathbb{R}^N} \Big(M\big[\big|\frac{\partial \pi}{\partial y_j}\big|^{q_j}\big](x)\Big)^{p_j/q_j}\,dx\Big)^{\frac{1}{p_j}} \Big(\int_{\mathbb{R}^N} \Big(M\Big[\big|E_l\big|^{s_l'}\Big](x)\Big)^{p_l'/s_l'}\,dx\Big)^{1/p_l'} \\
&\le C\Big(\sum_{j=1}^N \big\|\frac{\partial \pi}{\partial y_j}\big\|_{L^{p_j}(\mathbb{R}^N)}\Big)\Big(\sum_{l=1}^N \big\|E_l\big\|_{L^{p_l'}(\mathbb{R}^N)}\Big),
\end{aligned}$$

which concludes the proof of the theorem. □

We have come to the main result of this section, namely a compactness theorem for $(p_1,\dots,p_N)$-harmonic maps. This result can be viewed as an anisotropic version of a result of Toro and Wang [44] for p-harmonic maps, and our proof proceeds along the lines of [44].

Theorem 7.3. *Suppose $(u_\varepsilon)_{0<\varepsilon\le 1} \subset W^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{S}^{m-1})$ is a sequence of $(p_1,\dots,p_N)$-harmonic maps such that*

$$u_\varepsilon \rightharpoonup u \quad \text{in } W^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{S}^{m-1}) \text{ as } \varepsilon \to 0.$$

Then u is a $(p_1,\dots,p_N)$-harmonic map from Ω into $\mathbb{S}^{m-1}$.

Proof. Each u_ε is a weak solution of

$$-\sum_{l=1}^N \frac{\partial}{\partial x_l}\Big(\big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2}\frac{\partial u_\varepsilon}{\partial x_l}\Big) = f_\varepsilon, \quad f_\varepsilon := \sum_{l=1}^N \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l} u_\varepsilon.$$

Clearly, as u_ε is uniformly bounded in $W^{1,(p_1,\dots,p_N)}(\Omega;\mathbb{R}^m)$ and $\big|u_\varepsilon\big| = 1$ a.e. in Ω, we have that f_ε is uniformly bounded in $L^1(\Omega)$. Thus the above system fits into the theory developed previously in this paper.

As in [25, 44], the main point of the proof is exploit that the term f_ε has a particular structure due to the constraint $\left|u_\varepsilon\right| = 1$ a.e. in Ω, which implies that it in fact belongs to the Hardy space $\mathcal{H}^1_{\rm loc}(\Omega)$ and not just $L^1(\Omega)$. Indeed, observe that, for any $i = 1, \dots, N$,

$$\begin{aligned} f_{\varepsilon,i} &= \sum_{k=1}^m \sum_{l=1}^N \frac{\partial}{\partial x_l} u_{\varepsilon,k} \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2}\big(u_{\varepsilon,i}\frac{\partial}{\partial x_l}u_{\varepsilon,k} - u_{\varepsilon,k}\frac{\partial}{\partial x_l}u_{\varepsilon,i}\big) \\ &= \sum_{k=1}^m B_{\varepsilon,k}\cdot E_{\varepsilon,i,k}, \end{aligned}$$

where the vector fields $B_{\varepsilon,k} = (B_{\varepsilon,k})_{l=1}^N$ and $E_{\varepsilon,i,k} = (E_{\varepsilon,i,k})_{l=1}^N$ are defined by $(B_{\varepsilon,k})_l = \frac{\partial}{\partial x_l}u_{\varepsilon,k}$, $l = 1, \dots, N$, and

$$(E_{\varepsilon,i,k})_l = \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2}\big(u_{\varepsilon,i}\frac{\partial}{\partial x_l}u_{\varepsilon,k} - u_{\varepsilon,k}\frac{\partial}{\partial x_l}u_{\varepsilon,i}\big), \quad l = 1, \dots, N.$$

Clearly, $\operatorname{curl} B_{\varepsilon,k} = 0$. Let us show that $E_{\varepsilon,i,k}$ is divergence free:

$$\begin{aligned} \operatorname{div} E_{\varepsilon,i,k} &= \sum_{l=1}^N \frac{\partial}{\partial x_l}(E_{\varepsilon,i,k})_l \\ &= \sum_{l=1}^N \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2}\frac{\partial}{\partial x_l}u_{\varepsilon,i}\frac{\partial}{\partial x_l}u_{\varepsilon,k} + \sum_{l=1}^N \frac{\partial}{\partial x_l}\Big(\big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2}\frac{\partial}{\partial x_l}u_{\varepsilon,k}\Big)u_{\varepsilon,i} \\ &\quad - \sum_{l=1}^N \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2}\frac{\partial}{\partial x_l}u_{\varepsilon,k}\frac{\partial}{\partial x_l}u_{\varepsilon,i} - \sum_{l=1}^N \frac{\partial}{\partial x_l}\Big(\big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2}\frac{\partial}{\partial x_l}u_{\varepsilon,i}\Big)u_{\varepsilon,k} \\ &= \sum_{l=1}^N \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l} u_{\varepsilon,k}u_{\varepsilon,i} - \sum_{l=1}^N \big|\frac{\partial u}{\partial x_l}\big|^{p_l} u_{\varepsilon,i}u_{\varepsilon,k} = 0. \end{aligned}$$

According to Theorem 7.2, $E_{\varepsilon,i,k}\cdot B_{\varepsilon,k}$ is then bounded in $\mathcal{H}^1_{\rm loc}(\Omega)$.

Adapting the methods and results in Subsection 3.4, we can without loss of generality assume in the following that as $\varepsilon \to 0$,

$$\begin{gathered} u_\varepsilon \to u \text{ a.e. in } \Omega \text{ and } \frac{\partial u_\varepsilon}{\partial x_l} \to \frac{\partial u}{\partial x_l} \text{ a.e. in } \Omega, \\ \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2}\frac{\partial u_\varepsilon}{\partial x_l} \rightharpoonup \big|\frac{\partial u}{\partial x_l}\big|^{p_l-2}\frac{\partial u}{\partial x_l} \text{ in } L^{p_l'}(\Omega;\mathbb{R}^m), \end{gathered} \tag{7.6}$$

for $l = 1, \dots, N$. Therefore,

$$f_{\varepsilon,i} \to f_i := \sum_{k=1}^m \sum_{l=1}^N \frac{\partial}{\partial x_l}u_k\big|\frac{\partial u}{\partial x_l}\big|^{p_l-2}\big(u_i\frac{\partial}{\partial x_l}u_k - u_k\frac{\partial}{\partial x_l}u_i\big) \quad \text{a.e. in } \Omega,$$

and $f_i \in L^1(\Omega)$, $i = 1, \dots, N$. Of course, the main difficulty is to improve this a.e. convergence to the convergence

$$\int_\Omega f_\varepsilon\cdot\phi\,dx \to \int_\Omega f\cdot\phi\,dx \quad \text{for any } \phi \in W_0^{1,(p_1,\dots,p_N)} \cap L^\infty.$$

For each $i = 1, \dots, N$, by Theorem 7.2, $f_{\varepsilon,i}$ is bounded in $\mathcal{H}^1_{\rm loc}(\Omega)$, and for any compact $K \subset \Omega$ we have the bound

$$\begin{aligned} \|f_{\varepsilon,i}\|_{\mathcal{H}^1(K)} &\le C \sum_{k=1}^m \Big(\sum_{l=1}^N \big\|\frac{\partial}{\partial x_l} u_{\varepsilon,k}\big\|_{L^{p_l}(\Omega)}\Big) \\ &\quad \times \Big(\sum_{l=1}^N \big\| \big|\frac{\partial u}{\partial x_l}\big|^{p_l-2} \big(u_i \frac{\partial}{\partial x_l} u_k - u_k \frac{\partial}{\partial x_l} u_i\big)\big\|_{L^{p_l'}(\Omega)}\Big) \le C, \end{aligned} \tag{7.7}$$

where the last constant is independent of ε since $\frac{\partial u_\varepsilon}{\partial x_l}$ is bounded in $L^{p_l}(\Omega; \mathbb{R}^m)$, $l = 1, \dots, N$.

Let $\eta \in C_c^\infty(\Omega)$, $\int_\Omega \eta\, dx \neq 0$, and introduce

$$\begin{gathered} A_{\varepsilon,i} = \int_\Omega \eta f_{\varepsilon,i}\, dx \Big/ \int_\Omega \eta\, dx \in \mathbb{R}, \quad i = 1, \dots, N, \\ F_{\varepsilon,i} = \eta\, (f_{\varepsilon,i} - A_{\varepsilon,i})\,, \quad i = 1, \dots, N\,. \end{gathered}$$

Note that $\int_\Omega F_{\varepsilon,i}\, dx = 0$. Now we extend all relevant functions defined on Ω to $\mathbb{R}^N$ by setting them to zero off Ω.

According to Semmes [37, Proposition 1.92], $F_{\varepsilon,i}$ is bounded in $\mathcal{H}^1(\mathbb{R}^N)$ and if $K = \operatorname{supp}(\eta)$ then

$$\|F_{\varepsilon,i}\|_{\mathcal{H}^1(\mathbb{R}^N)} \le C\big(1 + \|F_{\varepsilon,i}\|_{L^1(\mathbb{R}^N)} + \|f_{\varepsilon,i}\|_{\mathcal{H}^1(K)}\big), \quad i = 1, \dots, N,$$

where the right-hand side is bounded by a constant independent of ε, thanks to (7.7). Observe that by (1.13) and the last part of (7.6) we have

$$\begin{aligned} A_{\varepsilon,i} &= \frac{\int_\Omega \sum_{l=1}^N \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2} \frac{\partial}{\partial x_l} u_{\varepsilon,i} \frac{\partial}{\partial x_l}\eta\, dx}{\int_\Omega \eta\, dx} \\ &\to \frac{\int_\Omega \sum_{l=1}^N \big|\frac{\partial u}{\partial x_l}\big|^{p_l-2} \frac{\partial}{\partial x_l} u_i \frac{\partial}{\partial x_l}\eta\, dx}{\int_\Omega \eta\, dx} =: A_i, \end{aligned}$$

for $i = 1, \dots, N$. Hence $F_{\varepsilon,i} \to F_i := \eta(f_i - A_i)$ a.e. in $\mathbb{R}^N$ and, as mentioned before, $F_{\varepsilon,i}$ is bounded in $\mathcal{H}^1(\mathbb{R}^N)$. Thanks to a theorem of Jones and Journé [26], this implies that $F_{\varepsilon,i} \overset{\star}{\rightharpoonup} F_i$ in $\mathcal{H}^1(\mathbb{R}^N)$, that is,

$$\int_{\mathbb{R}^N} F_{\varepsilon,i} \Psi\, dx \to \int_{\mathbb{R}^N} F_i \Psi\, dx, \quad \forall \Psi \in VMO(\mathbb{R}^N).$$

Now we have all the necessary tools at our disposal for concluding the proof of the theorem. Let $\phi \in W_0^{1,(p_1,\dots,p_N)}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ and choose $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ on $K = \operatorname{supp}(\phi)$. Then

$$\begin{aligned} &\int_\Omega \sum_{l=1}^N \big|\frac{\partial u_\varepsilon}{\partial x_l}\big|^{p_l-2} \frac{\partial u_\varepsilon}{\partial x_l} \cdot \frac{\partial \phi}{\partial x_l}\, dx \\ &= \int_\Omega \sum_{i=1}^N f_{\varepsilon,i} \phi_i\, dx \\ &= \int_{\mathbb{R}^N} \sum_{i=1}^N F_{\varepsilon,i} \phi_i\, dx + \int_{\mathbb{R}^N} \sum_{i=1}^N \eta A_{\varepsilon,i} \phi_i\, dx. \end{aligned}$$

Sending $\varepsilon \to 0$ yields

$$\begin{aligned}\int_\Omega \sum_{l=1}^N \big|\frac{\partial u}{\partial x_l}\big|^{p_l-2}\frac{\partial u}{\partial x_l}\cdot\frac{\partial \phi}{\partial x_l}\,dx &= \int_{\mathbb{R}^N}\sum_{i=1}^N F_i\phi_i\,dx + \int_{\mathbb{R}^N}\sum_{i=1}^N \eta A_i\phi_i\,dx \\ &= \int_\Omega \sum_{i=1}^N f_i\phi_i\,dx = \int_\Omega \sum_{l=1}^N \big|\frac{\partial u}{\partial x_l}\big|^{p_l} u\cdot\phi\,dx.\end{aligned}$$

Hence u is a $(p_1,\dots,p_N)$-harmonic map. □

Acknowledgments. This work was partially supported by the European network HYKE, contract HPRN-CT-2002-00282. M. Bendahmane was supported in part by The Gulbenkian Foundation and FCT of Portugal. K. H. Karlsen was supported in part by an Outstanding Young Investigators Award from the Research Council of Norway.

References

[1] E. Acerbi and N. Fusco. An approximation lemma for $W^{1,p}$ functions. In *Material instabilities in continuum mechanics (Edinburgh, 1985–1986)*, Oxford Sci. Publ., pages 1–5. Oxford Univ. Press, New York, 1988.

[2] E. Acerbi and N. Fusco. Partial regularity under anisotropic (p, q) growth conditions. *J. Differential Equations*, 107(1):46–67, 1994.

[3] M. Bendahmane and K. H. Karlsen. Nonlinear anisotropic elliptic and parabolic equations in $\mathbf{R}^N$ with advection and lower order terms and locally integrable data. *Potential Anal.*, 22(3):207–227, 2005.

[4] M. Bendahmane, M. Langlais, and M. Saad. On some anisotropic reaction-diffusion systems with L^1-data modeling the propagation of an epidemic disease. *Nonlinear Anal.*, 54(4):617–636, 2003.

[5] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):241–273, 1995.

[6] A. Bensoussan and L. Boccardo. Nonlinear systems of elliptic equations with natural growth conditions and sign conditions. *Appl. Math. Optim.*, 46(2-3):143–166, 2002. Special issue dedicated to the memory of Jacques-Louis Lions.

[7] D. Blanchard and F. Murat. Renormalised solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness. *Proc. Roy. Soc. Edinburgh Sect. A*, 127(6):1137–1152, 1997.

[8] L. Boccardo, J. I. Diaz, D. Giachetti, and F. Murat. Existence of a solution for a weaker form of a nonlinear elliptic equation. In *Recent advances in nonlinear elliptic and parabolic problems (Nancy, 1988)*, volume 208 of *Pitman Res. Notes Math. Ser.*, pages 229–246. Longman Sci. Tech., Harlow, 1989.

[9] L. Boccardo, T. Gallouët, and P. Marcellini. Anisotropic equations in L^1. *Differential Integral Equations*, 9(1):209–212, 1996.

[10] L. Boccardo, T. Gallouët, and J. L. Vazquez. Solutions of nonlinear parabolic equations without growth restrictions on the data. *Electron. J. Differential Equations*, pages No. 60, 20 pp. (electronic), 2001.

[11] L. Boccardo, D. Giachetti, J. I. Diaz, and F. Murat. Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms. *J. Differential Equations*, 106(2):215–237, 1993.

[12] L. Boccardo, P. Marcellini, and C. Sbordone. L^∞-regularity for variational problems with sharp nonstandard growth conditions. *Boll. Un. Mat. Ital. A (7)*, 4(2):219–225, 1990.

[13] Y. M. Chen. The weak solutions to the evolution problems of harmonic maps. *Math. Z.*, 201(1):69–74, 1989.

[14] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes. Compensated compactness and Hardy spaces. *J. Math. Pures Appl. (9)*, 72(3):247–286, 1993.

[15] G. Dal Maso and F. Murat. Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems. *Nonlinear Anal.*, 31(3-4):405–412, 1998.

[16] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):741–808, 1999.
[17] G. Dolzmann, N. Hungerbühler, and S. Müller. Non-linear elliptic systems with measure-valued right hand side. *Math. Z.*, 226(4):545–574, 1997.
[18] G. Dolzmann, N. Hungerbühler, and S. Müller. The p-harmonic system with measure-valued right hand side. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14(3):353–364, 1997.
[19] G. Dolzmann, N. Hungerbühler, and S. Müller. Uniqueness and maximal regularity for non-linear elliptic systems of n-Laplace type with measure valued right hand side. *J. Reine Angew. Math.*, 520:1–35, 2000.
[20] L. C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1990.
[21] L. C. Evans. Partial regularity for stationary harmonic maps into spheres. *Arch. Rational Mech. Anal.*, 116(2):101–113, 1991.
[22] M. Fuchs and J. Reuling. Non-linear elliptic systems involving measure data. *Rend. Mat. Appl. (7)*, 15(2):311–319, 1995.
[23] M. Giaquinta. Growth conditions and regularity, a counterexample. *Manuscripta Math.*, 59(2):245–248, 1987.
[24] R. Hardt and F.-H. Lin. Mappings minimizing the L^p norm of the gradient. *Comm. Pure Appl. Math.*, 40(5):555–588, 1987.
[25] F. Hélein. *Harmonic maps, conservation laws and moving frames*, volume 150 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, second edition, 2002. Translated from the 1996 French original, With a foreword by James Eells.
[26] P. W. Jones and J.-L. Journé. On weak convergence in $H^1(\mathbf{R}^d)$. *Proc. Amer. Math. Soc.*, 120(1):137–138, 1994.
[27] R. Landes. On the existence of weak solutions of perturbated systems with critical growth. *J. Reine Angew. Math.*, 393:21–38, 1989.
[28] R. Landes. Testfunctions for elliptic systems and maximum principles. *Forum Math.*, 12(1):23–52, 2000.
[29] F. Li and H. Zhao. Anisotropic parabolic equations with measure data. *J. Partial Differential Equations*, 14(1):21–30, 2001.
[30] F.-Q. Li. Anisotropic elliptic equations in L^m. *J. Convex Anal.*, 8(2):417–422, 2001.
[31] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, 1969.
[32] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, New York, 1996.
[33] S. Luckhaus. Convergence of minimizers for the p-Dirichlet integral. *Math. Z.*, 213(3):449–456, 1993.
[34] S. M. Nikol'skiĭ. An imbedding theorem for functions with partial derivatives considered in different metrics. *Izv. Akad. Nauk SSSR Ser. Mat.*, 22:321–336, 1958. English translation: Amer. Math. Soc. Transl. 90 (1970), 27–44.
[35] J.-M. Rakotoson. Uniqueness of renormalized solutions in a T-set for the L^1-data problem and the link between various formulations. *Indiana Univ. Math. J.*, 43(2):685–702, 1994.
[36] R. Schoen and K. Uhlenbeck. A regularity theory for harmonic maps. *J. Differential Geom.*, 17(2):307–335, 1982.
[37] S. Semmes. A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller. *Comm. Partial Differential Equations*, 19(1-2):277–319, 1994.
[38] J. Shatah. Weak solutions and development of singularities of the SU(2) σ-model. *Comm. Pure Appl. Math.*, 41(4):459–469, 1988.
[39] L. N. Slobodeckiĭ. Generalized Sobolev spaces and their application to boundary problems for partial differential equations. *Leningrad. Gos. Ped. Inst. Učen. Zap.*, 197:54–112, 1958. English translation: Amer. Math. Soc. Transl. (2) 57 (1966), 207–275.
[40] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
[41] B. Stroffolini. Global boundedness of solutions of anisotropic variational problems. *Boll. Un. Mat. Ital. A (7)*, 5(3):345–352, 1991.

[42] B. Stroffolini. Some remarks on the regularity of anisotropic variational problems. *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)*, 17:229–239, 1993.
[43] G. Talenti. Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. *Ann. Mat. Pura Appl. (4)*, 120:160–184, 1979.
[44] T. Toro and C. Wang. Compactness properties of weakly p-harmonic maps into homogeneous spaces. *Indiana Univ. Math. J.*, 44(1):87–113, 1995.
[45] M. Troisi. Teoremi di inclusione per spazi di sobolev non isotropi. *Ricerche. Mat.*, 18:3–24, 1969.
[46] N. S. Trudinger. An imbedding theorem for $H_0(G, \Omega)$ spaces. *Studia Math.*, 50:17–30, 1974.
[47] S. Zhou. A note on nonlinear elliptic systems involving measures. *Electron. J. Differential Equations*, pages No. 08, 6 pp. (electronic), 2000.

Mostafa Bendahmane
Centre of Mathematics for Applications, University of Oslo
P.O. Box 1053, Blindern, N–0316 Oslo, Norway
E-mail address: mostafab@math.uio.no

Kenneth Hvistendahl Karlsen
Centre of Mathematics for Applications, University of Oslo
P.O. Box 1053, Blindern, N–0316 Oslo, Norway
E-mail address: kennethk@math.uio.no
URL: http://www.math.uio.no/∼kennethk/