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EXISTENCE OF SOLUTIONS OF AN INTEGRAL EQUATION OF CHANDRASEKHAR TYPE IN THE THEORY OF RADIATIVE TRANSFER

JOSEFA CABALLERO, ANGELO B. MINGARELLI, KISHIN SADARANGANI

Abstract. We give an existence theorem for some functional-integral equations which includes many key integral and functional equations that arise in nonlinear analysis and its applications. In particular, we extend the class of characteristic functions appearing in Chandrasekhar's classical integral equation from astrophysics and retain existence of its solutions. Extensive use is made of measures of noncompactness and abstract fixed point theorems such as Darbo's theorem.

1. Introduction

The study of Chandrasekhar's integral equation [7]

$$x(t) = 1 + x(t)\int_0^1 \frac{t}{t+s}\varphi(s)\,x(s)\,ds$$

has been a subject of much investigation since its appearance around fifty years ago. It arose originally in connection with scattering through a homogeneous semi-infinite plane atmosphere and has since been used to model diverse forms of scattering via the H-functions of Chandrasekhar. These in turn are used to write down specific solutions of the integral equation. The problem of approximating such solutions is still much in vogue today and many efficient methods of calculation of these functions have been found, e.g., see [5] for details and [9] for an update on the method. Insofar as the theoretical question of the existence of solutions is concerned, we note that it is known that in some cases as many as two solutions may exist to one and the same equation, [[6], Chapter 2]. We show that an abstract framework exists in which Chandrasekhar's integral equation above takes part as a special case. Indeed, we show that for said equation, the mere continuity of the characteristic function $\varphi(t)$ along with $\varphi(0) = 0$ will guarantee the existence of at least one solution of (3.3). Recall that normally one assumes that $\varphi(t)$ is an even polynomial, [7].

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Associated with the usual equation (3.3) is the modified integral equation (1.1) first suggested by Chandrasekhar [7, Chapter 5, Sect. 38], namely

$$\frac{1}{x(t)} = (1-2\psi_0)^{1/2} + \int_0^1 \frac{s}{t+s}\psi(s)\,x(s)\,ds, \tag{1.1}$$

where $\psi_0 = \int_0^1 \psi(x)\,dx$. Theoretical results, see (3.10) below, give that $0 < (1-2\psi_0)^{1/2} \le 1$. The usefulness of (1.1) lies in part that it is better suited for numerical approximations than the original one, (3.3). It is known that (1.1) allows for two solutions each of one distinct sign, see [5]. Multiplying (1.1) by $x(t)/(1-2\psi_0)$ and rearranging terms we find the modified form

$$y(t) = \frac{1}{1-2\psi_0} - \int_0^1 \frac{s}{t+s}\psi(s)\,y(t)\,y(s)\,ds, \tag{1.2}$$

where $y(t) = x(t)/(1-2\psi_0)$, provided $\psi_0 \neq 1/2$, the non-critical case, in which case (1.1) and (1.2) are equivalent.

On the other hand, one could also start the process with equation (1.2). In this case, the existence of at least one real solution of (1.2) is a consequence of our abstract theorem below, and this for basically any choice of a characteristic function $\psi(s)$ in the sense that no additional assumption of the characteristic function at $s = 0$ is required in contrast to the original equation.

Multidimensional (matrix) generalizations of Chandrasekhar's H-equation can be found in [13] and the references therein. In this paper we study the existence of solutions of certain functional integral equations (so, possibly containing delays) which contain as particular cases many important integral and functional equations, for example: the nonlinear Volterra integral equation, and the integral equation of Chandrasekhar which gives rise to solutions expressible in terms of Chandrasekhar's H-functions (see [7] for more details). The main tool used in our research is the fixed-point theorem for the product of two operators which satisfy the Darbo condition with respect to a measure of noncompactness in the Banach algebra of continuous functions on the interval $[0, a]$. Applications to the theory of radiative transfer are provided at the end of Section 3, while specific applications to other integral equations such as those mentioned above are given in Section 4.

2. Notation and auxiliary facts

We recall basic results which we will need further on. Assume that E is a real Banach space with norm $\|\cdot\|$ and zero element, 0. Denote by $B(x, r)$ the closed ball centered at x with radius r and by B_r the ball $B(0, r)$. For X a nonempty subset of E we denote by $\overline{X}$, $Conv X$ the closure and the convex closure of X, respectively. We denote the standard algebraic operations on sets by the symbols λX and $X + Y$. Finally, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

Definition 2.1 ([3])**.** A function $\mu : \mathfrak{M}_E \to [0, \infty)$ is said to be a *regular measure of noncompactness* in the space E if it satisfies the following conditions:

(1) $\ker \mu = 0 \Longleftrightarrow X \in \mathfrak{N}_E$.
(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.
(3) $\mu(\overline{X}) = \mu(Conv X) = \mu(X)$.
(4) $\mu(\lambda X) = |\lambda|\mu(X)$, for $\lambda \in \mathbb{R}$.
(5) $\mu(X + Y) \le \mu(X) + \mu(Y)$.

(6) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.
(7) If $\{X_n\}_n$ is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Further facts concerning measures of noncompactness and their properties may be found in [3]. Now, let us assume that Ω is a nonempty subset of a Banach space E and $T : \Omega \to E$ is a continuous operator mapping bounded subsets of Ω to bounded ones. Moreover, let μ be a regular measure of noncompactness in E.

Definition 2.2 ([3])**.** We say that T satisfies the Darbo condition with a constant Q with respect to a measure of noncompactness μ provided

$$\mu(TX) \le Q \cdot \mu(X)$$

for each $X \in \mathfrak{M}_E$ such that $X \subset \Omega$.

If $Q < 1$, then T is called a *contraction* with respect to the measure μ (always assumed to be a measure of noncompactness in the sequel).

For our purposes we will need the following fixed point theorem [3].

Lemma 2.3. *Let N be a nonempty, bounded, closed, convex subset of the Banach space E and let $T : N \to N$ be a contraction with respect to a measure μ. Then T has a fixed point in the set N.*

In what follows we will work in the classical Banach space $C[0, a]$ consisting of all real functions defined and continuous on the interval $[0, a]$.This space is equipped with the standard (uniform) norm

$$\|x\| = \max\{|x(t)| :\ t \in [0, a]\}.$$

Obviously, the space $C[0, a]$ also has the structure of a Banach algebra. Now we present the definition of a special measure of noncompactness in $C[0, a]$ which will be used in the sequel, a measure that was introduced and studied in [3]. To do this let us fix a nonempty bounded subset X of $C[0, a]$. For $\varepsilon > 0$ and $x \in X$ denote by $w(x, \varepsilon)$ the *modulus of continuity* of x defined by

$$w(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, a],\ |t - s| \le \varepsilon\}$$

Further, let us put

$$w(X, \varepsilon) = \sup\{w(x, \varepsilon) : x \in X\}$$
$$w_0(X) = \lim_{\varepsilon\to 0} w(X, \varepsilon),$$

It can be shown (see [4]) that the function $\mu(X) = w_0(X)$ is a regular measure of noncompactness in the space $C[0, a]$. Moreover, the following theorem ([4]) holds, a result which is essential in the proof of our main result.

Lemma 2.4. *Assume that Ω is a nonempty, bounded, convex, closed subset of $C[0, a]$ and the operators F and G transform continuously the set Ω into $C[0, a]$ in such a way that $F(\Omega)$ and $G(\Omega)$ are bounded. Moreover, assume that the operator $T = F \cdot G$ transforms Ω into itself. If the operators F and G each satisfy the Darbo condition on the set Ω (with respect to the measure of noncompactness w_0) with constant Q_1 and Q_2, respectively, then the operator T satisfies the Darbo condition on Ω with the constant*

$$\|F(\Omega)\|Q_2 + \|G(\Omega)\|Q_1.$$

In particular, if $\|F(\Omega)\|Q_2 + \|G(\Omega)\|Q_1 < 1$ *then* T *is a contraction with respect to* w_0 *and so has at least one fixed point in* Ω.

3. Main Result

In this section, we will study the solvability of the functional-integral equation

$$x(t) = f\Big(t, \int_0^t v(t,s,x(s))\,ds,\ x(\alpha(t))\Big) \cdot g\Big(t, \int_0^a x(t)\,u(t,s,x(s))\,ds,\ x(\beta(t))\Big), \tag{3.1}$$

for $x \in C[0,a]$. The methods used will be shown to be sufficiently general to allow applications to complex functional integral equations that include Chandrasekhar's H-functions as solutions (see [7, Chapter 5]).

In what follows we will assume that the functions involved in (3.1) verify the following conditions:

(i) $f, g : [0,a]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous and there exist nonnegative constants c_1, c_2, d_1, d_2 such that

$$|f(t,0,x)| \le c_1 + d_1|x|$$
$$|g(t,0,x)| \le c_2 + d_2|x|$$

(ii) The functions $f(t,y,x), g(t,y,x)$ satisfy a Lipschitz condition with respect to the variables y and x with constants $k, k' \ge 0$ respectively, i.e.,

$$|f(t,y_1,x) - f(t,y_2,x)| \le k|y_1 - y_2|$$
$$|g(t,y_1,x) - g(t,y_2,x)| \le k|y_1 - y_2|,$$

for all $t \in [0,a]$, and $y_1, y_2, x \in \mathbb{R}$, and

$$|f(t,y,x_1) - f(t,y,x_2)| \le k'|x_1 - x_2|$$
$$|g(t,y,x_1) - g(t,y,x_2)| \le k'|x_1 - x_2|,$$

for all $t \in [0,a]$ and $x_1, x_2, y \in \mathbb{R}$.

(iii) $u, v : [0,a]\times[0,a]\times\mathbb{R} \to \mathbb{R}$ are continuous.

(iv) $\alpha, \beta : [0,a] \to [0,a]$ are continuous and satisfy,

$$|\alpha(t_1) - \alpha(t_2)| \le |t_1 - t_2|,$$
$$|\beta(t_1) - \beta(t_2)| \le |t_1 - t_2|,$$

for all $t_1, t_2 \in [0,a]$.

(v) (Sublinear nonlinearity) There exist nonnegative constants α_1, β_1 ,α_2 and β_2 such that

$$|v(t,s,x)| \le \alpha_1 + \beta_1|x|, \quad |u(t,s,x)| \le \alpha_2 + \beta_2|x|.$$

for all $t, s \in [0,a]$ and $x \in \mathbb{R}$.

(vi) The inequality

$$\big[k(\tilde\alpha + \tilde\beta r)\cdot a + (c+dr)\big]\big[k(\tilde\alpha + \tilde\beta r)\cdot r\cdot a + (c+dr)\big] \le r$$

has a positive solution r_0, where $\tilde\alpha = \max\{\alpha_1,\alpha_2\}, \tilde\beta = \max\{\beta_1,\beta_2\}, c = \max\{c_1,c_2\}$ and $d = \max\{d_1,d_2\}$.

(vii)

$$k'\big[k(\tilde\alpha + \tilde\beta r_0)\cdot a\cdot(1+r_0) + 2(c+dr_0)\big] < 1.$$

On this basis we have the following result.

Theorem 3.1. *Under the tacit assumptions (i)-(vii) above, the functional-integral equation*

$$x(t) = f\Big(t, \int_0^t v(t,s,x(s))\,ds, x(\alpha(t))\Big) \cdot g\Big(t, \int_0^a x(t)\,u(t,s,x(s))\,ds,\ x(\beta(t))\Big), \tag{3.2}$$

has at least one solution $x \in C[0,a]$.

Remark: Assumption (v) is essentially a *sublinear nonlinearity* assumption on the kernels u, v appearing in (3.1). I n order to handle a quadratic type of nonlinearity as can occur in, say, the integral equation of Chandrasekhar (see [7])

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \varphi(s)\,x(s)\,ds \tag{3.3}$$

we need to show that our technique can be used so as to include this class of important integral equations.

We note that usually the existence of solutions of (3.3) is derived under the additional assumption that the so-called *characteristic* function φ appearing in (3.3) is an even polynomial in s, (cf., [7, Chapter 5]). For such characteristic functions it is known that the resulting solutions can be expressed in terms of Chandrasekhar's H-functions [7, Chapters 4 & 5].

In our case, we derive the existence of solutions of this equation (3.3) under the much weaker assumption of continuity of φ along with $\varphi(0) = 0$. The condition $\varphi(0) = 0$ is actually physically meaningful in some cases of radiative transfer (see [[7], p.102, eq.(74)]). In this context, there does remain an interesting question, that is, in the case of a general characteristic function, can the solutions we obtain be expressed as an infinite linear combination of classical H-functions?

Proof. To prove this result using Lemma 2.4 as our main tool, we need to define operators F and G on the space $C[0,a]$ in the following way:

$$(Fx)(t) = f\Big(t,\ \int_0^t v(t,s,x(s))\,ds,\ x(\alpha(t))\Big),$$
$$(Gx)(t) = g\Big(t,\ \int_0^a x(t)\,u(t,s,x(s))\,ds,\ x(\beta(t))\Big).$$

Next, we prove that the operators F and G transform the space $C[0,a]$ into itself. To this end we are going to prove that F, G are compositions of continuous functions defined on $[0,a]$; that is, the operator F can be expressed as the composition of the following functions:

$$\begin{array}{ccccc} [0,a] & \overset{Id\times\int v\times(x\circ\alpha)}{\rightarrow} & [0,a]\times\mathbb{R}\times\mathbb{R} & \overset{f}{\rightarrow} & \mathbb{R} \\ t & \longmapsto & \Big(t, \int_0^t v(t,s,x(s))\,ds,\ x(\alpha(t))\Big) & \longmapsto & Fx(t) \end{array}$$

Now, taking into account assumptions (i), (iii) and (iv) it follows that above functions are continuous, and therefore F transforms the Banach algebra $C[0,a]$ into itself. Similarly, one can prove that the operator G transforms $C[0,a]$ into itself.

The required operator T on $C[0,a]$ is defined by setting

$$Tx = (Fx)\cdot(Gx).$$

Obviously, T transforms $C[0,a]$ into itself. Also using assumptions (ii), (iv) and (v) we get that for every $t\in[0,a]$,

$$
\begin{aligned}
|Fx(t)| &= \Big|f\Big(t,\int_0^t v(t,s,x(s))\,ds,\ x(\alpha(t))\Big)\Big| \\
&\le \Big|f\Big(t,\int_0^t v(t,s,x(s))\,ds,\ x(\alpha(t))\Big)-f(t,0,x(\alpha(t)))\Big|+|f(t,0,x(\alpha(t)))| \\
&\le k\Big|\int_0^t v(t,s,x(s))\,ds\Big|+c_1+d_1|x(\alpha(t))| \\
&\le k(\alpha_1+\beta_1\|x\|)\cdot a+(c_1+d_1\|x\|).
\end{aligned}
\tag{3.4}
$$

On the other hand, by (ii), (iv), and (v) again, we have

$$
\begin{aligned}
|Gx(t)| &= \Big|g\Big(t,\int_0^a x(t)\,u(t,s,x(s))\,ds,\ x(\beta(t))\Big)\Big| \\
&\le \Big|g\Big(t,\int_0^a x(t)\,u(t,s,x(s))\,ds,\ x(\beta(t))\Big)-g(t,0,x(\beta(t)))\Big| \\
&\quad +|g(t,0,x(\beta(t)))| \\
&\le k\Big|\int_0^a x(t)\,u(t,s,x(s))\,ds\Big|+c_2+d_2|x(\beta(t))| \\
&\le k\|x\|(\alpha_2+\beta_2\|x\|)\cdot a+(c_2+d_2\|x\|).
\end{aligned}
\tag{3.5}
$$

Linking (3.4) and (3.5) we obtain

$$
\begin{aligned}
&|Tx(t)| \\
&=|Fx(t)|\cdot|Gx(t)| \\
&\le \big[k\,(\alpha_1+\beta_1\|x\|)\cdot a+(c_1+d_1\|x\|)\big]\big[k\,\|x\|\,(\alpha_2+\beta_2\|x\|)\cdot a+(c_2+d_2\|x\|)\big].
\end{aligned}
$$

Hence,

$$
\|Tx\|\le\Big[k(\tilde\alpha+\tilde\beta\|x\|)\,a+(c+d\|x\|)\Big]\Big[k\,\|x\|\,(\tilde\alpha+\tilde\beta\|x\|)\,a+(c+d\|x\|)\Big]
$$

Taking into account assumption (vi) we deduce that the operator T maps the ball $B_{r_0}\subset C[0,a]$ into itself.

Next, we show that the operator F is continuous on B_{r_0}. To do this fix $\varepsilon>0$ and take $x,y\in B_{r_0}$ such that $\|x-y\|\le\varepsilon$. Then, for $t\in[0,a]$ we get

$$
\begin{aligned}
&|Fx(t)-Fy(t)| \\
&=\Big|f\Big(t,\int_0^t v(t,s,x(s))\,ds,\ x(\alpha(t))\Big)-f\Big(t,\int_0^t v(t,s,y(s))\,ds,y(\alpha(t))\Big)\Big| \\
&\le\Big|f\Big(t,\int_0^t v(t,s,x(s))\,ds,\ x(\alpha(t))\Big)-f\Big(t,\int_0^t v(t,s,y(s))\,ds,x(\alpha(t))\Big)\Big| \\
&\quad+\Big|f\Big(t,\int_0^t v(t,s,y(s))\,ds,\ x(\alpha(t))\Big)-f\Big(t,\int_0^t v(t,s,y(s))\,ds,y(\alpha(t))\Big)\Big| \\
&\le k\int_0^t|v(t,s,x(s))-v(t,s,y(s))|ds+k'|x(\alpha(t))-y(\alpha(t))| \\
&\le k\cdot w(v,\varepsilon)\cdot a+k'\|x-y\| \\
&\le k\cdot w(v,\varepsilon)\cdot a+k'\varepsilon,
\end{aligned}
$$

where

$$w(v,\varepsilon)=\sup\{|v(t,s,x_1)-v(t,s,x_2)|:t,s\in[0,a],x_1,x_2\in[-r_0,r_0],|x_1-x_2|\le\varepsilon\}.$$

Using the fact that the function v is uniformly continuous on the bounded subset $[0,a]\times[0,a]\times[-r_0,r_0]$, we infer that $w(v,\varepsilon)\to 0$ as $\varepsilon\to 0$. Thus, the above estimate shows that the operator F is continuous on B_{r_0}. Similarly, one can infer that the operator G is continuous on B_{r_0} and consequently deduce T is a continuous operator on B_{r_0}.

Now, we prove that the operators F and G satisfy the Darbo condition with respect to the measure w_0, defined in Section 2, in the ball B_{r_0}. Take a nonempty subset X of B_{r_0} and $x\in X$. Then, for a fixed $\varepsilon>0$ and $t_1,t_2\in[0,a]$ such that $t_1\le t_2$ and $t_2-t_1\le\varepsilon$, we obtain

$$\begin{aligned}
&|Fx(t_2)-Fx(t_1)|\\
&=\Big|f\Big(t_2,\int_0^{t_2}v(t_2,s,x(s))\,ds,\,x(\alpha(t_2))\Big)-f\Big(t_1,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_1))\Big)\Big|\\
&\le\Big|f\Big(t_2,\int_0^{t_2}v(t_2,s,x(s))\,ds,\,x(\alpha(t_2))\Big)-f\Big(t_2,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_2))\Big)\Big|\\
&\quad+\Big|f\Big(t_2,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_2))\Big)-f\Big(t_1,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_1))\Big)\Big|\\
&\le k\Big|\int_0^{t_2}v(t_2,s,x(s))\,ds-\int_0^{t_1}v(t_1,s,\,x(s))\,ds\Big|\\
&\quad+\Big|f\Big(t_2,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_2))\Big)-f\Big(t_1,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_2))\Big)\Big|\\
&\quad+\Big|f\Big(t_1,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_2))\Big)-f\Big(t_1,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_1))\Big)\Big|\\
&\le k\Big[\int_0^{t_1}|v(t_2,s,x(s))-v(t_1,s,x(s))\,ds|+\int_{t_1}^{t_2}|v(t_2,s,x(s))|ds\Big]\\
&\quad+\Big|f\Big(t_2,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_2))\Big)-f\Big(t_1,\int_0^{t_1}v(t_1,s,x(s))\,ds,\,x(\alpha(t_2))\Big)\Big|\\
&\quad+k'|x(\alpha(t_2))-x(\alpha(t_1))|
\end{aligned}\tag{3.6}$$

At this point we introduce the notation:

$$\begin{gathered}
w_v(\varepsilon,\cdot,\cdot)=\sup\big\{|v(t,s,x)-v(t',s,x)|:t,t',s\in[0,a],|t-t'|\le\varepsilon\\
\text{and }x\in[-r_0,r_0]\big\},\\
L=\sup\big\{|v(t,s,x)|:\ t,s\in[0,a],x\in[-r_0,r_0]\big\},\\
w_f(\varepsilon,\cdot,\cdot)=\sup\big\{|f(t,x,y)-f(t',x,y)|:t,t'\in[0,a],|t-t'|\le\varepsilon,\\
x\in[-L\,r_0\,a,L\,r_0\,a],y\in[-r_0,r_0]\big\}.
\end{gathered}$$

Then, using (3.6) we obtain the estimate

$$|Fx(t_2)-Fx(t_1)|\le k\cdot[w_v(\varepsilon,\cdot,\cdot)\cdot a+L\cdot\varepsilon]+w_f(\varepsilon,\cdot,\cdot)+k'|x(\alpha(t_2))-x(\alpha(t_1))|.$$

Now, assumption (iv) allows us to deduce

$$w(Fx,\varepsilon)\le k\cdot[w_v(\varepsilon,\cdot,\cdot)\cdot a+L\cdot\varepsilon]+w_f(\varepsilon,\cdot,\cdot)+k'w(x,\varepsilon).$$

Thus, taking the supremum in X, then the limit as $\varepsilon \to 0$, and taking into account the uniform continuity of the functions f and v in bounded sets, we can deduce that

$$w_0(FX) \leq k' w_0(X). \tag{3.7}$$

Similarly, one can prove that

$$w_0(GX) \leq k' w_0(X). \tag{3.8}$$

Finally, linking (3.4)-(3.5), (3.7)-(3.8) and keeping in mind Lemma 2.4, we infer that the operator T satisfies the Darbo condition on B_{r_0} with respect to the measure w_0 with constant

$$Q = k'\Big[k(\tilde{\alpha} + \tilde{\beta} r_0) \cdot a \cdot (1 + r_0) + 2(c + dr_0)\Big],$$

(see assumption (vii)). Moreover, from assumption (vii) we deduce that the operator T is a contraction on B_{r_0}. Therefore, applying Darbo's theorem we get that T has at least one fixed point in B_{r_0}. Consequently, the functional-integral equation (3.1) has at least one solution in B_{r_0}. This completes the proof. □

Remark: Moreover, in going through the estimates leading to a solution of (4.1) we note that, in actuality, for this specific choice of f and g condition (vi) can be relaxed to

$$k\|\varphi\| a r^2 + c \leq r,$$

which, of course, shows that Chandrasekhar's equation has a real continuous solution in this setting where $k = 1$ and $c = 1$ provided the characteristic function φ and the interval $[0, a]$ are related by the inequality

$$4\|\varphi\| a < 1. \tag{3.9}$$

In [[7], Section 38, Corollary 1] Chandrasekhar proves that a necessary condition for a solution of equation (3.3) to be real is that, in the case $a = 1$, we have

$$\int_0^1 \varphi(s)\, ds \leq \frac{1}{2}. \tag{3.10}$$

But we have shown above that if (3.9) holds then the solution of (3.3) that we found must be *real* and continuous. Under assumption (3.9), however, it is easy to see that, when $a = 1$,

$$\int_0^1 \varphi(s)\, ds \leq \|\varphi\| < \frac{1}{4}.$$

This result is consistent with the stated one in (3.10) for the existence of a real solution. Indeed, when $a = 1$ it is easy to see that $r_0 = 1/\sqrt{\|\varphi\|}$ is a solution of the inequality $\|\varphi\| r^2 + 1 \leq r$. Since the fixed point of our operator T (i.e., our solution $x(t)$) must lie in the ball with radius r_0, it follows that our solution(s) lie in this ball, and so there holds the *a priori* estimate

$$\|x\| \leq \frac{1}{\sqrt{\|\varphi\|}}.$$

Such a result, in the case of a general characteristic function, does not appear in the literature (nor in [7]), and so is new. We therefore state this as a separate result.

Theorem 3.2. *Any solution $x(t)$ of Chandrasekhar's integral equation* (4.1) *on* $[0, a]$ *necessarily satisfies*

$$\|x\| \leq \frac{1}{\sqrt{\|\varphi\|}} \tag{3.11}$$

for any choice of the characteristic function $\varphi(t)$ subject to it only being continuous on $[0, a]$ and $\varphi(0) = 0$.

The bound on the right of (3.11) can likely be improved for specific classes of characteristic functions. Finally, we present an additional concrete example of a functional-integral equation where all the functions involved in the equation satisfy our conditions.

4. Applications

In this section we present some examples of classical integral and functional equations considered in nonlinear analysis which are particular cases of equation (3.1) and consequently, the existence of their solutions can be established using Theorem 3.1.

Example 4.1. First we note that equation (1) concerns the well-known functional equation of the first order with a possible delay of the form

$$x(t) = f_1\left(t,\, x(\alpha(t))\right),$$

see [11]. To obtain this example it is sufficient to put $f(t, y, x) = f_1(t, x)$ and $g(t, y, x) = 1$.

Example 4.2. Next, setting $g(t, y, x) \equiv 1$ and $f(t, y, x) = a(t) + y$, equation (3.1) reduces to the well-known nonlinear Volterra integral equation

$$x(t) = a(t) + \int_0^t v(t, s, x(s))\, ds.$$

Example 4.3. On the other hand, if we choose $f(t, y, x) \equiv 1$, $g(t, y, x) = 1 + y$, $u(t, s, y) = \frac{t}{t+s}\varphi(s)y$, and $\beta(t) = t$ in Theorem 3.1, equation (3.2) now takes the form

$$x(t) = 1 + x(t) \int_0^a \frac{t}{t+s}\varphi(s)x(s)\, ds, \tag{4.1}$$

and this is the famous quadratic integral equation of Chandrasekhar discussed above and considered in many papers and monographs (e.g., [2, 7]).

Remark. Applying our technique to the specific equation (4.1) we see that in order for all the assumptions (i)-(vii) to be satisfied in Theorem 3.1 we only need to impose the additional condition that the characteristic function φ defined in (3.2) is continuous and satisfies $\varphi(0) = 0$. This previous condition will ensure that the kernel $u(t, s, x)$ defined by

$$u(t, s, x) = \begin{cases} 0, & s = 0, t \geq 0, x \in \mathbb{R} \\ \frac{t}{t+s}\varphi(s)x, & s \neq 0, t \geq 0, x \in \mathbb{R} \end{cases}$$

is continuous on $[0, a] \times [0, a] \times \mathbb{R}$ in accordance with assumption (iii). To see this let $\varphi(0) = 0$ along with $u(0, 0, x) = 0$. Since φ is continuous at $s = 0$, given $\varepsilon > 0$ we can choose $\delta_1 > 0$ so small that $|\varphi(s)| < \sqrt{\varepsilon}$ whenever $|s| < \delta_1$. Next, let (t, s, x) be such that $\sqrt{t^2 + s^2 + x^2} < \delta_1$. Then $|u(t, s, x)| \leq |\varphi(s)||x| < \sqrt{\varepsilon}\delta_1 < \varepsilon$ provided we choose $\delta_1 < \sqrt{\varepsilon}$. Thus $u(t,s,x)$ is continuous at $(0,0,0)$, and clearly at every other point in $[0,a] \times [0,a] \times \mathbb{R}$.

Example 4.4. In addition, setting $f(t,x,y) \equiv 1$ and $g(t,y,x) = b(t) + y + x$, we obtain existence results for the functional integral equation

$$x(t) = b(t) + k\,x(ct) + \int_0^a x(t)\,u(t,s,x(s))\,ds, \tag{4.2}$$

where $k \in \mathbb{R}$ and $0 \le c \le 1$ are constants. This is an equation that includes the modified equation of radiative transfer (1.1) since we can fix the function $b(t) = 1/(1-2\psi_0)$, to be a constant function and $k = 0$. In this case, $u(t,s,y) = -s\psi(s)y/(t+s)$, and this function is automatically continuous at $s = 0$. Thus, it is not necessary to assume anything about the value of $\psi(x)$ at $x = 0$, in contrast with Example 4.3 and the arguments in the Remark above concerning equation (3.3).

Example 4.5. Let us take $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(t,y,x) = \frac{1}{9}y + \frac{1}{10}\sin x$ and $g(t,y,x) \equiv 1$. It is easy to prove that these functions are continuous and satisfy hypothesis (i) with $c_1 = \frac{1}{10}$, $d_1 = 0$, $c_2 = 1$ and $d_2 = 0$. In this case $c = \max\{c_1, c_2\} = 1$ and $d = \max\{d_1, d_2\} = 0$.

Also the functions f and g verify the Lipschitz condition with respect to the variables y and x with constants $k = \frac{1}{9}$ and $k' = \frac{1}{10}$, respectively. On the other hand, we define the continuous functions $v(t,s,x) = t \cdot s \arctan x$ and $u(t,s,x) \equiv 0$. It is clear that

$$|v(t,s,x)| \le |\arctan x| \le |x|$$

then v satisfies assumption (v) with $\alpha_1 = 0$ and $\beta_1 = 1$. Moreover, it is obvious that u satisfies the same hypothesis with $\alpha_2 = 0$ and $\beta_2 = 0$. Consequently, $\tilde{\alpha} = \max\{\alpha_1, \alpha_2\} = 0$ and $\tilde{\beta} = \max\{\beta_1, \beta_2\} = 1$.

Next, we take $\alpha(t) = 1/(1+t)$ and $\beta(t) = t$, each of which satisfies assumption (v). Taking into account the above estimates we obtain that the inequality of hypothesis (vi) has the form

$$\big(\frac{r}{9} + 1\big)\big(\frac{r^2}{9} + 1\big) \le r.$$

However, it is easy to see that there is a root r_0 of this inequality with $r_0 \in (0,3)$. For this value of r_0, we have that assumption (vii) is satisfied. Now taking into account all the functions defined previously, the functional-integral equation is

$$x(t) = \frac{t}{9}\int_0^t s \arctan x(s)\,ds + \frac{1}{10}\sin\big(\frac{1}{1+t}\big).$$

Applying the result obtained in Theorem 3.1, we deduce that this equation has at least one solution in $B_{r_0} \subset C[0,a]$.

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Josefa Caballero
Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain
E-mail address: jmena@ull.es

Angelo B. Mingarelli
School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, K1S 5B6, Canada
E-mail address: amingare@math.carleton.ca

Kishin Sadarangani
Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain
E-mail address: ksadaran@dma.ulpgc.es