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TRAVELLING WAVE SOLUTIONS IN DELAYED CELLULAR NEURAL NETWORKS WITH NONLINEAR OUTPUT

XIUXIANG LIU, PEIXUAN WENG, ZHITING XU

Abstract. This paper concerns the existence of travelling wave solutions of delayed cellular neural networks distributed in a 1-dimensional lattice with nonlinear output. Under appropriate assumptions, we prove the existence of travelling waves and extend some known results.

1. Introduction

Cellular neural networks (CNN) system were first proposed by Chua and Yang [1, 2] as an achievable alternative to fully-connected neural networks in electric circuit systems, so it is also called CY-CNN system. The structure of CNN is similar to the cellular automata that any cell in a CNN is connected only to its neighboring cells. In the recent years, the CNN approach has been applied to a broad scope of problems arising from, for example, image and video signal processing, robotic and biological visions etc. We refer the readers to [3, 4, 5] for some practical applications. An 1-dimensional CNN without inputs is given by

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + z + \alpha f(x_i(t)) + \beta f(x_{i+1}(t)), \tag{1.1}$$

for i in a 1-dimensional lattice $\mathbb{Z}$, where the positive coefficients α, β of the signal output function f constitute the so-called space-invariant template that measures the synaptic weights of self-feedback and neighborhood interaction. The quantity z is called a threshold or bias term and is related to the independent voltage sources in electric circuits. Due to the finite switching speed and finite velocity of signal transmission, the distributed delays may exist for CNN systems. One of the models with delay is proposed in [6] with $z = 0$ by

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + \alpha f(x_i(t)) + \beta \int_0^\tau K(u) f(x_{i+1}(t-u))\mathrm{d}u, \tag{1.2}$$

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where $\tau > 0$ is a constant, $K : [0, \tau] \to [0, +\infty)$ is a piece-wise continuous function satisfying

$$\int_0^\tau K(u)\mathrm{d}u = 1.$$

It is assumed that the self-feedback interaction is instantaneous and there exists delay in neighborhood interaction. A typical output function f in (1.1) or (1.2) is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ x & \text{if } |x| \leq 1, \\ -1 & \text{if } x \leq -1; \end{cases} \tag{1.3}$$

see for example [1, 2, 3, 4, 5, 6].

A travelling wave solution of (1.2) is defined as a solution of (1.2) with

$$x_i(t) = \phi(i - ct) := \phi(s), \quad \text{for all } i \in \mathbb{Z} \text{ and } t \in \mathbb{R}, \tag{1.4}$$

where $c \in \mathbb{R}$ is the wave speed. The profile equation for $\phi(s)$ can be written as

$$-c\phi'(s) = -\phi(s) + \alpha f(\phi(s)) + \beta \int_0^\tau K(u) f(\phi(s + 1 + u))\mathrm{d}u. \tag{1.5}$$

Assuming that the synaptic connection is sufficiently large so that

$$\alpha + \beta > 1, \tag{1.6}$$

then there are three equilibria of (1.5):

$$x^- = -(\alpha + \beta), \quad x^0 = 0, \quad x^+ = \alpha + \beta. \tag{1.7}$$

Recently, Weng and Wu [7] studied the deformation and existence of travelling wave solutions for (1.2) with (1.3), which satisfy different types of asymptotic boundary conditions. For instance one type of conditions are

$$\lim_{s \to -\infty} \phi(s) = x^0, \quad \lim_{s \to \infty} \phi(s) = x^+. \tag{1.8}$$

Ling [6] discussed the deformation and existence of travelling wave solutions of (1.2) with a nonlinear output function f on $[-1, 1]$, that is

$$f(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ \sin(\frac{\pi}{2} x) & \text{if } |x| \leq 1, \\ -1 & \text{if } x \leq -1. \end{cases} \tag{1.9}$$

This paper is a continuation of the work in [6, 7]. We consider the existence of travelling wave solutions of (1.2) with a more general output function f under the following hypotheses:

(H1) f is a continuous odd function on $(-\infty, +\infty)$ and satisfies:
  (1) $f(x) = 1$ for $x \geq 1$;
  (2) $f(0) = 0$ and f is differentiable at $x = 0$, $\mu = f'(0) \geq 1$;
  (3) f is non-decreasing and $f(x) \leq f'(0)x$ on $[0, 1]$.

Under the above hypotheses and (1.6), we see that x^-, x^0 and x^+ are still equilibria of (1.5), furthermore,

$$f'(0)x - f(x) = o(x) \quad \text{as } x \to 0. \tag{1.10}$$

In fact, noting that $f(0) = 0$, one has

$$\lim_{x\to 0} \frac{f'(0)x - f(x)}{x} = f'(0) - \lim_{x\to 0} \frac{f(x) - f(0)}{x} = 0,$$

hence $f'(0)x - f(x) = o(x)$.

2. Existence of Monotone Travelling Waves

In this section, we study the existence of travelling wave solutions of (1.2) under the assumption (H1). First of all, we define the characteristic function of (1.5) at $x^0 = 0$ by

$$\Delta(\lambda, c, x^0) = -c\lambda + 1 - \mu\alpha - \mu\beta \int_0^\tau K(u)\mathrm{e}^{\lambda(1+cu)}\mathrm{d}u. \tag{2.1}$$

The characteristic function (2.1) plays crucial roles in our study. The following lemma is needed.

Lemma 2.1. *Assume that $\alpha \geq 1$. There exist exactly a pair of numbers (c_*, λ_*) with $c_* < 0$, $\lambda_* = \lambda(c_*) > 0$ such that*

(i) $\Delta(\lambda_*, c_*, x^0) = 0$, $\frac{\partial}{\partial\lambda}\Delta(\lambda_*, c_*, x^0) = 0$;
(ii) *for* $c_* < c \leq 0$, $\Delta(\lambda, c, x^0) < 0$ *for any* $\lambda \in \mathbb{R}$;
(iii) *for any* $c < c_*$, *there exists* $\lambda_1 > 0$, $\varepsilon_1 > 0$ *such that* $\Delta(\lambda_1, c, x^0) = 0$, *and for any small* $\varepsilon \in (0, \varepsilon_1)$ *one has* $\Delta(\lambda_1 + \varepsilon, c, x^0) > 0$.

Proof. A simple calculation leads to

$$\begin{aligned}
\frac{\partial}{\partial c}\Delta(\lambda, c, x^0) &= -\lambda\Big(1 + \mu\beta \int_0^\tau K(u)u\mathrm{e}^{\lambda(1+cu)}\mathrm{d}u\Big),\\
\frac{\partial}{\partial\lambda}\Delta(\lambda, c, x^0) &= -c - \mu\beta \int_0^\tau K(u)(1+cu)\mathrm{e}^{\lambda(1+cu)}\mathrm{d}u,\\
\frac{\partial^2}{\partial\lambda^2}\Delta(\lambda, c, x^0) &= -\mu\beta \int_0^\tau K(u)(1+cu)^2\mathrm{e}^{\lambda(1+cu)}\mathrm{d}u < 0\,.
\end{aligned}$$

Note that,

$$\begin{gathered}
\Delta(0, c, x^0) = 1 - \mu(\alpha + \beta) < 0 \quad \text{for any } c \in \mathbb{R},\\
\lim_{c\to-\infty} \Delta(\lambda, c, x^0) = +\infty \quad \text{for any } \lambda > 0,\\
\Delta(\lambda, 0, x^0) = 1 - \mu\alpha - \mu\beta\mathrm{e}^{\lambda} < 0 \quad \text{for any } \lambda \in \mathbb{R},
\end{gathered}$$

and that $\Delta(\lambda, c, x^0)$ is a concave function of $\lambda \in \mathbb{R}$ for any given $c \in \mathbb{R}$. Therefore, there exist exactly a pair of numbers (c_*, λ_*) with $c_* < 0, \lambda_* = \lambda(c_*) > 0$ satisfying (i) and (iii).

Note that $\Delta(\lambda, 0, x^0) < 0$ for any $\lambda \in \mathbb{R}$. Furthermore, for any given $\lambda < 0$, we have $\frac{\partial\Delta}{\partial c} > 0$. Therefore, $\Delta(\lambda, c, x^0) = 0$ has no real roots for any $c \in (c_*, 0]$. This completes the proof. $\square$

We now consider the existence of monotone travelling waves of (1.2) for $c < c_*$. Our approach is based on monotone iteration, coupled with the concept of upper and lower solutions introduced below.

Definition 2.2. A function $V : \mathbb{R} \to \mathbb{R}$ is called an upper solution of (1.5) if it is differentiable almost everywhere (a.e.) and satisfies the inequality

$$-cV'(s) \geq -V(s) + \alpha f(V(s)) + \beta \int_0^\tau K(u) f(V(s+1+cu))\mathrm{d}u.$$

Similarly, a function $v : \mathbb{R} \to \mathbb{R}$ is called a lower solution of (1.5) if it is differentiable almost everywhere and satisfies the inequality

$$-cv'(s) \leq -v(s) + \alpha f(v(s)) + \beta \int_0^\tau K(u) f(v(s+1+cu))\mathrm{d}u.$$

For any $c < c_*$, we define following two functions:

$$V(s) = \begin{cases} x^+ & s \geq 0, \\ x^+ \mathrm{e}^{\lambda_1 s} & s \leq 0, \end{cases}$$

$$v(s) = \begin{cases} 0 & s \geq 0, \\ \eta(1 - \mathrm{e}^{\varepsilon s})\mathrm{e}^{\lambda_1 s} & s \leq 0, \end{cases}$$

where x^+ is defined in (1.7), λ_1, ε are as in Lemma 2.1 and $\eta \in (0, 1)$ is chosen small so that $V(s) \geq v(s)$ and to be decided in the following such that $v(s)$ is a lower solution. Clearly, we have $0 \leq v(s) \leq V(s) \leq x^+$ and $v(s) \not\equiv 0$ for $s \in \mathbb{R}$.

Lemma 2.3. *For any $c < c_* < 0$, V is an upper solution and v is a lower solution of* (1.5).

Proof. If $s \geq 0$, $V(s) = x^+$. Note that $f(u) \leq 1$ for any $u \in \mathbb{R}$, then we have

$$cV'(s) - V(s) + \alpha f(V(s)) + \beta \int_0^\tau K(u) f(V(s+1+cu))\mathrm{d}u \leq 0 - x^+ + \alpha + \beta = 0.$$

If $s \leq 0$, $V(s) = x^+\mathrm{e}^{\lambda_1 s}$. Note that $V(s) \leq x^+\mathrm{e}^{\lambda_1 s}$ for $s \in \mathbb{R}$. Therefore, if $x^+\mathrm{e}^{\lambda_1 s} > 1$, one has $f(V(s)) = 1 < x^+\mathrm{e}^{\lambda_1 s} \leq \mu x^+\mathrm{e}^{\lambda_1 s}$; if $x^+\mathrm{e}^{\lambda_1 s} \leq 1$, one also has $f(V(s)) \leq \mu V(s) = \mu x^+\mathrm{e}^{\lambda_1 s}$ from the assumption (3) in (H1). According to Lemma 2.1, we have

$$\begin{aligned} &cV'(s) - V(s) + \alpha f(V(s)) + \beta \int_0^\tau K(u) f(V(s+1+cu))\mathrm{d}u \\ &\leq c\lambda_1 x^+\mathrm{e}^{\lambda_1 s} - x^+\mathrm{e}^{\lambda_1 s} + \mu\alpha x^+\mathrm{e}^{\lambda_1 s} + \mu\beta \int_0^\tau K(u) x^+\mathrm{e}^{\lambda_1(s+1+cu)}\mathrm{d}u \\ &= -x^+\mathrm{e}^{\lambda_1 s}\Delta(\lambda_1, c, x^0) = 0. \end{aligned}$$

So $V(s)$ is an upper solution of (1.5).

Next we show that v is a lower solution of (1.5). That is,

$$cv'(s) - v(s) + \alpha f(v(s)) + \beta \int_0^\tau K(u) f(v(s+1+cu))\mathrm{d}u \geq 0. \tag{2.2}$$

If $s \geq 0$, it is obviously that (2.2) holds because

$$\begin{aligned} &cv'(s) - v(s) + \alpha f(v(s)) + \beta \int_0^\tau K(u) f(v(s+1+cu))\mathrm{d}u \\ =&0 - 0 + 0 + \begin{cases} \beta\int_{-\frac{s+1}{c}}^\tau K(u)\eta(1 - \mathrm{e}^{\varepsilon(s+1+cu)})\mathrm{e}^{\lambda_1(s+1+cu)}\mathrm{d}u & \text{if } -\frac{s+1}{c} \in [0, \tau) \\ 0 & \text{if } -\frac{s+1}{c} \geq \tau \end{cases} \\ \geq&0. \end{aligned}$$

If $s<0$, we will consider the following three cases, namely, $-(c\tau+1)<s<0$, $-1<s\le-(c\tau+1)$, and $s\le-1$.

Case 1. $-(c\tau+1)<s<0$. In this case, for any $u\in[0,\tau]$, one has $s+1+cu>0$ which implies $v(s+1+cu)=0$. Moreover, if $s<0, 0<\eta<1$, one has $0<\eta\mathrm{e}^{\lambda_1 s}(1-\mathrm{e}^{\varepsilon s})<1$. So we can obtain

$$\begin{aligned}
&cv'(s)-v(s)+\alpha f(v(s))+\beta\int_0^\tau K(u)f(v(s+1+cu))\mathrm{d}u\\
&=-\eta\mathrm{e}^{\lambda_1 s}(-c\lambda_1+1)+\eta\mathrm{e}^{(\lambda_1+\varepsilon)s}\,(-c(\lambda_1+\varepsilon)+1)\\
&\quad+\alpha f(\eta(1-\mathrm{e}^{\varepsilon s})\mathrm{e}^{\lambda_1 s})\\
&=-\eta\mathrm{e}^{\lambda_1 s}\Delta(\lambda_1,c,x^0)+\eta\mathrm{e}^{(\lambda_1+\varepsilon)s}\Delta(\lambda_1+\varepsilon,c,x^0)+G_1,
\end{aligned}$$

where

$$\begin{aligned}
G_1&=-\mu\beta\eta\int_0^\tau K(u)\mathrm{e}^{\lambda_1(s+1+cu)}[1-\mathrm{e}^{\varepsilon(s+1+cu)}]\mathrm{d}u-\alpha\mathrm{e}^{\lambda_1 s}(1-\mathrm{e}^{\varepsilon s})|o(\eta)|\\
&\ge\eta\Big(\mu\beta\int_0^\tau K(u)\mathrm{e}^{\lambda_1(s+1+cu)}\,[\mathrm{e}^{\varepsilon(s+1+cu)}-1\,]\mathrm{d}u-\alpha\mathrm{e}^{\lambda_1 s}(1-\mathrm{e}^{\varepsilon s})\frac{|o(\eta)|}{\eta}\Big).
\end{aligned}$$

Thus, $G_1\ge 0$ if η is small enough. ¿From Lemma 2.1, $\Delta(\lambda_1,c,x^0)=0$, $\Delta(\lambda_1+\varepsilon,c,\ x^0)>0$, hence (2.2) holds.

Case 2. $-1<s\le-(c\tau+1)$. In this case, for $u\in[0,-\frac{s+1}{c}]$, one has $s+1+cu\ge0$, and for $u\in(-\frac{s+1}{c},\tau]$ one has $s+1+cu<0$. Choose $\eta\in(0,1)$, we have

$$\begin{aligned}
&cv'(s)-v(s)+\alpha f(v(s))+\beta\int_0^\tau K(u)f(v(s+1+cu))\mathrm{d}u\\
&=-\eta\mathrm{e}^{\lambda_1 s}(-c\lambda_1+1)+\eta\mathrm{e}^{(\lambda_1+\varepsilon)s}\,[-c(\lambda_1+\varepsilon)+1]+\alpha f(\eta(1-\mathrm{e}^{\varepsilon s})\mathrm{e}^{\lambda_1 s})\\
&\quad+\beta\int_{-\frac{s+1}{c}}^\tau K(u)f(\eta(1-\mathrm{e}^{\varepsilon(s+1+cu)})\mathrm{e}^{\lambda_1(s+1+cu)})\mathrm{d}u\\
&=-\eta\mathrm{e}^{\lambda_1 s}\Delta(\lambda_1,c,x^0)+\eta\mathrm{e}^{(\lambda_1+\varepsilon)s}\Delta(\lambda_1+\varepsilon,c,x^0)+G_2\\
&=\eta\mathrm{e}^{(\lambda_1+\varepsilon)s}\Delta(\lambda_1+\varepsilon,c,x^0)+G_2,
\end{aligned}$$

where

$$\begin{aligned}
G_2&=\mu\beta\eta\int_0^\tau K(u)\mathrm{e}^{\lambda_1(s+1+cu)}[\mathrm{e}^{\varepsilon(s+1+cu)}-1]\mathrm{d}u-\alpha\mathrm{e}^{\lambda_1 s}(1-\mathrm{e}^{\varepsilon s})|o(\eta)|\\
&\quad+\beta(\mu\eta-|o(\eta)|)\int_{-\frac{s+1}{c}}^\tau K(u)\mathrm{e}^{\lambda_1(s+1+cu)}(1-\mathrm{e}^{\varepsilon(s+1+cu)})\mathrm{d}u\\
&=\mu\beta\eta\int_0^{-\frac{s+1}{c}} K(u)\mathrm{e}^{\lambda_1(s+1+cu)}[\mathrm{e}^{\varepsilon(s+1+cu)}-1]\mathrm{d}u-\alpha\mathrm{e}^{\lambda_1 s}(1-\mathrm{e}^{\varepsilon s})|o(\eta)|\\
&\quad-\beta|o(\eta)|\int_{-\frac{s+1}{c}}^\tau K(u)\mathrm{e}^{\lambda_1(s+1+cu)}(1-\mathrm{e}^{\varepsilon(s+1+cu)})\mathrm{d}u\\
&\ge\mu\beta\eta\int_0^{-\frac{s}{c}} K(u)\mathrm{e}^{\lambda_1(s+1+cu)}[\mathrm{e}^{\varepsilon(s+1+cu)}-1]\mathrm{d}u-\alpha\mathrm{e}^{\lambda_1 s}(1-\mathrm{e}^{\varepsilon s})|o(\eta)|\\
&\quad-\beta|o(\eta)|\int_{-\frac{s+1}{c}}^\tau K(u)\mathrm{e}^{\lambda_1(s+1+cu)}(1-\mathrm{e}^{\varepsilon(s+1+cu)})\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&\geq \mu\beta\eta\Big[\mathrm{e}^{\lambda_1}(\mathrm{e}^{\varepsilon}-1)\int_0^{-\frac{s}{c}} K(u)\mathrm{d}u - \alpha \mathrm{e}^{\lambda_1 s}(1-\mathrm{e}^{\varepsilon s})\frac{|o(\eta)|}{\eta} \\
&\quad - \frac{|o(\eta)|}{\eta}\int_{-\frac{s+1}{c}}^{\tau} K(u)\mathrm{e}^{\lambda_1(s+1+cu)}(1-\mathrm{e}^{\varepsilon(s+1+cu)})\mathrm{d}u\Big].
\end{aligned}
$$

Therefore, $G_2 \geq 0$ if η is small enough, and similar to case 1, we have (2.2).
Case 3. $s \leq -1$. In this case for any $u \in [0,\tau]$, $s+1+cu \leq 0$. Hence

$$
\begin{aligned}
&cv'(s) - v(s) + \alpha f(v(s)) + \beta \int_0^{\tau} K(u) f(v(s+1+cu))\mathrm{d}u \\
&= -\eta \mathrm{e}^{\lambda_1 s}(-c\lambda_1 + 1) + \eta \mathrm{e}^{(\lambda_1+\varepsilon)s}\left[-c(\lambda_1+\varepsilon)+1\right] + \alpha f(\eta \mathrm{e}^{\lambda_1 s}(1-\mathrm{e}^{\varepsilon s})) \\
&\quad + \beta \int_0^{\tau} K(u) f(\eta \mathrm{e}^{\lambda_1(s+1+cu)}(1-\mathrm{e}^{\varepsilon(s+1+cu)}))\mathrm{d}u \\
&= -\eta \mathrm{e}^{\lambda_1 s}\Delta(\lambda_1, c, x^0) + \eta \mathrm{e}^{(\lambda_1+\varepsilon)s}\Delta(\lambda_1+\varepsilon, c, x^0) + G_3 \\
&= \eta \mathrm{e}^{(\lambda_1+\varepsilon)s}\Delta(\lambda_1+\varepsilon, c, x^0) + G_3,
\end{aligned}
$$

where

$$
G_3 = -\Big[\alpha \mathrm{e}^{\lambda_1 s}(1-\mathrm{e}^{\varepsilon s}) + \beta \int_0^{\tau} K(u)\mathrm{e}^{\lambda_1(s+1+cu)}(1-\mathrm{e}^{\varepsilon(s+1+cu)})\mathrm{d}u\Big]|o(\eta)|.
$$

So we can choose η small enough such that

$$
\eta \mathrm{e}^{(\lambda_1+\varepsilon)s}\Delta(\lambda_1+\varepsilon, c, x^0) + G_3 > 0.
$$

Hence (2.2) still holds in this case. According to the above discussion, we know that $v(s)$ is a lower solution of (1.5). This completes the proof. $\square$

Now we let $\mathbf{C} = \mathbf{C}(\mathbb{R}, [x^0, x^+])$, and

$$
\begin{aligned}
S_1 = \big\{\phi \in \mathbf{C} : &\text{(i) } \phi(s) \text{ is nondecreasing for } s \text{ in } \mathbb{R}; \\
&\text{(ii) } \lim_{s\to-\infty}\phi(s) = x^0,\ \lim_{s\to\infty}\phi(s) = x^+.\big\}.
\end{aligned}
$$

Assume that $c < c_* < 0$. Consider the following equivalent form of equation (1.5),

$$
\frac{\mathrm{d}\phi(s)}{\mathrm{d}s} + \gamma\phi(s) = F(\phi)(s), \tag{2.3}
$$

where

$$
F(\phi)(s) = (\gamma + \frac{1}{c})\phi(s) - \frac{\alpha}{c} f(\phi(s)) - \frac{\beta}{c}\int_0^{\tau} K(u) f(\phi(s+1+cu))\mathrm{d}u.
$$

Note that $c < 0$ and f is non-decreasing, so we can choose $\gamma > -\frac{1}{c} > 0$ such that $F(\phi)(s) \geq F(\psi)(s)$ provided that $\phi(s) \geq \psi(s)$ for $s \in \mathbb{R}$. It is easy to show that (2.3) is equivalent to

$$
\phi(s) = \mathrm{e}^{-\gamma s}\int_{-\infty}^{s} \mathrm{e}^{\gamma u} F(\phi)(u)\mathrm{d}u.
$$

Define an operator $T:\ S_1 \to \mathbf{C}$ by

$$
(T\phi)(s) = \mathrm{e}^{-\gamma s}\int_{-\infty}^{s} \mathrm{e}^{\gamma u} F(\phi)(u)\mathrm{d}u, \quad \phi \in S_1,\ s \in \mathbb{R}. \tag{2.4}
$$

Then we have the following result.

Lemma 2.4. *Assume that $c < c_* < 0$. Then T defined in (2.4) satisfies*

(1) *if $\phi \in S_1$, then $T\phi \in S_1$;*

(2) *if ϕ is an upper (resp. a lower) solution of* (1.5)*, then $\phi(s) \geq (T\phi)(s)$ (resp. $\phi(s) \leq (T\phi)(s)$) for $s \in \mathbb{R}$;*
(3) *if $\phi(s) \geq \psi(s)$ for $s \in \mathbb{R}$, then $(T\phi)(s) \geq (T\psi)(s)$ for $s \in \mathbb{R}$;*
(4) *if ϕ is an upper (a lower) solution of* (1.5)*, then $T\phi$ is also an upper (a lower) solution of* (1.5).

Proof. (1) is a direct verification by L'Hospital rule. On the other hand, the monotonicity of F leads to the conclusion (3). In the following, we only show that (2) and (4) hold. In fact, if $\phi(s)$ is an upper solution of (1.5), then

$$\frac{\mathrm{d}\phi(s)}{\mathrm{d}s} + \gamma\phi(s) \geq F(\phi)(s).$$

This leads to

$$\frac{\mathrm{d}(\mathrm{e}^{\gamma s}\phi(s))}{\mathrm{d}s} \geq \mathrm{e}^{\gamma s}F(\phi)(s).$$

Integrating the inequality above from $-\infty$ to s , we obtain (2). Noting that $F(\phi)(s) \geq F((T\phi)(s))$ from (2), we have

$$\frac{\mathrm{d}(T\phi)(s))}{\mathrm{d}s} + \gamma(T\phi)(s) = F(\phi)(s) \geq F((T\phi)(s)).$$

This means that $(T\phi)(s)$ is also an upper solution of (1.5). The proof is similar if $\phi(s)$ is a lower solution. Thus the proof is complete. □

Consider the iterative scheme

$$V_0 = V \quad \text{and} \quad V_n = TV_{n-1}, \quad n = 1, 2, \cdots .$$

By Lemma 2.2, we have

$$x^0 \leq v(s) \leq V_n(s) \leq V_{n-1}(s) \leq \cdots \leq V(s) \leq x^+.$$

By Lebesgue's dominated convergence theorem, the limit exists which allows defining the function $\phi(s) = \lim_{n\to\infty} V_n(s) \geq v(s)$ exists and is a fixed point of T. Therefore, ϕ is a solution of (1.2) and satisfies

$$\lim_{s\to-\infty} \phi(s) = x^0, \quad \lim_{s\to\infty} \phi(s) = x^+.$$

So we can obtain the following existing theorem of travelling waves.

Theorem 2.5. *For any $c < c_* < 0$, there exists a wave solution $\phi(s)$ of* (1.2) *which is increasing and satisfies*

$$\lim_{s\to-\infty} \phi(s) = x^0, \ \lim_{s\to\infty} \phi(s) = x^+.$$

Note that f is an odd function, let $\psi = -\phi$, then (1.5) is changed to

$$-c\psi'(s) = -\psi(s) + \alpha f(\psi(s)) + \beta \int_0^\tau K(u) f(\psi(s+1+u))\mathrm{d}u.$$

which is exactly of the same form as (1.5), so we have the following result.

Theorem 2.6. *For any $c < c_* < 0$, there exists a wave solution $\phi(s)$ of* (1.2) *which is decreasing and satisfies*

$$\lim_{s\to-\infty} \phi(s) = x^0, \quad \lim_{s\to\infty} \phi(s) = x^-.$$

In the rest of this section, we will discuss the existence of monotone travelling waves of (1.2) for $c > 0$. By the facts

$$\Delta(0, c, x^0) = 1 - \mu(\alpha + \beta) < 0; \quad \lim_{\lambda \to -\infty} \Delta(\lambda, c, x^0) = +\infty;$$
$$\frac{\partial \Delta}{\partial \lambda}(\lambda, c, x^0) < 0, \lambda \in \mathbb{R}; \quad \frac{\partial^2 \Delta}{\partial \lambda^2}(\lambda, c, x^0) < 0, \lambda \in \mathbb{R}.$$

Thus, we know, for any fixed $c > 0$, the equation $\Delta(\lambda, c, x^0) = 0$ has a unique real root $\lambda_2 = \lambda_2(c) < 0$. Furthermore, there is $\varepsilon_2 > 0$ such that for $0 < \varepsilon < \varepsilon_2$, one has

$$\Delta(\lambda_2 - \varepsilon, c, x^0) > 0. \tag{2.5}$$

We note that (1.5) becomes

$$\phi'(s) = \frac{1}{c}(\phi(s) - \alpha - \beta) \tag{2.6}$$

if $\phi(s) \geq 1$ for large $|s|$, and

$$\phi(s) = (\phi(0) - \alpha - \beta)\mathrm{e}^{\frac{s}{c}} + \alpha + \beta.$$

Therefore, (2.6) has no monotone solution satisfying (1.8), so we consider monotone solutions with boundary conditions

$$\lim_{s \to -\infty} \phi(s) = x^+, \quad \lim_{s \to \infty} = x^0.$$

Let

$$S_2 = \{\phi \in \mathbf{C} : \text{(i)}\phi(s) \text{ is nonincreasing for } s \in \mathbb{R};$$
$$\text{(ii)} \lim_{s \to -\infty} \phi(s) = x^+, \lim_{s \to \infty} \phi(s) = x^0.\},$$
$$\bar{V}(s) = \begin{cases} x^+, & s \leq 0, \\ x^+\mathrm{e}^{\lambda_2 s}, & s \geq 0, \end{cases}$$

and

$$\bar{v}(s) = \begin{cases} 0, & s \leq 0, \\ \eta(1 - \mathrm{e}^{-\varepsilon s})\mathrm{e}^{\lambda_2 s}, & s \geq 0, \end{cases}$$

where λ_2, ε are given in (2.5). We can show that $\bar{V}(s)$ is an upper solution and $\bar{v}(s)$ is a lower solution of (1.5) while η is appropriately chosen, with the argument similar to that for the situation where $c < c_* < 0$. Let

$$H(\phi)(s) = (\frac{1}{c} - \gamma)\phi(s) - \frac{\alpha}{c} f(\phi(s)) - \frac{\beta}{c}\int_0^\tau K(u) f(\phi(s + 1 + cu))\mathrm{d}u.$$

We choose $\gamma > 1/c$ such that $H(\phi)(s) \leq H(\psi)(s)$ provided that $\phi(s) \geq \psi(s)$ for $s \in \mathbb{R}$. Note that (1.5) is equivalent to

$$\phi(s) = -\mathrm{e}^{\gamma s}\int_s^\infty \mathrm{e}^{-\gamma u} H(\phi)(u)\mathrm{d}u.$$

Define an operator $Q : S_2 \to \mathbf{C}$ by

$$(Q\phi)(s) = -\mathrm{e}^{\gamma s}\int_s^\infty \mathrm{e}^{-\gamma u} H(\phi)(u)\mathrm{d}u, \quad \phi \in S_2, s \in \mathbb{R}. \tag{2.7}$$

Similar to the proof of Lemma 2.3 we have the following result.

Lemma 2.7. *Let Q be defined in* (2.7). *Then*

(1) *if $\phi \in S_2$, then $Q\phi \in S_2$;*

(2) *if ϕ is an upper (a lower) solution of* (1.5)*, then $\phi(s) \geq (Q\phi)(s)$ ($\phi(s) \leq (Q\phi)(s)$) for $s \in \mathbb{R}$;*
(3) *if $\phi(s) \geq \psi(s)$ for $s \in \mathbb{R}$, then $(Q\phi)(s) \geq (Q\psi)(s)$ for $s \in \mathbb{R}$;*
(4) *if ϕ is an upper (a lower) solution of* (1.5)*, then $Q\phi$ is also an upper (a lower) solution of* (1.5).

Then we can show the existence of a monotone solution in S_2 of (1.5) by monotone iteration method, with the argument similar to that of the situation where $c < c_* < 0$. In particular, we have the following.

Theorem 2.8. *For any $c > 0$, we have the following conclusions.*

(1) *There exists a wave solution $\phi(s)$ of* (1.2) *which is decreasing and satisfies*
$$\lim_{s\to-\infty} \phi(s) = x^+, \quad \lim_{s\to\infty} \phi(s) = x^0.$$
(2) *There exists a wave solution $\phi(s)$ of* (1.2) *which is increasing and satisfying*
$$\lim_{s\to-\infty} \phi(s) = x^-, \quad \lim_{s\to\infty} \phi(s) = x^0.$$

Finally, we shall briefly discuss the existence of monotone waves of CNN model with some explicit output function f.

Example 2.9. Let the output function f be defined as (1.3). Obviously, the assumption (H1) is satisfied with $\mu = 1$. Then (1.2) has monotone waves by Theorem 2.1-2.3, which leads to [6, Theorem 2.1-2.2].

Example 2.10. Let the output function f be defined as (1.9) [see [7]). Then the assumption (H1) is satisfied with $\mu = \frac{\pi}{2} > 1$. Hence (1.2) has monotone waves by Theorem 2.1-2.3.

Example 2.11. Let the output function f be defined by
$$f(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ 2x - x^2 & \text{if } 0 \leq x \leq 1, \\ 2x + x^2 & \text{if } -1 \leq x \leq 0, \\ -1 & \text{if } x \leq -1. \end{cases}$$
Obviously the assumption (H1) is satisfied with $\mu = 2$, which leads to (1.2) having monotone waves by Theorem 2.1-2.3.

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References

[1] L. O. Chua, L. Yang; Cellular neural networks: theory, *IEEE Trans. Circuits Systems*, 35 (1988): 1257-1272.
[2] L. O. Chua, L. Yang; Cellular neural networks: applications, *IEEE Trans. Circuits Systems,* 35 (1988): 1273-1290.
[3] C. H. Hsu, S. S. Lin; Existence and multiplicity of travelling waves in a lattice dynamcal systems, *J. Diff. Eqs.,* 164 (2000): 431-450.
[4] C. H. Hsu, S. S. Lin, W. Shen; Traveling waves in cellular neural networks, *Int. J. Bifur. Chaos*, 9 (1999): 1307-1319.
[5] C. H. Hsu, S. Y. Yang; On Camel-like travelling wave solutions in cellular neural networks. *J. Diff. Eqs.*, 196 (2004): 481-514.

[6] J. X. Ling; Existence of travelling waves for delay population model on patches with age-structure and deformation of travelling waves for delay CNN model with nonlinear output. Master Degree Thesis, *South China Normal University*, Guangzhou, 2003.

[7] P. X. Weng, J. H. Wu; Deformation of travelling waves in delayed cellular neural networks. *Int. J. Bifur. Chaos*, 13 (2003): 797-813.

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