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NONLINEAR ELASTIC MEMBRANES INVOLVING THE P-LAPLACIAN OPERATOR

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Abstract. This paper concerns an optimization problem related to the Poisson equation for the p-Laplace operator, subject to homogeneous Dirichlet boundary conditions. Physically the Poisson equation models, for example, the deformation of a nonlinear elastic membrane which is fixed along the boundary, under load. A particular situation where the load is represented by a characteristic function is investigated.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. This paper is concerned with an optimization problem related to the Poisson boundary-value problem

$$\begin{aligned} -\Delta_p u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here $p \in (1, \infty)$, and Δ_p stands for the usual p-Laplacian; that is, $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$. Let $f_0 \in L^q(\Omega)$, with $q = p/(p-1)$, and let $\mathcal{R}$ be the class of rearrangements of f_0. We are interested in finding

$$\sup_{f \in \mathcal{R}} \int_\Omega f\, u_f\, dx \tag{1.2}$$

where u_f is the (unique) solution of (1.1).

The p-Laplace operator arises in various physical contexts: non Newtonian fluids, reaction diffusion problems, non linear elasticity, electrochemical machining, elastic-plastic torsional creep, etc., see [1], [10], and references therein. For a theoretical develop of the theory of the p-Laplacian we refer to the monograph [9]. The case of $p = 2$ is the most important and easier to discuss: it corresponds to a first approximation, the linear case. For non ideal materials, it is often appropriate to involve a power of the gradient $|\nabla u|$ to describe the law governing the model. For example, problem (1.1) models a nonlinear elastic membrane under load f. The solution u_f stands for the deformation of the membrane from the rest position. Therefore, the functional $\int_\Omega f\, u_f\, dx$ measures the average deformation, with respect

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to the measure $f\,dx$, of the membrane. Thus, any solution to (1.2) determines an *optimal* load chosen from the class $\mathcal{R}$.

Our interest in (1.2) spans questions such as existence, uniqueness (in case Ω is a ball), and qualitative properties of maximizers. In case of $p=2$, the problem is well understood, see [3], [4], [5], [6], [11]. In this case the functional $\int_\Omega f\,u_f\,dx$ is weakly sequentially continuous and strictly convex, say on $L^2(\Omega)$, so the classical results of R. Burton are available to be applied to prove existence and some qualitative properties of the maximizers. However, in the case $p \neq 2$, we will use a method which does not need the convexity of the functional. The existence in a similar situation has been discussed in [7].

2. Preliminaries

In this section we collect some well known results. Let us begin with the definition of a weak solution of (1.1).

Definition. A function $u \equiv u_f \in W_0^{1,p}(\Omega)$ is a weak solution of (1.1) provided

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx = \int_\Omega f v\,dx, \qquad \forall\, v \in W_0^{1,p}(\Omega).$$

It is a standard result that (1.1) has a unique weak solution u_f, for which the following equations hold

$$\int_\Omega f u_f\,dx = \int_\Omega |\nabla u_f|^p\,dx = \frac{1}{p-1}\sup_{u\in W_0^{1,p}(\Omega)}\int_\Omega \big(pf\,u - |\nabla u|^p\big)dx. \tag{2.1}$$

Definition. Suppose $f:(X,\Sigma,\mu)\to\mathbb{R}^+$ and $g:(X',\Sigma',\mu')\to\mathbb{R}^+$ are measurable functions. We say f and g are rearrangements of each other if and only if

$$\mu(\{x\in X|\ f(x)\geq\alpha\}) = \mu'(\{x\in X'|\ g(x)\geq\alpha\}), \quad \forall\alpha\geq 0.$$

Henceforth we fix $f_0 \in L^q_+(\Omega)$, with $q = p/(p-1)$. The set of all rearrangements of f_0 is denoted by $\mathcal{R}$. Thus, for any $f\in\mathcal{R}$, we have

$$\mathcal{L}_N(\{x\in\Omega|\ f(x)\geq\alpha\}) = \mathcal{L}_N(\{x\in\Omega|\ f_0(x)\geq\alpha\}), \quad \forall\alpha\geq 0,$$

where $\mathcal{L}_N$ denotes the N-dimensional Lebesgue measure. For $f:\Omega\to\mathbb{R}^+$, f^Δ and f^* denote the decreasing and Schwarz rearrangements of f, respectively. Recall that f^Δ is defined on $(0,\mathcal{L}_N(\Omega))$, and f^* is defined on B, the ball centered at the origin with volume equal to $\mathcal{L}_N(\Omega)$.

Lemma 2.1. *Let $q = p/(p-1)$, $f\in L^q_+(\Omega)$, $g\in L^p_+(\Omega)$. Suppose that every level set of g (that is, sets of the form $g^{-1}(\{\alpha\})$), has measure zero. Then there exists an increasing function ϕ such that $\phi\circ g$ is a rearrangement of f.*

Lemma 2.2. *Suppose $\zeta\in L^p_+(\Omega)$, and $f\in L^q_+(\Omega)$. Suppose there exists an increasing function ϕ such that $\phi\circ\zeta\in\mathcal{R}(f)$, the set of rearrangements of f. Then $\phi\circ\zeta$ is the unique maximizer of the linear functional $\int_\Omega h\,\zeta\,dx$, relative to $h\in\overline{\mathcal{R}}(f)$, where $\overline{\mathcal{R}}(f)$ denotes the weak closure of $\mathcal{R}(f)$ in $L^q(\Omega)$.*

Lemma 2.3. *Suppose $g\in L^p_+(\Omega)$. Then there exists $\hat f\in\mathcal{R}(f_0)$ which maximizes the linear functional $\int_\Omega hg\,dx$, relative to $h\in\overline{\mathcal{R}}(f_0)$; that is,*

$$\int_\Omega \hat f g\,dx = \sup_{h\in\overline{\mathcal{R}}(f_0)}\int_\Omega hg\,dx.$$

For the proof of Lemma 2.1, see [5, Lemma 2.4], and for Lemmas 2.2 and 2.3, see [4, Lemma 2.4].

Next we recall a well known rearrangement inequality. If $u \in W_0^{1,p}(\Omega)$ is non-negative and if u^* denotes the Schwarz rearrangement of u, then $u^* \in W_0^{1,p}(\Omega)$ and the inequality

$$\int_B |\nabla u^*|^p \, dx \le \int_\Omega |\nabla u|^p \, dx \tag{2.2}$$

holds. The case of equality in (2.2) has been considered in [2]. The following result can be deduced from Lemma 3.2, Theorem 1.1, and Lemma 2.3(v), in [2].

Theorem 2.4. *Let $u \in W_0^{1,p}(\Omega)$ be non-negative, and suppose equality holds in (2.2). Then $u^{-1}(\alpha,\infty)$ is a translate of ${u^*}^{-1}(\alpha,\infty)$, for every $\alpha \in [0,M]$, where M is the essential supremum of u over Ω, modulo sets of measure zero. Moreover, if*

$$\mathcal{L}_N(\{x \in \Omega | \ \nabla u(x) = 0, \ \ 0 < u(x) < M\}) = 0, \tag{2.3}$$

then $u(x) = u(x - x_0)$, for some $x_0 \in \mathbb{R}^N$; that is, u is a translation of u^*.

3. Main Results

We begin with the following result.

Theorem 3.1. *The maximization problem (1.2) is solvable; that is, there exists $\hat{f} \in \mathcal{R}(f_0)$ such that*

$$\int_\Omega \hat{f}\hat{u} \, dx = \sup_{f \in \mathcal{R}(f_0)} \int_\Omega f u_f \, dx,$$

where $\hat{u} = u_{\hat{f}}$.

Proof. Let

$$I = \sup_{f \in \mathcal{R}(f_0)} \int_\Omega f u_f \, dx.$$

We first show that I is finite. Consider $f \in \mathcal{R}(f_0)$; then from (2.1) followed by Hölder's inequality we find

$$\int_\Omega |\nabla u_f|^p \, dx = \int_\Omega f u_f \, dx \le \|f\|_q \, \|u_f\|_p. \tag{3.1}$$

Since $\|f\|_q = \|f_0\|_q$, it follows from (3.1) and the Poincaré inequality that I is finite.

Let $\{f_i\}$ be a maximizing sequence and let $u_i = u_{f_i}$. From (3.1) it is clear that $\{u_i\}$ is bounded in $W^{1,p}(\Omega)$, hence it has a subsequence (still denoted $\{u_i\}$) that converges weakly to $u \in W_0^{1,p}(\Omega)$. We also infer that $\{u_i\}$ converges strongly to u in $L^p(\Omega)$. On the other hand, since $\{f_i\}$ is bounded in $L^q(\Omega)$, it must contain a subsequence (still denoted $\{f_i\}$) converging weakly to $\eta \in L^q(\Omega)$. Note that $\eta \in \overline{\mathcal{R}}$, the weak closure of $\mathcal{R}$ in $L^q(\Omega)$. Thus, using the weak lower semi-continuity of the $W_0^{1,p}(\Omega)$-norm and (2.1) we obtain

$$\begin{aligned} I = \lim_{i\to\infty} \int_\Omega f_i u_i \, dx &= \frac{1}{p-1} \lim_{i\to\infty} \int_\Omega \big(p f_i u_i - |\nabla u_i|^p\big) \, dx \\ &\le \frac{1}{p-1} \int_\Omega \big(p\eta \, u - |\nabla u|^p\big) \, dx. \end{aligned} \tag{3.2}$$

Note that from Lemma 2.3 we infer the existence of $\hat{f} \in \mathcal{R}(f_0)$ that maximizes the linear functional $\int_\Omega hu\,dx$, relative to $h \in \overline{\mathcal{R}}(f_0)$. As a consequence we obtain

$$\int_\Omega \eta\, u\, dx \le \int_\Omega \hat{f}\, u\, dx. \tag{3.3}$$

Applying (2.1), (3.2) and (3.3) we find

$$I \le \frac{1}{p-1}\int_\Omega \big(p\hat{f}u - |\nabla u|^p\big)\,dx \le \frac{1}{p-1}\int_\Omega \big(p\hat{f}\hat{u} - |\nabla \hat{u}|^p\big)\,dx = \int_\Omega \hat{f}\hat{u}\,dx \le I.$$

Recall that $\hat{u} = u_{\hat{f}}$. Thus $\hat{f}$ is a solution to (1.2), as desired. $\square$

The next issue addressed is the so called Euler-Lagrange equation for solutions of (1.2).

Theorem 3.2. *Suppose $\hat{f}$ is a solution of* (1.2) *with f_0 non negative. Then there exists an increasing function ϕ such that*

$$\hat{f} = \phi \circ \hat{u} \tag{3.4}$$

almost everywhere in Ω, where $\hat{u} = u_{\hat{f}}$. Equation (3.4) *is referred to as the Euler-Lagrange equation for $\hat{f}$.*

To prove Theorem 3.2 we need some preparations. Let us begin with the following result.

Lemma 3.3. *Suppose $\hat{f}$ and $\hat{u}$ are as in Theorem 3.2. Then $\hat{f}$ maximizes the linear functional $\int_\Omega h\hat{u}\,dx$, relative to $h \in \mathcal{R}(f_0)$.*

Proof. Since $\hat{f}$ is a solution of (1.2), the following inequality holds for every $f \in \mathcal{R}(f_0)$

$$\int_\Omega \hat{f}\hat{u}\,dx \ge \int_\Omega f u_f\,dx. \tag{3.5}$$

Next, applying (2.1) to the right hand side of (3.5) yields

$$\int_\Omega \hat{f}\hat{u}\,dx \ge \frac{1}{p-1}\int_\Omega \big(pf\hat{u} - |\nabla \hat{u}|^p\big)\,dx, \tag{3.6}$$

for every $f \in \mathcal{R}(f_0)$. We also have

$$\int_\Omega \hat{f}\hat{u}\,dx = \frac{1}{p-1}\int_\Omega \big(p\hat{f}\hat{u} - |\nabla \hat{u}|^p\big)\,dx. \tag{3.7}$$

Combination of (3.6) and (3.7) implies

$$\int_\Omega \hat{f}\hat{u}\,dx \ge \int_\Omega f\hat{u}\,dx,$$

for every $f \in \mathcal{R}(f_0)$. This completes the proof. $\square$

In what follows we shall write $\inf_{x\in S} f(x)$ ($\sup_{x\in S} f(x)$) for the essential inferior (superior) of $f(x)$ in S.

Lemma 3.4. *Let $\hat{f}$ and $\hat{u}$ be as in Theorem 3.2, and let $S(\hat{f}) = \{x \in \Omega |\ \hat{f}(x) > 0\}$. Set*

$$\gamma = \inf_{x\in S(\hat{f})} \hat{u}(x), \quad \delta = \sup_{x\in \Omega\setminus S(\hat{f})} \hat{u}(x).$$

Then $\gamma \ge \delta$.

Proof. To derive a contradiction assume $\gamma < \delta$. Let us fix $\gamma < \xi_1 < \xi_2 < \delta$. Since $\xi_1 > \gamma$, there exists a set $A \subset S(\hat{f})$, with positive measure, such that $\hat{u} \le \xi_1$ on A. Similarly, $\xi_2 < \delta$ implies that there exists a set $B \subset \Omega \setminus S(\hat{f})$, with positive measure, such that $\hat{u} \ge \xi_2$ on B. Without loss of generality we may assume that $\mathcal{L}_N(A) = \mathcal{L}_N(B) > 0$ (otherwise we consider suitable subsets of A and B having the same measures). Next, consider a measure preserving map $T : A \to B$, see [12]. Using T we define a particular rearrangement of $\hat{f}$, denoted $\overline{f}$.

$$\overline{f}(x) = \begin{cases} \hat{f}(Tx), & x \in A \\ \hat{f}(T^{-1}x) & x \in B \\ \hat{f}(x) & \Omega \setminus (A \cup B). \end{cases}$$

Thus

$$\begin{aligned} \int_\Omega \overline{f}\hat{u}\,dx - \int_\Omega \hat{f}\hat{u}\,dx &= \int_{A\cup B} \overline{f}\hat{u}\,dx - \int_{A\cup B} \hat{f}\hat{u}\,dx \\ &= \int_B \overline{f}\hat{u}\,dx - \int_A \hat{f}\hat{u}\,dx \\ &\ge \xi_2 \int_B \overline{f}dx - \xi_1 \int_A \hat{f}dx \\ &= (\xi_2 - \xi_1) \int_A \hat{f}\,dx > 0. \end{aligned}$$

Therefore, $\int_\Omega \overline{f}\hat{u}\,dx > \int_\Omega \hat{f}\hat{u}\,dx$, which contradicts the maximality of $\hat{f}$ (see Lemma 3.3). $\square$

Proof of theorem 3.2. Notice that from (1.1) and [8, Lemma 7.7], it is clear that the level sets of $\hat{u}$, restricted to $S(\hat{f})$, have measure zero. Therefore applying Lemma 2.1, we infer existence of an increasing function $\tilde{\phi}$ such that $\tilde{\phi} \circ \hat{u}$ is a rearrangement of $\hat{f}$ relative to the set $S(\hat{f})$. Equivalently, $\tilde{\phi} \circ \hat{u}$, restricted to $S(\hat{f})$, is a rearrangement of $\hat{f}^\Delta$, restricted to the interval $(0, s)$, where $s = \mathcal{L}_N(S(\hat{f}))$. Now, define

$$\phi(t) = \begin{cases} \tilde{\phi}(t) & t \ge \gamma \\ 0 & t < \gamma, \end{cases}$$

where $\gamma = \inf_{S(\hat{f})} \hat{u}(x)$. Note that, since $\tilde{\phi}$ is non-negative, ϕ is an increasing function. Moreover, $\phi \circ \hat{u}$ is a rearrangement of $\hat{f}^\Delta$, on $(0, \omega)$, where $\omega = \mathcal{L}_N(\Omega)$. Thus $\phi \circ \hat{u} \in \mathcal{R}(f_0)$, hence we can apply Lemma 2.2 to deduce that $\phi \circ \hat{u}$ is the unique maximizer of the linear functional $\int_\Omega h\hat{u}\,dx$, relative to $h \in \mathcal{R}(f_0)$. This obviously implies $\hat{f} = \phi \circ \hat{u}$, almost everywhere in Ω. $\square$

Remark. The function ϕ in above can be extended to all of $\mathbb{R}$. Thus from (3.4) we infer $S(\hat{f}) = \hat{u}^{-1}\big(\phi^{-1}(0, \infty)\big)$. Since ϕ is increasing the set $\phi^{-1}(0, \infty)$ is either the interval (γ, ∞) or $[\gamma, \infty)$. In both cases, since the level sets of $\hat{u}$ on $S(\hat{f})$ have measure zero, we can write $S(\hat{f}) = \{\hat{u} > \gamma\}$. If we assume $f_0 \in L^\infty(\Omega)$ then we have the continuity of the solution $\hat{u}$ (see [15]). In this situation the boundary of $S(\hat{f})$, denoted $\partial S(\hat{f})$, satisfies

$$\partial S(\hat{f}) \subset \{\hat{u} = \gamma\}, \tag{3.8}$$

thanks to the continuity of $\hat{u}$. Note that $\mathcal{L}_N(S(f_0)) = \mathcal{L}_N(S(\hat{f}))$. Therefore, if $\mathcal{L}_N(S(f_0)) < \mathcal{L}_N(\Omega)$, it follows that γ is strictly positive.

An example of interest is the following.

Example. Suppose $f_0 = \chi_{E_0}$, where χ_{E_0} is the characteristic function of the measurable set $E_0 \subset \Omega$, and let $\mathcal{L}_N(E_0) < \mathcal{L}_N(\Omega)$. Denoting a solution of (1.2) by $\hat{f}$, it is clear that $\hat{f} = \chi_{\hat{E}}$, for some measurable set $\hat{E} \subset \Omega$, having the same measure as E_0. From the last Remark we infer that $\hat{u} = u_{\hat{f}}$ is constant on $\partial\hat{E}$. Also, $\partial\hat{E}$ does not intersect $\partial\Omega$. So physically speaking, in order to maximize the average deformation of the nonlinear elastic membrane under uniform loads (given by the appropriate rearrangement class) it is best to place the load away from the boundary (independently of the geometry of the membrane). We will return to this example in the last section.

We now address the question of uniqueness in a ball.

Theorem 3.5. *Suppose Ω is a ball centered at the origin. Then the maximization problem* (1.2) *with f_0 non negative and essentially bounded has a unique solution, namely, f_0^*.*

For the rest of this section Ω is always a ball. We need the following result.

Lemma 3.6. *If $f \geq 0$, then*
$$\frac{1}{p}\int_\Omega |\nabla u_f^*|^p\,dx + \frac{1}{q}\int_\Omega f^* u_{f^*}\,dx \geq \int_\Omega f^* u_f^*\,dx. \tag{3.9}$$

Proof. From the variational characterization of u_{f^*} the following inequality is clear
$$\begin{aligned}\frac{1}{p}\int_\Omega |\nabla u_f^*|^p\,dx + \frac{1}{q}\int_\Omega f^* u_{f^*}\,dx &\geq \frac{1}{p}\int_\Omega |\nabla u_{f^*}|^p\,dx - \int_\Omega f^* u_{f^*}\,dx \\ &\quad + \frac{1}{q}\int_\Omega f^* u_{f^*}\,dx + \int_\Omega f^* u_f^*\,dx.\end{aligned}$$
Since the first three terms on the right hand side of the above inequality drop out thanks to (2.1), we obtain (3.9). $\square$

Proof of Theorem 3.5. Suppose f is a solution of (1.2). Then, from Lemma 3.6 we have
$$\frac{1}{p}\int_\Omega |\nabla u_f^*|^p\,dx + \frac{1}{q}\int_\Omega f^* u_{f^*}\,dx \geq \int_\Omega f^* u_f^*\,dx.$$
Applying the Hardy-Littlewood inequality $\int_\Omega f^* u_f^*\,dx \geq \int_\Omega f u_f\,dx$ to the right hand side of the above inequality we find
$$\frac{1}{p}\int_\Omega |\nabla u_f^*|^p\,dx + \frac{1}{q}\int_\Omega f^* u_{f^*}\,dx \geq \int_\Omega f u_f\,dx = \frac{1}{p}\int_\Omega f u_f\,dx + \frac{1}{q}\int_\Omega f u_f\,dx. \tag{3.10}$$
Since f is a solution of (1.2) we have $\int_\Omega f u_f\,dx \geq \int_\Omega f^* u_{f^*}\,dx$. Moreover, an application of (2.1) yields $\int_\Omega f u_f\,dx = \int_\Omega |\nabla u_f|^p\,dx$. Therefore, (3.10) implies
$$\int_\Omega |\nabla u_f^*|^p\,dx \geq \int_\Omega |\nabla u_f|^p\,dx. \tag{3.11}$$
Hence, from (2.2), we obtain equality in (3.11).

Next we show that $u_f = u_f^*$. According to Theorem 2.4, we only need to show that (2.3) holds. Let us consider $x \in \Omega$ such that $0 < u(x) < \max_\Omega u(x)$, and set $S = \{z \in \Omega :\ u(z) \geq u(x)\}$, which is a closed ball by Theorem 2.4. If we define

$v(z) = u(z) - u(x)$, we have $-\Delta_p v(z) = -\Delta_p u(z) \geq 0$, since f is non-negative. Since v vanishes on ∂S, by the strong maximum principle [16, Theorem 5] we have $v > 0$ in S. Therefore, $u(z) > u(x)$ for all $z \in S$. Hence x must be a boundary point of S. So, by the Hopf boundary lemma [16, Theorem 5] we derive $\frac{\partial u}{\partial \nu}(x) = \frac{\partial v}{\partial \nu}(x) \neq 0$, where ν stands for the outward unit normal to ∂S at x. Thus (2.3) holds, as desired. Finally, from (3.4), we deduce that f coincides with its Schwarz rearrangement, so $f = f^* = f_0^*$. This completes the proof of the theorem. □

4. Domain derivative

This section is devoted to the example mentioned earlier. We have seen that if $\hat{f} = \chi_{\hat{D}}$ is a solution of (1.2), then $\hat{u} = u_{\hat{f}}$ is constant on the free boundary $\partial\hat{D}$. A natural question arises: Does the same result hold if $\chi_{\hat{D}}$ is *any* critical point of the functional $\int_\Omega f u_f$, relative to the class of rearrangements of $\chi_{\hat{D}}$? We give an affirmative answer to this question under some restrictions on $\hat{D}$. In order to put things in the right context we need to introduce the notion of domain derivative [13], [14], that is specialized to our situation.

Let D be an open smooth subset of Ω with $\operatorname{dist}(D, \partial\Omega) > 0$. Let V be a regular (smooth) vector field with support in Ω. Define $D^t = (Id + tV)(D)$, with small $t \in R^+$ such that $D^t \subset \Omega$. Here Id denotes the identity map. Note that for small t, the operator $Id + tV$ is a diffeomorphism. In particular, D^t is an open and smooth set. If $D\Delta D^t$ denotes the familiar symmetric difference of D and D^t, then

$$\mathcal{L}_N(D\Delta D^t) \leq ct, \tag{4.1}$$

where c is a positive constant independent of t. As a consequence of (4.1), the function $\chi_{D^t} - \chi_D$ tends to zero in $L^q(\Omega)$ (for any $q > 1$) as t tends to zero. Let us define

$$I(D) = \int_D u\,dx,$$

where u satisfies

$$-\Delta_p u = \chi_D \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

Also

$$I(D^t) = \int_{D^t} u^t\,dx,$$

where u^t satisfies

$$-\Delta_p u^t = \chi_{D^t} \quad \text{in } \Omega, \quad u^t = 0 \quad \text{on } \partial\Omega. \tag{4.3}$$

For the sake of the following definition we introduce $\mathcal{V}$ to be the set of all regular vector fields with support in Ω.

Definition. We say that D (as above) is a critical point of the functional I provided

$$dI(D; V) = c\,dVol(D; V),$$

for some constant c and every $V \in \mathcal{V}$. Here

$$\mathrm{dI}(D; V) := \lim_{t \to 0^+} \frac{I(D^t) - I(D)}{t},$$

$$\mathrm{dVol}(D; V) := \lim_{t \to 0^+} \frac{\mathcal{L}_N(D^t) - \mathcal{L}_N(D)}{t}.$$

Of course, if we consider measure preserving vector fields V then $dVol(D; V) = 0$. We are now ready to state the main result of this section.

Theorem 4.1. *Suppose D is an open smooth subset of Ω. Suppose $\operatorname{dist}(D, \partial\Omega) > 0$, and D is a critical point of I, relative to Γ. Then u, the solution of (4.2), is constant on ∂D.*

Lemma 4.2. *Let u and u^t be the solutions of (4.2) and (4.3), respectively. Then $u^t \to u$, in $W_0^{1,p}(\Omega)$, as $t \to 0^+$.*

Proof. Let us recall the well known inequality (see, for example, [15])

$$C(|X|^{p-2}X - |Y|^{p-2}Y, X - Y) \geq \begin{cases} |X-Y|^p, & p \geq 2 \\ \frac{|X-Y|^2}{(|X|+|Y|)^{2-p}}, & p \leq 2, \end{cases} \tag{4.4}$$

where X and Y are vectors in $\mathbb{R}^n$, $|X|$ (similarly $|Y|$) denotes the Euclidean length of X, C is a positive constant, and $(\cdot, \cdot)$ stands for the usual dot product in $\mathbb{R}^n$. Let us consider two cases
Case 1: $p \geq 2$: Using (4.4) we have

$$\|\nabla u^t - \nabla u\|_p^p \leq C \int_\Omega (|\nabla u^t|^{p-2}\nabla u^t - |\nabla u|^{p-2}\nabla u) \cdot (\nabla u^t - \nabla u)\, dx.$$

Using (4.2) and (4.3) we can rewrite the above inequality as

$$\|\nabla u^t - \nabla u\|_p^p \leq C \int_\Omega (\chi_{D^t} - \chi_D)(u^t - u)\, dx.$$

So by applying the Hölder's inequality followed by the Poincaré inequality we obtain

$$\|\nabla u^t - \nabla u\|_p^{p-1} \leq \tilde{C}\Big(\int_\Omega |\chi_{D^t} - \chi_D|^q\, dx\Big)^{1/q}.$$

¿From the above inequality, the assertion of the lemma follows.
Case 2: $p \leq 2$: Let us begin with the following observation

$$\begin{aligned} \|\nabla u^t - \nabla u\|_p^p &= \int_\Omega \frac{|\nabla u^t - \nabla u|^p}{(|\nabla u^t| + |\nabla u|)^{\frac{p(2-p)}{2}}}\ (|\nabla u^t| + |\nabla u|)^{\frac{p(2-p)}{2}}\, dx \\ &\leq \Big(\int_\Omega \frac{|\nabla u^t - \nabla u|^2}{(|\nabla u^t| + |\nabla u|)^{(2-p)}}\, dx\Big)^{p/2} \Big(\int_\Omega (|\nabla u^t| + |\nabla u|)^p\, dx\Big)^{(2-p)/2}, \end{aligned}$$

which follows from the Hölder inequality, since $2/p > 1$. Note that (u^t) is bounded in $W_0^{1,p}(\Omega)$. Thus from the above inequality we find

$$\|\nabla u^t - \nabla u\|_p^p \leq C \left(\int_\Omega \frac{|\nabla u^t - \nabla u|^2}{(|\nabla u^t| + |\nabla u|)^{(2-p)}}\, dx\right)^{p/2}. \tag{4.5}$$

Now applying (4.4) to the right hand side of (4.5), the assertion of the lemma can be confirmed using similar arguments as in the ending part of Case 1. □

Proof of Theorem 4.1. Let us begin with the identity

$$I(D^t) - I(D) = \int_{D^t} (u^t - u)\, dx + \int_{D^t} u\, dx - \int_D u\, dx. \tag{4.6}$$

Following [13], we define

$$u'(x) = \lim_{t \to 0^+} \frac{u^t(x) - u(x)}{t}. \tag{4.7}$$

Moreover, we have

$$\int_{D^t} u\,dx - \int_D u\,dx = \int_D \Big[u(x+tV)\big|\det\big(\delta_{ij} + t\frac{\partial V_i}{\partial x_j}\big)\big| - u(x)\Big]\,dx.$$

Since t is small, we find

$$\begin{aligned}\int_{D^t} u\,dx - \int_D u\,dx &= \int_D [(u(x) + t\nabla u\cdot V + o(t))(1 + t\nabla\cdot V + o(t)) - u(x)]\,dx\\ &= t\int_D (\nabla u\cdot V + u\nabla\cdot V)\,dx + o(t)\\ &= t\int_D \nabla\cdot(uV)\,dx + o(t)\\ &= t\int_{\partial D} u(V\cdot\nu)\,d\sigma + o(t),\end{aligned} \tag{4.8}$$

where ν is the outward unit normal to ∂D and $d\sigma$ is the surface measure. Inserting (4.8) and (4.7) into (4.6) yields

$$\lim_{t\to 0^+}\frac{I(D^t) - I(D)}{t} = \int_D u'\,dx + \int_{\partial D} u(V\cdot\nu)\,d\sigma. \tag{4.9}$$

Now multiply (4.2) by u^t, (4.3) by u, subtract the new equations, and finally integrate over Ω. We find

$$\int_\Omega \frac{|\nabla u^t|^{(p-2)} - |\nabla u|^{(p-2)}}{t}\,\nabla u^t\cdot\nabla u\,dx = \frac{1}{t}\Big[\int_{D^t} u\,dx - \int_D u\,dx\Big] + \int_D \frac{u-u^t}{t}\,dx. \tag{4.10}$$

Since u^t tends to u in $W_0^{1,p}(\Omega)$ as t tends to zero, and also

$$\frac{d}{dt}|\nabla u^t|^{(p-2)} = (p-2)|\nabla u|^{(p-4)}\nabla u\cdot\nabla u', \quad \text{at } t=0,$$

taking the limit of (4.10), when t tends to zero, we find

$$(p-2)\int_\Omega |\nabla u|^{(p-2)}\nabla u\cdot\nabla u'\,dx = \int_{\partial D} u(V\cdot\nu)\,d\sigma - \int_D u'\,dx. \tag{4.11}$$

If we multiply (4.2) by u', and integrate we find (recall that the support of V is in Ω)

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u'\,dx = \int_D u'\,dx.$$

Inserting this equation in (4.11) we find

$$\int_D u'\,dx = \frac{1}{p-1}\int_{\partial D} u(V\cdot\nu)\,d\sigma.$$

Finally inserting the latter estimate into (4.9) yields

$$dI(D,V) = q\int_{\partial D} u(V\cdot\nu)\,d\sigma.$$

Recalling the formula for the derivative of the volume, that is,

$$\mathrm{dVol}(D,V) = \int_{\partial D} (V\cdot\nu)\,d\sigma,$$

and the fact that D is a critical point of I, we derive

$$\mathrm{dI}(D,V) = c\,\mathrm{dVol}(D,V) \Leftrightarrow u(x) = \text{constant}, \quad \text{on } \partial D.$$

This obviously completes the proof of the theorem. □

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