Electronic Journal of Differential Equations, Vol. 2006(2006), No. 56, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

PERIODIC SOLUTIONS FOR SOME PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

RACHID BENKHALTI, ABDELHAI ELAZZOUZI, KHALIL EZZINBI

Abstract. In this work, we study the existence of periodic solutions for partial neutral functional differential equation. We assume that the linear part is not necessarily densely defined and satisfies the Hille-Yosida condition. In the nonhomogeneous linear case, we prove that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of a periodic solution. In nonlinear case, we use the concept of boundedness and ultimate boundedness to prove the existence of periodic solutions.

1. Introduction

The aim of this work is to study the existence of a periodic solution for the partial neutral functional differential equation

$$\begin{gathered} \frac{d}{dt}\mathcal{D}(u_t) = A\mathcal{D}(u_t) + F(t, u_t) \quad \text{for } t \geq 0 \\ u_0 = \varphi, \quad \varphi \in C := C([-r, 0]; X), \end{gathered} \tag{1.1}$$

where A is not necessarily densely defined linear operator on a Banach space X. We suppose that A satisfies the Hille-Yosida condition, which means that there exist $\overline{M} \geq 1$, $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{\overline{M}}{(\lambda - \omega)^n} \quad \text{for } n \in \mathbb{N},\ \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$. Here C is the space of continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology, and $\mathcal{D} : C \to X$ is a bounded linear operator which is given by

$$\mathcal{D}\varphi := \varphi(0) - \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta) \quad \text{for } \varphi \in C,$$

for a mapping $\eta : [-r, 0] \to \mathcal{L}(X)$ of bounded variation and non atomic at zero, which means that

$$\operatorname*{var}_{[-\epsilon,0]}(\eta) \to 0 \qquad \text{as } \epsilon \to 0.$$

2000 *Mathematics Subject Classification.* 34C25, 34D40, 34K40, 34K60.
Key words and phrases. Integral solutions; Hille-Yosida condition; boundedness; ultimate boundedness; condensing map; Hale and Lunel's fixed point theorem.

Submitted November 14, 2005. Published April 28, 2006.
Research is supported by grant 03-030 RG/MATHS/AF/AC from TWAS.

$\mathcal{L}(X)$ is the space of bounded linear operators from X into X. For every $t \geq 0$, as usual, the history function $u_t \in C$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in [-r, 0].$$

F is a continuous function from $\mathbb{R}^+ \times C$ into X which is periodic in t.

The theory of functional differential equations of neutral type has been developed recently by several authors, for instance we refer to [1, 2, 3, 4, 6, 8, 18, 19, 25, 26, 27]. In [26] and [27], the authors studied neutral partial functional differential-difference equation defined on the unit circle S, which is a model for a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions

$$\frac{d}{dt}[u(.,t) - qu(.,t-r)] = k\frac{\partial^2}{\partial x^2}[u(.,t) - qu(.,t-r)] + \zeta(u_t) \quad \text{for } t \geq 0, \tag{1.2}$$

where $x \in S$, k is a positive constant, ζ is a continuous function and $0 \leq q < 1$. The phase space is $C([-r, 0], H^1(S))$. In [18, 19], the author studied the qualitative behavior of solutions of equation (1.2), and obtained several results about stability, attractiveness of solutions and bifurcation of solutions near an equilibrium. The idea of studying partial neutral functional differential equations with operators satisfying Hille-Yosida condition, begins with [2], where the authors studied the following class of equation

$$\frac{d}{dt}[u(t) - Gu(t-r)] = A[u(t) - Gu(t-r)] + P(u_t) + Qu(t-r),$$

where A satisfies the Hille-Yosida condition, G and Q are bounded linear operators from X into X and P is a bounded linear operator from C into X. It has been proved in particular, that the solutions generate a locally Lipschitz continuous integrated semigroup. In [5], the authors studied the existence, uniqueness and regularity of solutions of (1.1). They obtained several results concerning dissipativeness and existence of global attractor.

One of the most attractive areas of the qualitative theory of partial neutral functional differential equations is the existence of periodic solutions. Naturally, fixed point theorems play a significant role in the investigation of the existence of periodic solutions. In finite dimensional spaces, many works are devoted to this subject. In [11] and [16], using Horn's fixed point theorem, the authors proved that if the solutions of an n-dimensional periodic ordinary differential equation are bounded and ultimately bounded, then the system has a periodic solution. In [10], the authors gave several criteria for the existence of periodic solutions of functional differential equations with infinite delay, they obtained the existence of periodic solutions by using Sadovskii's fixed point theorem. In [20] and [22] the authors used Horn's fixed point theorem to prove the existence of periodic solutions for functional differential equations with finite delay. Recently, the authors in [7], studied the following partial neutral functional differential equation

$$\frac{d}{dt}\mathcal{D}(u_t) = A\mathcal{D}(u_t) + L(u_t) + g(t) \quad \text{for } t \geq 0, \tag{1.3}$$

where A satisfies the Hille-Yosida condition, L is a bounded linear operator from C into X and g is a continuous function for $\mathbb{R}^+$ to X. They established a variation of constants formula for equation (1.3). This formula is used to prove the existence of bounded, periodic and almost periodic solution when the solution semigroup of equation (1.3) with $g = 0$ is hyperbolic. Recall that the main approach to prove the existence of periodic solutions, is to consider the Poincaré map $\mathcal{P}$ which is defined by

$$\mathcal{P}\varphi = u_\omega(.,\varphi),$$

where $u(.,\varphi)$ is the solution of (1.1). Then one establishes the existence of fixed points of $\mathcal{P}$ which are the initial values of periodic solutions.

In [15, 23], the authors used the Poincaré map and they proved the existence of periodic solutions for nonlinear partial functional differential equations of retarded type which correspond to $\mathcal{D}\varphi = \varphi(0)$, they used the boundedness and the ultimate boundedness of solutions to get a periodic solution by using Horn's fixed point theorem which requires the compactness of the solution operator. For partial neutral functional differential equations, the Poincaré map $\mathcal{P}$ is not compact, and fixed point theorems requiring compactness couldn't be used. We consider the case where F is linear with respect to the second argument, we show that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of a periodic solution. To achieve this goal, we use Chow and Hale's fixed point theorem for affine maps [12] to prove that the Poincaré map $\mathcal{P}$ has at least one fixed point. For the nonlinear case, we use the boundedness and the ultimate boundedness and we prove the existence of periodic solutions by using Hale and Lunel's fixed point theorem which is an extension of Horn's fixed point theorem for condensing maps.

The work is organized as follows: in section 2, we give some definitions and results about the solutions of (1.1). In section 3, we discuss the existence of periodic solutions where F is linear with respect to the second argument. In section 4, we study the existence of periodic solutions in the nonlinear case, we assume that solutions are bounded and ultimate bounded. Finally, we propose some applications for some partial neutral functional differential equations with diffusion.

2. Existence and estimation of solutions

Throughout this work, we suppose that

(H0) A satisfies the Hille-Yosida condition.

The following results concern the existence of integral solutions of (1.1).

Definition 2.1 ([3, 5])**.** A continuous function u from $[-r,T]$ to X with $T>0$, is an integral solution of (1.1) if

(i) $\int_0^t \mathcal{D}(u_s)ds \in D(A)$ for $t\in[0,T]$,
(ii) $\mathcal{D}(u_t) = \mathcal{D}\varphi + A\int_0^t \mathcal{D}(u_s)ds + \int_0^t F(s,u_s)ds$ for $t\in[0,T]$,
(iii) $u_0=\varphi$.

From the closedness property of A, one can see that if u is an integral solution of (1.1), then $\mathcal{D}(u_t)\in\overline{D(A)}$ for all $t\in[0,T]$. In particular, $\mathcal{D}\varphi\in\overline{D(A)}$. It has been proved in [3], that the condition $\mathcal{D}\varphi\in\overline{D(A)}$ is enough for the existence of integral solutions of (1.1). The part A_0 of the operator A in $\overline{D(A)}$ is defined by

$$D(A_0) = \{x\in D(A) : Ax\in\overline{D(A)}\},$$
$$A_0x = Ax \quad \text{for } x\in D(A_0).$$

Lemma 2.2. [9] *A_0 generates a strongly continuous semigroup $(T_0(t))_{t\ge0}$ on $\overline{D(A)}$.*

For the existence of the integral solutions, we assume that

(H1) F is continuous and Lipschitzian with respect to the second argument: There exists a positive constant μ such that
$$|F(t,\phi)-F(t,\psi)| \le \mu|\phi-\psi| \quad \text{for } \phi,\psi\in C,\ t\ge 0.$$

Theorem 2.3 ([3, Theorem 2]). *Assume that (H0) and (H1) hold. Then, for all $\varphi\in C$ such that $\mathcal{D}\varphi\in\overline{D(A)}$, there exists a unique integral solution u of* (1.1) *on $[0,+\infty)$. Moreover, u is given by*
$$\mathcal{D}(u_t)=T_0(t)\mathcal{D}\varphi+\lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B_\lambda F(s,u_s)ds \quad \text{for } t\ge 0, \tag{2.1}$$
where $B_\lambda=\lambda R(\lambda,A)$ for $\lambda>\omega$.

In the sequel, integral solutions will be called solutions.

Proposition 2.4. *Assume that (H0) and (H1) hold. Let u and v be solutions of* (1.1) *on $[-r,T]$ for $T>0$. Then, there exist positive constants N and $\widetilde{N}$ such that*
$$|u_t-v_t|\le Ne^{\widetilde{N}t}|u_0-v_0| \quad \text{for } t\in[0,T]. \tag{2.2}$$

This is an immediate consequence of the following fundamental lemma.

Lemma 2.5 ([3, Lemma 5]). *There are positive constants a,b and c such that for any continuous function $h:\mathbb{R}^+\to X$, the solution w of the difference equation*
$$\begin{gathered}\mathcal{D}(w_t)=h(t) \quad \text{for } t\ge 0\\ w_0=\varphi.\end{gathered}$$
satisfies the estimate
$$|w_t(.,\varphi)|\le \exp(at)\Big[b|w_0|+c\sup_{s\in[0,t]}|h(s)|\Big] \quad \text{for } t\ge 0. \tag{2.3}$$

Proof of Proposition 2.4. Let u and v be two solutions of (1.1) on $[-r,T]$, for some $T>0$. Then, for $t\in[0,T]$
$$\mathcal{D}(u_t-v_t)=T_0(t)\mathcal{D}(u_0-v_0)+\lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B_\lambda(F(s,u_s)-F(s,v_s))ds. \tag{2.4}$$
Let g be defined by the right hand side of (2.4). Then, by assumption (H1), we deduce that there exist positive constants k_1 and k_2 such that
$$|g(t)|\le k_1|u_0-v_0|+k_2\int_0^t|u_\xi-v_\xi|d\xi \quad \text{for } t\in[0,T].$$
Using estimate (2.3), we obtain that
$$|u_t-v_t|\le \widetilde{k_1}|u_0-v_0|+\widetilde{k_2}\int_0^t|u_\xi-v_\xi|d\xi \quad \text{for } t\in[0,T],$$
for some positive constants $\widetilde{k_1}$ and $\widetilde{k_2}$. Using Gronwall's Lemma, one obtains the estimate (2.2). □

Consequently, we have the local boundedness of the solutions.

Corollary 2.6. *Assume that (H0) and (H1) hold. Then, the solutions of* (1.1) *are locally bounded, in the sense that for each $B_0>0$ and $T_0>0$, there exists a constant $\overline{B}_0>0$, such that $|\varphi|\le B_0$ implies that $|u(t,\varphi)|\le\overline{B}_0$ for $t\in[0,T_0]$.*

To study the qualitative behavior of solutions, we need to make additional assumptions on the following difference equation

$$\begin{gathered}\frac{d}{dt}\mathcal{D}(w_t) = 0 \quad \text{for } t \geq 0\\ w_0 = \varphi.\end{gathered} \tag{2.5}$$

The following definition was given for neutral functional differential equation in finite dimensional spaces, for more details we refer to [17].

Definition 2.7. [5] The operator $\mathcal{D}$ is stable if there exist positive constants β and γ such that the solution of the homogeneous difference equation (2.5) with $w_0 = \varphi \in \{\psi \in C; \quad \mathcal{D}\psi = 0\}$, satisfies the following estimate

$$|w_t(.,\varphi)| \leq \gamma \exp(-\beta t)|\varphi| \quad \text{for } t \geq 0.$$

Example 2.8. The operator $\mathcal{D}$ defined by

$$\mathcal{D}\varphi = \varphi(0) - q\varphi(-r)$$

is stable if and only if $|q| < 1$.

Theorem 2.9 ([5, Lemma 2.9])**.** *If the operator $\mathcal{D}$ is stable. Then, there are positive constants a, b, c and d such that for any continuous function $h : \mathbb{R}^+ \to X$, the solution w of the difference equation*

$$\begin{gathered}\mathcal{D}(w_t) = h(t) \quad \textit{for } t \geq 0\\ w_0 = \varphi \in C,\end{gathered}$$

satisfies the estimate

$$|w_t(.,\varphi)| \leq e^{-at}\Big(b|\varphi| + c\sup_{s\in[0,t]}|h(s)|\Big) + d\sup_{s\in[\max\{0,t-r\},t]}|h(s)| \quad \textit{for } t \geq 0.$$

The Kuratowski's measure of noncompactness. of bounded sets K on a Banach space Y is defined by

$$\alpha(K) = \inf\{\epsilon > 0 : K \text{ has a finite cover of ball of diameter less than } \epsilon\}.$$

Lemma 2.10. [21] *Let A_1 and A_2 be bounded sets of a Banach space Y. Then*
- (i) $\alpha(A_1) \leq \operatorname{dia}(A_1)$, *where* $\operatorname{dia}(A_1) = \sup_{x,y\in A_1}|x-y|$,
- (ii) $\alpha(A_1) = 0$ *if and only if A_1 is relatively compact in Y,*
- (iii) $\alpha(A_1 \cup A_2) = \max\{\alpha(A_1), \alpha(A_2)\}$.

Let $\mathcal{K} : Y \to Y$ be a closed linear operator with a dense domain $D(\mathcal{K})$ in a Banach space Y. We denote by $\sigma(\mathcal{K})$ the spectrum of $\mathcal{K}$.

Definition 2.11 ([25])**.** The essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{K})$ of $\mathcal{K}$ is the set of all $\lambda \in \mathbb{C}$ such that at least one of the following holds:
- (i) The range $\operatorname{Im}(\lambda I - \mathcal{K})$ is not closed,
- (ii) the generalized eigenspace $M_\lambda(\mathcal{K}) = \cup_{n\geq 1}\ker(\lambda I - \mathcal{K})^n$ of λ is infinite dimensional,
- (iii) λ is a limit point of $\sigma(\mathcal{K})$, that is $\lambda \in \overline{\sigma(\mathcal{K})/\{\lambda\}}$.

For a bounded linear operator $\mathcal{K}$ on Y, the Kuratowski measure of non-compactness of $\mathcal{K}$ is defined by

$$|\mathcal{K}|_\alpha = \inf\{\epsilon > 0 : \alpha(\mathcal{K}(B)) \leq \epsilon\alpha(B) \text{ for every bounded subset } B \text{ of } Y\}.$$

The essential radius $r_{\mathrm{ess}}(\mathcal{K})$ is given by
$$r_{\mathrm{ess}}(\mathcal{K}) = \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{ess}}(\mathcal{K})\}.$$
The computation of essential radius is given by the following Nussbaum's formula.

Lemma 2.12 ([24])**.**
$$r_{\mathrm{ess}}(\mathcal{K}) = \lim_{n\to+\infty}(|\mathcal{K}^n|_\alpha)^{1/n}.$$

Definition 2.13 ([17])**.** A continuous mapping $P : Y \to Y$ is said to be an α-contraction if P maps bounded sets into bounded sets and if there exists a constant $k \in (0,1)$ such that
$$\alpha(P(B)) \le k\alpha(B),$$
for every bounded subset B of Y.

Definition 2.14 ([17])**.** A continuous mapping $P : Y \to Y$ is a condensing map on Y if P maps bounded sets into bounded sets and
$$\alpha(P(B)) < \alpha(B),$$
for every bounded subset B of Y such that $\alpha(B) > 0$.

Let C_0 be the phase space of Equation (1.1) defined by
$$C_0 = \{\varphi \in C : \mathcal{D}\varphi \in \overline{D(A)}\}.$$
For each $t \ge 0$, we define the linear operator $\mathcal{U}(t)$ on C_0 by
$$\mathcal{U}(t)\varphi = x_t(.,\varphi),$$
where $x(.,\varphi)$ is the solution of the equation
$$\begin{gathered} \frac{d}{dt}\mathcal{D}(u_t) = A\mathcal{D}(u_t) \quad \text{for } t \ge 0 \\ u_0 = \varphi \in C. \end{gathered} \tag{2.6}$$
Without loss of generality, we assume that

(H2) $(T_0(t))_{t\ge0}$ is exponentially stable, which means that there exist $\alpha_0 > 0$ and $M_0 \ge 1$ such that
$$|T_0(t)| \le M_0 e^{-\alpha_0 t} \quad \text{for } t \ge 0.$$

Otherwise, we can replace A by $A - \delta I$, where $\delta > 0$ can be chosen such that the semigroup generated by the part of $A - \delta I$ on $\overline{D(A)}$ is exponentially stable.
We assume that

(H3) $\mathcal{D}$ is stable.

The following fundamental lemma plays a crucial role for the existence of periodic solutions.

Lemma 2.15 ([5, Proposition 2.11])**.** *Assume that (H0), (H2) and (H3) hold. Then, $(\mathcal{U}(t))_{t\ge0}$ is an exponentially stable semigroup on C_0, that is there exist $\eta > 0$ and $M \ge 1$ such that*
$$|\mathcal{U}(t)| \le Me^{-\eta t} \quad \textit{for } t \ge 0.$$

For $\varphi \in C_0$, we introduce the new norm on C_0 by

$$|\varphi|_\eta = \sup_{t \geq 0} e^{\eta t} |\mathcal{U}(t)\varphi|,$$

where η is the positive constant given in Lemma 2.15. Clearly,

$$|\varphi| \leq |\varphi|_\eta \leq M|\varphi|,$$

which implies that $|.|_\eta$ and $|.|$ are equivalent norms on C_0. As an immediate result, we have the following result.

Corollary 2.16. *Assume that (H0), (H2) and (H3) hold. Then*

$$|\mathcal{U}(t)|_\eta \leq e^{-\eta t} \quad \text{for } t \geq 0.$$

Proof. For every $t \geq 0$, one has,

$$\begin{aligned}
|\mathcal{U}(t)\varphi|_\eta &= \sup_{s \geq 0} e^{\eta s} |\mathcal{U}(s)\mathcal{U}(t)\varphi|, \\
&= e^{-\eta t} \sup_{s \geq 0} e^{\eta(t+s)} |\mathcal{U}(s+t)\varphi|, \\
&\leq e^{-\eta t} \sup_{s \geq 0} e^{\eta s} |\mathcal{U}(s)\varphi| = e^{-\eta t} |\varphi|_\eta,
\end{aligned}$$

which implies $|\mathcal{U}(t)|_\eta \leq e^{-\eta t}$ for $t \geq 0$. □

(H4) $T_0(t)$ is compact on $\overline{D(A)}$ whenever $t > 0$.

Theorem 2.17 ([5, Theorem 5.2])**.** *Assume that (H0), (H1), (H2), (H3) and (H4) hold. Then the solution $u(.,\varphi)$ of* (1.1) *is decomposed as follows:*

$$u_t(.,\varphi) = \mathcal{U}(t)\varphi + \mathcal{W}(t)\varphi \quad \text{for } t \geq 0,$$

where $\mathcal{W}(t)$ is a compact operator on C_0, for each $t \geq 0$.

3. Existence of periodic solutions in nonhomogeneous linear case

In this section, we assume that F takes the form

$$F(t,\varphi) = L(t,\varphi) + f(t) \quad \text{for } t \geq 0,\ \varphi \in C,$$

where L is a continuous function from $\mathbb{R}^+ \times C$ into X, linear with respect to the second argument and f is a continuous function from $\mathbb{R}$ into X. Equation (1.1) becomes

$$\begin{gathered}
\frac{d}{dt}\mathcal{D}(u_t) = A\mathcal{D}(u_t) + L(t,u_t) + f(t) \quad \text{for } t \geq 0, \\
u_0 = \varphi \in C,
\end{gathered} \tag{3.1}$$

For the existence of periodic solutions, we assume that

(H5) L and f are ω-periodic in t.

Theorem 3.1. *Assume that (H0), (H2), (H3), (H4) and (H5) hold. If Equation (3.1) has a bounded solution on $\mathbb{R}^+$, then it has an ω-periodic solution.*

For the proof, we use Chow and Hale's fixed point theorem which gives sufficient conditions for affine maps to have fixed points.

Theorem 3.2 ([12])**.** *Let Y be a Banach space and $P : Y \to Y$ be an affine map which is defined by*

$$Px = Sx + y,$$

where S is a bounded linear operator on Y and y is given in Y. If $\operatorname{Im}(I - S)$ *is closed and there exists $x_0 \in Y$ such that $(P^n(x_0))_{n\geq 0}$ is bounded, then P has at least one fixed point.*

Proof of Theorem 3.1. Define the Poincaré map $\mathcal{P} : C_0 \to C_0$ by

$$\varphi \to u_\omega(., \varphi) = u_\omega(., 0, \varphi, L, f),$$

where $u(., 0, \varphi, L, f)$ is the solution of (3.1). By the uniqueness of solutions with respect to the initial data, $u_t(., 0, \varphi, L, f)$ is decomposed as follows

$$u_t(., 0, \varphi, L, f) = u_t(., 0, \varphi, L, 0) + u_t(., 0, 0, L, f) \quad \text{for } t \geq 0.$$

Therefore, the Poincaré map $\mathcal{P}$ is affine, $\mathcal{P}\varphi = \mathcal{P}_0\varphi + \psi$, where $\mathcal{P}_0\varphi = u_\omega(., 0, \varphi, L, 0)$ and $\psi = u_\omega(., 0, 0, L, f)$. We claim that $r_{\text{ess}}(\mathcal{P}_0) < 1$. In fact, by Theorem 2.17, $\mathcal{P}_0$ is decomposed as follows

$$\mathcal{P}_0\varphi = \mathcal{U}(\omega)\varphi + \mathcal{W}(\omega)\varphi,$$

where $\mathcal{W}(\omega)$ is a compact operator on C_0. We deduce that $\alpha(\mathcal{P}_0) \leq \alpha(\mathcal{U}(\omega))$. By Corollary 2.16, we have

$$\alpha(\mathcal{P}_0) \leq \exp(-\eta\omega) < 1.$$

Using Lemma 2.12, we obtain that $r_{\text{ess}}(\mathcal{P}_0) < 1$ which implies that 1 is not in the essential spectrum of $\mathcal{P}_0$. Consequently, $\operatorname{Im}(I - \mathcal{P}_0)$ is closed. Let y be the bounded solution of Equation (3.1) on $\mathbb{R}^+$. Then,

$$\{\mathcal{P}^n y_0, n \in \mathbb{N}\} = \{y_{n\omega}, n \in \mathbb{N}\},$$

which gives that $(\mathcal{P}^n y_0)_{n\geq 0}$ is bounded in C_0. By Theorem 3.2, we deduce that $\mathcal{P}$ has at least one fixed point, which gives an ω-periodic solution of (3.1). □

4. Boundedness, ultimate boundedness and periodicity

In this section, we study the existence of periodic solutions where the solutions are bounded and ultimate bounded.

Definition 4.1. The solutions of (1.1) are bounded if for each $B_1 > 0$, there exists a constant $\overline{B}_1 > 0$, such that $|\varphi| \leq B_1$ implies that $|u(t, \varphi)| \leq \overline{B}_1$, for $t \geq 0$.

Definition 4.2. The solutions of (1.1) are ultimate bounded if there is a bound $B > 0$ such that for each $B_2 > 0$, there exists a constant $k > 0$ such that $|\varphi| \leq B_2$ and $t \geq k$ imply that $|u(t, \varphi)| \leq B$.

Recall that in [15], the authors have used the concept of boundedness and ultimate boundedeness to prove the existence of a periodic solution for partial functional differential equations of retarded type which correspond to $\mathcal{D}\varphi = \varphi(0)$. The relationship between the local boundedness, the boundedness and the ultimate boundedeness is given below.

Proposition 4.3. *The local boundedness and ultimate boundedness of solutions of* (1.1) *imply the boundedness of the solutions.*

Proof. Let B be given by the ultimate boundedness, then for any $B_1 > 0$, there exists a constant $k > 0$ such that $|\varphi| \leq B_1$ and $t \geq k$ imply that $|u(t,\varphi)| \leq B$. Local boundedness of solutions gives that there exists a constant $B_2 > B$ such that $|\varphi| \leq B_1$ implies that $|u(t,\varphi)| < B_2$, for $t \in [0,k]$. It follows that for any positive constant B_1, there exists a constant $B_2 > B$ such that $|\varphi| \leq B_1$ implies that $|u(t,\varphi)| < B_2$, for $t \geq 0$. □

Proposition 4.4. *Under assumptions (H0)–(H4), the Poincaré map $\mathcal{P}$ is an α-contraction on C_0.*

Proof. By Theorem 2.17, $\mathcal{P}$ is decomposed as

$$\mathcal{P}\varphi = \mathcal{U}(\omega)\varphi + \mathcal{W}(\omega)\varphi,$$

where $\mathcal{W}(\omega)$ is a compact operator on C_0. Let Ω a bounded set in C_0. Then

$$\alpha(\mathcal{P}(\Omega)) \leq \alpha(\mathcal{U}(\omega)(\Omega)).$$

Corollary 2.16 implies

$$\alpha(\mathcal{P}(\Omega)) < \exp(-\eta\omega)\alpha(\Omega) \quad \text{for any bounded set } \Omega \text{ in } C_0,$$

which gives that $\mathcal{P}$ is an α-contraction map on C_0. □

In the following, we assume that

(H6) F is ω-periodic in t.

Theorem 4.5. *Assume that (H0), (H1), (H2), (H3), (H4) and (H6) hold. If the solutions of* (1.1) *are ultimately bounded, then* (1.1) *has an ω-periodic solution.*

For the proof, we use Hale and Lunel's fixed point theorem which is an extension of the well known Horn's fixed point theorem for condensing maps.

Theorem 4.6 ([17, Hale and Lunel's fixed point theorem])**.** *Suppose $S_0 \subseteq S_1 \subseteq S_2$ are convex bounded subsets of a Banach space Y, such that S_0, S_2 are closed and S_1 is open in S_2. Let P be a condensing map on Y such that $P^j(S_1) \subseteq S_2$, for $j \geq 0$, and there is a number $N(S_1)$ such that $P^k(S_1) \subseteq S_0$, for $k \geq N(S_1)$, then P has a fixed point.*

Proof of Theorem 4.5. By Corollary 2.6 and Proposition 4.3, we know that the solutions of (1.1) are bounded and ultimate bounded. Let B be the bound in the definition of ultimate boundedness. By the boundedness of solutions, there exists a constant $B_1 > B$ such that for $|\varphi| \leq B$ and $t \geq 0$, one has $|u(t,\varphi)| < B_1$. Moreover, there exists a constant $B_2 > B_1$ such that for $|\varphi| \leq B_1$ and $t \geq 0$, then $|u(t,\varphi)| < B_2$. By using the ultimate boundedness of solutions of (1.1), we can see that there exists a positive integer $m = m(B_1)$ such that for $|\varphi| \leq B_1$ and $t \geq m\omega$, we have $|u(t,\varphi)| < B$. On the other hand,

$$\mathcal{P}^j\varphi = u_{j\omega}(.,\varphi) \quad \text{for } j \in \mathbb{N}.$$

Let $k = \left[\frac{r}{\omega}\right] + m + 1$, where $[t]$ denotes the integer part of t. Then for $j \geq k$ and $|\varphi| \leq B_1$, one has

$$|\mathcal{P}^j(\varphi)| = |u_{j\omega}(.,\varphi)| \leq B, \tag{4.1}$$

and for $j \in \{1, 2, \dots, k-1\}$ and $|\varphi| \leq B_1$,

$$|\mathcal{P}^j(\varphi)| = |u_{j\omega}(.,\varphi)| \leq B_2. \tag{4.2}$$

We define the sets

$$\begin{aligned} S_0 &= \{\varphi \in C_0 : |\varphi| \le B\}, \\ S_1 &= \{\varphi \in C_0 : |\varphi| < B_1\}, \\ S_2 &= \{\varphi \in C_0 : |\varphi| \le B_2\}. \end{aligned}$$

Clearly, S_0, S_1 and S_2 are convex bounded subsets of the Banach space C_0. Moreover $S_0 \subseteq S_1 \subseteq S_2$, S_0 and S_2 are closed and S_1 is open in S_2, and

$$P^j(S_1) \subseteq S_2 \quad \text{for all } j \ge 0.$$

In fact, inequality (4.1) gives that there exists a positive integer $k = k(S_1)$ such that

$$P^j(S_1) \subseteq S_0 \subseteq S_2 \quad \text{for } j \ge k,$$

and from inequality (4.2), we deduce that

$$P^j(S_1) \subseteq S_2 \quad \text{for } j \in \{1, 2, \dots, k-1\}.$$

By Proposition 4.4, $\mathcal{P}$ is an α-contraction map on C_0. Consequently, Theorem 4.6 gives that the Poincaré map $\mathcal{P}$ has at least one fixed point which gives an ω-periodic solution of (1.1). □

5. Applications

Nonhomogeneous linear case. To illustrate our previous results, we consider the following model of continuous circular array of resistively coupled transmission lines which is taken from [27]

$$\begin{aligned} \frac{\partial}{\partial t}[u(t,x) - qu(t-r,x)] &= \frac{\partial^2}{\partial x^2}[u(t,x) - qu(t-r,x)] + a_1(t)u(t-r,x) \\ &\quad + h_1(t,x) \quad \text{for } t \ge 0,\ x \in [0,\pi], \\ [u(t,x) - qu(t-r,x)]_{x=0,\pi} &= 0 \quad \text{for } t \ge 0, \\ u(\theta,x) &= \varphi_0(\theta,x) \quad \text{for } \theta \in [-r,0],\ x \in [0,\pi], \end{aligned} \tag{5.1}$$

where $a_1 : \mathbb{R} \to \mathbb{R}$, $h_1 : \mathbb{R} \times [0,\pi] \to \mathbb{R}$ and $\varphi_0 : [-r,0] \times [0,\pi] \to \mathbb{R}$ are continuous functions and $0 < q < 1$. Let $X = C([0,\pi];\mathbb{R})$ be the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology. In order to rewrite (5.1) in abstract form, we introduce the linear operator $A : D(A) \subset X \to X$ defined by

$$\begin{aligned} D(A) &= \{y \in C^2([0,\pi];\mathbb{R}) : y(0) = y(\pi) = 0\}, \\ Ay &= y''. \end{aligned}$$

Lemma 5.1 ([13]). *$(0,+\infty) \subset \rho(A)$ and $|(\lambda I - A)^{-1}| \le \frac{1}{\lambda}$ for $\lambda > 0$.*

The above lemma implies that assumption $(\mathbf{H}_0)$ is satisfied. Moreover, one has

$$\overline{D(A)} = \{y \in X : y(0) = y(\pi) = 0\}.$$

Let $\mathcal{D} : C \to X$ and $L : \mathbb{R} \times C \to X$ be the bounded linear operators defined respectively by

$$\begin{aligned} \mathcal{D}\varphi &:= \varphi(0) - q\varphi(-r), \\ L(t,\varphi) &= a_1(t)\ \varphi(-r) \quad \text{for } t \in \mathbb{R},\ \varphi \in C([-r,0];X). \end{aligned}$$

Let $f : \mathbb{R} \longrightarrow X$ be given by

$$f(t)(x) = h_1(t,x) \quad \text{for } t \in \mathbb{R},\ x \in [0,\pi].$$

Then, Equation (5.1) takes the abstract form (3.1). Since $0 < q < 1$, the operator $\mathcal{D}$ is stable. The part A_0 of A in $\overline{D(A)}$ is defined by

$$D(A_0) = \{y \in C^2([0,\pi];\mathbb{R}) : y(0) = y(\pi) = y''(0) = y''(\pi) = 0\},$$
$$A_0 y = y''.$$

Lemma 5.2. [10] *A_0 generates a compact strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$ such that*

$$|T_0(t)| \leq e^{-t} \quad \text{for } t \geq 0.$$

Therefore, the assumptions (H2) and (H4) hold. Consequently, for any $\varphi \in C$ such that

$$\mathcal{D}\varphi \in \{y \in X : \, y(0) = y(\pi) = 0\}.$$

Equation (3.1) has a unique solution u on $[-r,+\infty)$. To establish the existence of a periodic solution of (3.1), we suppose that

- (H7) a_1 and h_1 are ω-periodic in t.
- (H8) There exists a positive constant $\beta \in (0,1)$ such that $|a_1|_\infty \leq (1-q)\beta$, where $|a_1|_\infty = \sup_{s\in\mathbb{R}} |a_1(s)|$.

Proposition 5.3. *Assume that (H7) and (H8) hold. Then* (3.1) *has an ω-periodic solution.*

Proof. We will first show that (1.1) has a bounded solution on $\mathbb{R}^+$. Let $\rho = \frac{1}{1+q}\big(1+\frac{|f|_\infty}{1-\beta}\big)$ and $\varphi \in C$ such that $|\varphi| < \rho$. Then $|\varphi(0) - q\varphi(-r)| < (1+q)\rho$. We claim that

$$|u(t) - qu(t-r)| \leq (1+q)\rho \quad \text{for all } t \geq 0. \tag{5.2}$$

We proceed by contradiction. Let t_0 be the first time such that (5.2) is not true. Then,

$$t_0 = \inf\{t > 0 : |u(t) - qu(t-r)| > (1+q)\rho\}.$$

By continuity, one can see that

$$|u(t_0) - qu(t_0 - r)| = (1+q)\rho,$$

and there exists a positive constant $\varepsilon > 0$ such that

$$|u(t) - qu(t-r)| > (1+q)\rho \quad \text{for } t \in (t_0, t_0+\varepsilon).$$

Using the variation-of-constants formula (2.1), we get that

$$|u(t_0) - qu(t_0-r)| \leq e^{-t_0}(1+q)\rho + \int_0^{t_0} e^{-(t_0-s)}[|a_1|_\infty|u(s+\theta)|d\theta + |f|_\infty]ds.$$

Since $|u(t) - qu(t-r)| \leq (1+q)\rho$ for $t \leq t_0$, then

$$|u(t)| \leq (1+q)\rho + q|u(t-r)| \quad \text{for } t \in [-r, t_0].$$

$|\varphi| < \rho$, then we can see that

$$|u(t)| \leq \frac{1+q}{1-q}\rho \quad \text{for } t \in [-r, t_0],$$

and

$$|u(t_0) - qu(t_0-r)| \leq e^{-t_0}(1+q)\rho + (1-e^{-t_0})[\frac{1+q}{1-q}|a_1|_\infty\rho + |f|_\infty].$$

Using hypotheses (H8), we obtain

$$|u(t_0) - qu(t_0-r)| \leq e^{-t_0}(1+q)\rho + (1-e^{-t_0})(\beta(1+q)\rho + |f|_\infty),$$

consequently,

$$|u(t_0) - qu(t_0 - r)| \le (1+q)\rho - (1 - e^{-t_0})((1-\beta)(1+q)\rho - |f|_\infty),$$
$$|u(t_0) - qu(t_0 - r)| \le (1+q)\rho - (1 - e^{-t_0})(1-\beta) < (1+q)\rho.$$

By continuity, there exists a positive ε_0 such that

$$|u(t) - qu(t-r)| < (1+q)\rho \quad \text{for } t \in (t_0, t_0 + \varepsilon_0),$$

which gives a contradiction and we deduce that

$$|u(t) - qu(t-r)| \le (1+q)\rho \quad \text{for } t \ge 0.$$

Let $t \in [0, r]$. Then

$$|u(t)| \le (1+q)\rho + q\rho \le (1+q)(1+q)\rho,$$

and for $t \in [r, 2r]$,

$$|u(t)| \le (1+q)(1+q+q^2)\rho.$$

We proceed by steps, then for $t \in [(n-1)r, nr]$, we have

$$|u(t)| \le (1+q)(1+q+q^2+\dots+q^n)\rho.$$

Consequently,

$$|u(t)| \le (1+q)\rho \sum_{n\ge 0} q^n = \frac{1+q}{1-q}\rho \quad \text{for all } t \ge 0.$$

Then, Equation (3.1) has a bounded solution u on $\mathbb{R}^+$. By Theorem 3.1, we deduce that Equation (3.1) has an ω-periodic solution. □

Nonlinear case. We consider the nonlinear equation

$$\begin{gathered}
\frac{\partial}{\partial t}[u(t,x) - qu(t-r,x)] = \frac{\partial^2}{\partial x^2}[u(t,x) - qu(t-r,x)] + a_2(t) g_1(u(t-r,x)) \\
+ h_2(t,x) \quad \text{for } t \ge 0,\ x \in [0,\pi], \\
[u(t,x) - qu(t-r,x)]_{x=0,\pi} = 0 \quad \text{for } t \ge 0, \\
u(\theta, x) = \varphi_0(\theta, x) \quad \text{for } \theta \in [-r, 0],\ x \in [0,\pi],
\end{gathered} \tag{5.3}$$

where $g_1 : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and a_2, $\varphi_0 : [-r, 0] \times [0,\pi] \to \mathbb{R}$ are continuous functions and $0 < q < 1$. We define the function $F : \mathbb{R} \times C \to X$ by

$$F(t,\varphi)(x) = a_2(t) g_1(\varphi(-r)(x)) + h_2(t,x) \quad \text{for } t \in \mathbb{R},\ x \in [0,\pi],\ \varphi \in C.$$

Then, (5.3) takes the abstract form (1.1).

We assume that

- (H9) a_2, h_2 are ω-periodic in t.
- (H10) g_1 is bounded on $\mathbb{R}$.

Proposition 5.4. *Assume that (H9) and (H10) hold. Then, the solutions of* (1.1) *are ultimately bounded.*

Proof. Since $0 < q < 1$, then the operator $\mathcal{D}$ is stable. By Theorem 2.5, we deduce that there exist positive constants $\overline{a}$, $\overline{b}$ and $\overline{c}$ such that

$$|u_t(.,\varphi)| \le \overline{a} e^{-\overline{b}t}\big(|\varphi| + \sup_{s\in[0,t]} |h(s)|\big) + \overline{c} \sup_{s\in[\max\{0,t-r\},t]} |h(s)|, \tag{5.4}$$

where
$$h(t) = T_0(t)\mathcal{D}\varphi + \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B_\lambda F(s,u_s)ds \quad \text{for } t \geq 0.$$
Using Assumption (H10) and the fact that $|T_0(t)| \leq e^{-t}$ for $t \geq 0$, we obtain that there exist positive constants $\widetilde{a}$ and $\widetilde{b}$, such that
$$|h(t)| \leq \widetilde{a}e^{-t}|\varphi| + \widetilde{b} \quad \text{for } t \geq 0,$$
which implies for $t > r$ that
$$\sup_{s\in[t-r,t]}|h(s)| \leq \widetilde{a}e^{r-t}|\varphi| + \widetilde{b} \quad \text{for } \varphi \in C.$$
Using the estimate (5.4), we obtain
$$|u_t(.,\varphi)| \leq ae^{-bt}|\varphi| + c \quad \text{for } t > r \ \ \varphi \in C,$$
for some positive constants a, b and c. Consequently, there exists a positive constant $\widetilde{K}$ such that
$$\limsup_{t\to+\infty}|u(t,\varphi)| < \widetilde{K} \quad \text{for } \varphi \in C,$$
and we deduce that the solutions of (1.1) are ultimately bounded. $\square$

Consequently by Theorem 4.5, we obtain the following result.

Proposition 5.5. *Assume that (H9) and (H10) hold. Then* (1.1) *has an ω-periodic solution.*

Acknowledgments. The authors would like to thank the anonymous referee for his/her careful reading of the original version.

References

[1] M. Adimy and K. Ezzinbi, Local existence and linearized stability for partial functional differential equations, Dynamic Systems and Applications, Vol. 7, no. 3, 389-404, (1998).
[2] M. Adimy and K. Ezzinbi, A class of linear partial neutral functional differential equations with nondense domain, Journal of Differential Equations, Vol. 147, no. 2, 285-332, (1998).
[3] M. Adimy and K. Ezzinbi, Existence and linearized stability for partial neutral functional differential equations with nondense domains, Differential Equations and Dynamical Systems, Vol. 7, 371-417, (1999).
[4] M. Adimy and K. Ezzinbi, Strict solutions of nonlinear hyperbolic neutral differential equations, Applied Mathematics Letters, Vol. 12, Issue 1, January, 107-112, (1999).
[5] M. Adimy and K. Ezzinbi, Existence and stability of solutions for a class of partial neutral functional differential equations, Hiroshima Mathematical Journal, Vol. 34 , no. 3, (2004).
[6] M. Adimy, K. Ezzinbi and M. Laklach, Local existence and global continuation for a class of partial neutral functional differential equations, C. R. Acad.Sci. Paris, t. 330, Serie I, 952-962, (2002).
[7] M. Adimy, K. Ezzinbi and M. Laklach, Spectral decomposition for partial neutral functional differential equations, Canadian Applied Mathematics Quarterly, Vol. 9, no. 1, 1-34, Spring (2001).
[8] M. Adimy, H. Bouzahir and K. Ezzinbi, Existence and stability for some partial neutral functional differential equations with infinite delay, Journal of Mathematical Analysis and Applications, Vol. 294, no. 2, 438-461, (2004).
[9] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, Vector Valued Laplace Transforms and Cauchy Problems, Monographs in Mathematics. Vol. 96, Birkhäuser-Verlag, (2001).
[10] R. Benkhalti, H. Bouzahir and K. Ezzinbi, Existence of a periodic solution for some partial functional differential equations with infinite delay, Journal of Mathematical Analysis and Applications, Vol. 256, 257-280, (2001).

[11] T. Burton, Stability and Periodic Solutions of Ordinary Differential Equation and Functional Differential Equations. Academic Press, New York, 197-308, (1985).
[12] S. N. Chow and J. K. Hale, Strongly limit-compact maps, Funkcioj Ekvacioj, Vol. 17, 31-38, (1974).
[13] G. Da Prato and E. Sinestrari, Differential operators with nondense domains, Annali Scuola Normale Superiore di Pisa, Vol. 14, no. 2, 285-344, (1987).
[14] K. J. Engel, R. Nagel, One-Parameter Semigroups of Linear Evolution Equations, Graduate Texts in Mathematics, Springer-Verlag, Vol. 194, (2000).
[15] K. Ezzinbi and J. Liu, Periodic solutions of non-densely defined delay evolution equations, Journal of Applied Mathematics and Stochastic Analysis, Vol. 15, no. 2, 113-123, (2002).
[16] J. Haddock, Liapunov functions and boundedness and global existence of solutions, Applicable Analysis, 321-330, (1972).
[17] J. K. Hale and S. Verduyn-Lunel, Introduction to Functional Differential Equations, Applied Mathematical Sciences, Vol. 99, Springer-Verlag, New York, (1993).
[18] J. K. Hale, Partial neutral functional differential equations, Rev. Roumaine Math. Pure Appli., Vol. 39, no. 4, 339-344, (1994).
[19] J. K. Hale, Coupled oscillators on a circle, Dynamical phase transitions (São Paulo, 1994), Resenhas, Vol. 1, no. 4, 441-457, (1994).
[20] J. K. Hale and O. Lopes, Fixed point theorems and dissipative processes, Journal of Differential Equations, Vol. 13, 391-402, (1966).
[21] V. Lakshmikantham, S. Leela, Differential and Integral Inequalities, Vol. 1, Academic Press, (1969).
[22] J. Liang and Fa-Lun Huang, Horn type theorem in Fréchet spaces and applications, Chin. Ann. Math. 12B, 131-136, (1991).
[23] J. Liu, Bounded and periodic solutions of finite delay evolution equations, Nonlinear analysis, Vol. 34, 101-111, (1998).
[24] R. D. Nussbaum, The radius of essential spectrum, Duke Math. J., Vol. 37, 473-478, (1970).
[25] J. Wu, Theory and Applications of Partial Functional Differential Equations, Applied Mathematical Sciences, Springer-Verlag, Vol. 119, (1996).
[26] J. Wu and H. Xia, Self-sustained oscillations in a ring array of coupled lossless transmission lines, Journal of Differential Equations, Vol. 124, no. 1, 247-278, (1996).
[27] J. Wu and H. Xia, Rotating waves in neutral partial functional differential equations, Journal of Dynamics and Differential Equations, Vol. 11, no. 2, 209-238, (1999).

Rachid Benkhalti
Pacific Lutheran University, Department of Mathematics, Tacoma, Washington, 98447, USA
E-mail address: benkhar@plu.edu

Abdelhai Elazzouzi
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P. 2390 Marrakesh, Morocco
E-mail address: a.elazzouzi@ucam.ac.ma

Khalil Ezzinbi
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P. 2390 Marrakesh, Morocco
E-mail address: kezzinbi@ictp.it, ezzinbi@ucam.ac.ma