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RESONANCES GENERATED BY ANALYTIC SINGULARITIES ON THE DENSITY OF STATES MEASURE FOR PERTURBED PERIODIC SCHRÖDINGER OPERATORS

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Abstract. We consider a perturbation of a periodic Shrödinger operator P_0 by a potential $W(hx)$, $(h \searrow 0)$. We study singularities of the density of states measure and we obtain lower bound for the counting function of resonances.

1. Introduction

In this paper we present a lower bound for the counting function of resonances for the perturbed periodic Shrödinger operator

$$P(h) = P_0 + W(hy) \ , \ P_0 = -\triangle + V \quad (h \searrow 0).$$

Here V is C^∞ function, real valued and Γ-periodic with respect to a lattice $\Gamma = \oplus_{i=1}^n \mathbb{Z} e_i$ in $\mathbb{R}^n$. The potential W is real valued and satisfies the hypothesis

(H1) There exist positive constants a and C such that W extends analytically to

$$\Gamma(a) := \{z \in \mathbf{C}^n : |\Im(z)| \leq a\langle \Re(z)\rangle\}$$

and

$$|W(z)| \leq C\langle z\rangle^{-N}, \text{ uniformly on } z \in \Gamma(a),\ N > n, \tag{1.1}$$

where $\langle z\rangle = (1+|z|^2)^{1/2}$. Here $\Re(z)$, $\Im(z)$ denote respectively the real part and the imaginary part of z.

Let $\Gamma^* = \oplus_{i=1}^n \mathbf{Z} e_i^*$ be the dual lattice of Γ, where $\{e_j^*\}_{j=1}^n$ is the basis satisfying $(e_j, e_k^*) = 2\pi\delta_{jk}$. Set $E = \{x = \sum_{j=1}^n t_j e_j,\ t_j \in [-1/2, 1/2[\}$, and $E^* = \{x = \sum_{j=1}^n t_j e_j^*,\ t_j \in [-1/2, 1/2[\}$. We use the usual flat metrics on $\mathbf{T} := \mathbb{R}^n/\Gamma$ and $\mathbf{T}^* := \mathbb{R}^n/\Gamma^*$, when we integrate or do local considerations we identify $\mathbf{T}$ (resp. $\mathbf{T}^*$) with E (resp. E^*).

For $k \in \mathbb{R}^n$, we define the operator P_k on $L^2(\mathbf{T})$ by

$$P_k := (D_y + k)^2 + V(y).$$

Let $\lambda_1(k) \leq \lambda_2(k) \leq \dots$ be the Floquet eigenvalues of P_k (enumerated according to their multiplicities). It is well known (see [4]) that $\lambda_p(k)$ are continuous functions

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of k for any fixed p. Moreover $\lambda_p(k)$ is an analytic function in k near any point $k_0 \in T^*$, where $\lambda_p(k_0)$ is a simple eigenvalue of P_{k_0}.

We consider the self-adjoint operator $P_0 = -\Delta + V(y)$ on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$. By Bloch-Floquet theory, it is known that

$$\sigma(P_0) = \sigma_{\rm ac}(P_0) = \cup_{p\geq 1}\Lambda_p, \quad \text{where } \Lambda_p = \lambda_p(\mathbf{T}^*).$$

Let us introduce the density of states measure

$$\rho(\lambda) := \frac{1}{(2\pi)^n}\sum_{p\geq 1}\int_{\{k\in E^*;\ \lambda_p(k)\leq\lambda\}} dk. \tag{1.2}$$

Since the spectrum of P_0 is absolutely continuous, the measure ρ is absolutely continuous with respect to the Lebesgue measure $d\lambda$. Therefore, the density of states $\frac{d\rho}{d\lambda}$ of P_0, is locally integrable.

For $f \in C_0^\infty(\mathbf{R})$, we set

$$\langle \mu, f\rangle = \int [f(W(x)) - f(0)]dx, \tag{1.3}$$

$$\langle \omega, f\rangle = \frac{1}{(2\pi)^n}\sum_j \int_{E^*}\int_{\mathbb{R}^n_x} [f(W(x)+\lambda_j(k)) - f(\lambda_j(k))]dx\, dk, \tag{1.4}$$

Proposition 1.1 ([1])**.** *The functionals operators ω and μ are distributions of order ≤ 1. Moreover, in $\mathcal{D}'(\mathbb{R})$, we have*

$$\omega = d\rho * \mu. \tag{1.5}$$

Definition 1.2. We say that $\lambda \in \sigma(P_0)$ is a simple energy level if it is a simple eigenvalue of P_k, for every $k \in F(\lambda) := \{k \in \mathbf{T}^*; \lambda \in \sigma(P_k)\}$.

We use also the following hypothesis

(H2) There exists an open bounded interval I such that for all $\lambda \in I$ and all $k_0 \in \mathbb{R}^n/\Gamma^*$ with $\lambda_p(k_0) = \lambda$, the eigenvalue $\lambda_p(k_0)$ is simple and $d_k\lambda_p(k_0) \neq 0$.

We use $\operatorname{sing\,supp}_a(\omega)$ for analytic singular support of ω. Under assumptions (H1) and (H2) in [2] it was proved that if $E \in \operatorname{sing\,supp}_a(\omega) \cap I$ then for every h-independent complex neighborhood Ω of E, there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ large enough such that for $h \in]0, h_0[$,

$$\#\{z \in \Omega; z \in \operatorname{Res} P(h)\} \geq C_\Omega h^{-n}.$$

This result is based on the trace formula in the periodic case [2, 5].

Since (1.5) for ω, the analytic singular support of ω depends on both $\operatorname{sing\,supp}_a(\mu)$ and $\operatorname{sing\,supp}_a(d\rho)$. The question is to find some criteria to determine if $e_0 = \lambda_p(k_0)$ belongs to the $\operatorname{sing\,supp}_a(d\rho)$.

If $e_0 = \lambda_p(k_0)$ is a simple eigenvalue in a neighborhood of k_0 then $\lambda_p(k)$ is a smooth function there. Moreover if e_0 is non critical then e_0 is not in the analytic singular support of ρ (see Lemma 2.1).

The distribution ρ can be singular for a variety of reasons. If $e_0 = \lambda_p(k_0)$ is a critical value, we expect in general that e_0 will belong to the analytic singular support of ρ. Multiple eigenvalues can also give rise to analytic singularities of ρ. We recall that, the case when $e_0 = \lambda_p(k_0)$ is a non-degenerate extremum was studied by Dimassi and Mnif in [1]. They studied also the case of bands crossing when $n = 2$.

In this paper we are interested to more general situations. We first study the case when $e_0 = \lambda_p(k_0)$ is a non-degenerate critical point and we prove that in this situation e_0 belongs to the analytic singular support of ρ. We note that this result generalizes the case when e_0 is a non-degenerate extremum point established in [1, Theorem 1]. In the case when e_0 is a degenerate critical point one gives a positive answer to the question if e_0 is an extremum. This result encloses the case of finite number of extremum at the same level. Finally we look for resonances near a singularity of ρ generated by bands crossing at e_0. This study is devoted to the case $n = 3$.

The paper is presented as follows: Section 2: Lower bound of the number of resonances near a critical non-degenerate point. Section 3: Lower bound of the number of resonances near a degenerate critical point. Section 4: Lower bound of the number of resonances near a conic singularity of the density of states.

2. Lower bound of the number of resonances near a critical non-degenerate point

Let O be an open bounded set in $\mathbb{R}^n$ with analytic boundary almost every where, and let U be a complex neighborhood of O. Let $x \to \varphi(x)$ be analytic on U and real valued for all x in O. Let us introduce the real function

$$I(e) := \int_{\{x\in O,\,\varphi(x)\le e\}} dx.$$

Lemma 2.1 ([3]). *If $\nabla\varphi(x) \neq 0$ near every $x \in \Sigma_{e_0} := \{x \in O : \varphi(x) = e_0\}$ and if the sets ∂O and Σ_{e_0} intersect transversely, then $I(e)$ is analytic near e_0.*

The next lemma generalizes the result in [1, Lemma 2], where the authors consider the case of non-degenerate extremum.

Lemma 2.2. *If φ has a non-degenerate critical point at x_0 with $\varphi(x_0) = e_0$ and if $\nabla\varphi(x) \neq 0$ for all $x \in \Sigma_{e_0}\backslash\{x_0\}$, then there exists an open interval J neighborhood of e_0, such that $I(e)$ is analytic on $J\backslash\{e_0\}$ and has a C^2 singularity at e_0.*

Proof. Under the assumption $\nabla\varphi(x) \neq 0$ for all $x \in \Sigma_{e_0}\backslash\{x_0\}$ and since φ has a non-degenerate critical point at x_0 there exists an open interval J neighborhood of e_0 such that for all $e \in J\backslash\{e_0\}$ we have $\nabla\varphi(x) \neq 0$ near every $x \in \Sigma_e := \{x \in O : \varphi(x) = e\}$. Hence by Lemma 2.1 $I(e)$ is analytic on $J\backslash\{e_0\}$. One now studies the behavior of I at e_0. Let $(k, n-k)$ be the signature of the hessian form of φ at x_0. The case $k = 0$ or $k = n$ corresponds to e_0 non-degenerate extremum which is studied in [1]. Here we focus our study on the case of saddle point. By Morse lemma, for all $\epsilon > 0$ small enough, there exist a neighborhood Ω of x_0 and a local analytic diffeomorphism $D : \Omega \to B(0,\epsilon)$ such that

$$I_\epsilon(e) := \int_{\{x\in\Omega,\,\varphi(x)\le e\}} dx = \int_{\{x\in B(0,\epsilon),\,\sum_{i=1}^k x_i^2-\sum_{i=k+1}^n x_i^2\le e-e_0\}} \mathrm{Jac}(D^{-1}(x))dx.$$

We introduce the notation: $x = (X_+, X_-)$ with $X_+ = (x_1, \dots, x_k)$ and $X_- = (x_{k+1}, \dots, x_n)$. $B_{k,\epsilon} = \{X \in \mathbb{R}^k : \|X\| < \epsilon\}$.

Up to an analytic correction of $I_\epsilon(e)$, we can suppose that

$$I_\epsilon(e) = \int_{\{x=(X_+,X_-)\in B_{k,\epsilon}\times B_{n-k,\epsilon},\,\sum_{i=1}^k x_i^2-\sum_{i=k+1}^n x_i^2\le e-e_0\}} \mathrm{Jac}(D^{-1}(x))dx.$$

Let $x = \epsilon y$ and $E = (e - e_0)/\epsilon^2$, we have

$$\begin{aligned} I_\epsilon(e) &= \epsilon^n J(\epsilon, E) \\ &:= \epsilon^n \int_{\{y=(Y_+,Y_-)\in B_{k,1}\times B_{n-k,1},\, \sum_{i=1}^k y_i^2 - \sum_{i=k+1}^n y_i^2 \le E\}} \mathrm{Jac}(D^{-1}(\epsilon y))dy. \end{aligned}$$

To prove that I_ϵ has a C^2 singularity at e_0 we prove that $J(\epsilon, .)$ has a C^2 singularity at $E = 0$. On the other hand, we can see that for E small enough, $J(., E)$ is analytic near $\epsilon = 0$. Therefore, it is sufficient to prove that $E = 0$ is a C^2 singularity for $J(0, .)$. We have

$$J(0, E) = \frac{2^{n/2}}{\sqrt{|\det(\mathrm{Hess}(\varphi)(x_0)|}} \int_{\{y=(Y_+,Y_-)\in B_{k,1}\times B_{n-k,1},\, \sum_{i=1}^k y_i^2 - \sum_{i=k+1}^n y_i^2 \le E\}} dy.$$

Using polar coordinates we get

$$J(0, E) = C_n \int_{\{0\le r_1\le 1,\, 0\le r_2\le 1,\, r_1^2 - r_2^2 \le E\}} r_1^{k-1} r_2^{n-k-1} dr_1 dr_2,$$

where

$$C_n = \frac{2^{\frac{n}{2}}\, \mathrm{Vol}(S^{k-1})\, \mathrm{Vol}(S^{n-k-1})}{\sqrt{|\det(\mathrm{Hess}(\varphi)(x_0)|}}$$

For $E > 0$,

$$\begin{aligned} J(0, E) &:= f_r(E) \\ &= C_n\Big[\int_0^{\sqrt{E}} \int_0^1 r_1^{k-1} r_2^{n-k-1} dr_2 dr_1 + \int_{\sqrt{E}}^1 \int_{\sqrt{r_1^2 - E}}^1 r_1^{k-1} r_2^{n-k-1} dr_2 dr_1\Big] \\ &= C_n\Big[\frac{1}{k(n-k)} + \int_1^{\sqrt{E}} \frac{(r_1^2 - E)^{\frac{n-k}{2}}}{n-k} r_1^{k-1} dr_1\Big]. \end{aligned}$$

For $E < 0$, we write

$$J(0, E) := f_l(E) = C_n \int_{\{0\le r_1\le 1 \ ,\ 0\le r_2\le 1\ ;\ r_2^2 \ge r_1^2 - E\}} r_1^{k-1} r_2^{n-k-1} dr_1 dr_2\,.$$

In the same way as above we obtain

$$f_l(E) = -C_n \int_1^{\sqrt{-E}} \frac{(r_2^2 + E)^{\frac{k}{2}}}{k} r_2^{n-k-1} dr_2.$$

Computing the second derivatives, we get for $n > 4$: If $n - k \ne 2$, then

$$\frac{d^2 f_r}{dE^2}(0) = -C_n \frac{n-k-2}{4(n-4)}.$$

If $k \ne 2$, then

$$\frac{d^2 f_l}{dE^2}(0) = C_n \frac{k-2}{4(n-4)}.$$

If $n - k = 2$, then

$$\frac{d^2 f_r}{dE^2}(0) = 0 \quad \text{and} \quad \frac{d^2 f_l}{dE^2}(0) = \frac{C_n}{4}.$$

If $k = 2$, then

$$\frac{d^2 f_r}{dE^2}(0) = -\frac{C_n}{4} \quad \text{and} \quad \frac{d^2 f_l}{dE^2}(0) = 0.$$

So, for all $n > 4$, we have

$$\frac{d^2 f_r}{dE^2}(0) \neq \frac{d^2 f_l}{dE^2}(0).$$

On the other hand, for $n \leq 4$: If $n-k \neq 2$, then

$$\lim_{E \to 0^+} \frac{d^2 f_r}{dE^2}(E) = \infty\,.$$

If $k \neq 2$, then

$$\lim_{E \to 0^-} \frac{d^2 f_l}{dE^2}(E) = \infty\,.$$

If $k = 2$ and $n-k = 2$, then

$$\frac{d^2 f_r}{dE^2}(0) = -\frac{1}{4} \quad \text{and} \quad \frac{d^2 f_l}{dE^2}(0) = \frac{1}{4}.$$

Hence, for all n, $J(0,.)$ has a C^2 singularity at 0. □

The following result is a consequence of Lemma 2.1, Lemma 2.2 and the representation (1.2) of ρ.

Lemma 2.3. *Let e_0 be a simple eigenvalue of P_0. We assume that:*

(i) *There exist i_0 and k_0 such that $\lambda_{i_0}(k_0) = e_0$, $\nabla\lambda_{i_0}(k_0) = 0$.*

(ii) *$\nabla\lambda_{i_0}(k) \neq 0$, for all $k \in \lambda_{i_0}^{-1}(\{e_0\})$, $k \neq k_0$ and $\nabla\lambda_i(k) \neq 0$ for all $k \in \lambda_i^{-1}(\{e_0\})$, $i \neq i_0$.*

Then there exists an open interval J neighborhood of e_0 such that the density of states measure ρ is analytic on $J\backslash\{e_0\}$ and has a C^2 singularity at e_0.

Therefore, by [2, Theorem 1.6], we obtain the following result.

Theorem 2.4. *Let e_0 and J be as in Lemma 2.3, I satisfying (H2) and let $E \in (e_0 + \operatorname{sing\,supp_a}(\mu)) \cap I$ be such that $(E - \operatorname{supp}(\mu)) \subset J$. Then for all h-independent complex neighborhood Ω of E, there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that for $h \in]0, h_0[$,*

$$\#\{z \in \Omega; z \in \operatorname{Res} P(h)\} \geq C_\Omega h^{-n}.$$

3. Lower bound for the number of resonances near a degenerate critical point

Let K be a compact set in $\mathbb{R}^n$, we consider $C(K,\mathbb{R})$ the space of continuous real functions on K, with the norm $\|\varphi\|_\infty = \sup_{x\in K} |\varphi(x)|$. Let us introduce the real valued function $\mathcal{H}_e : C(K,\mathbb{R}) \to \mathbb{R}$,

$$\varphi \mapsto \int_{\{x \in K,\, \varphi(x) \leq e\}} dx\,.$$

Lemma 3.1. *Let $\varphi \in C(K,\mathbb{R})$ such that $\varphi^{-1}(\{e\})$ is a finite set. $\mathcal{H}_e$ is continuous at φ.*

Proof. Without loss of generality, we can take $\varphi^{-1}(\{e\})$ reduced to $\{x_0\}$. Let $\epsilon > 0$, by the continuity of φ on K and the fact that $\varphi(x) \neq e$ for all $x \in K_\epsilon = K\backslash B(x_0, \epsilon)$ which is a compact set, we have the statement:

$$\text{There exists } \alpha(\epsilon) > 0 \text{ such that } |\varphi(x) - e| > \alpha(\epsilon), \text{ for all } x \in K_\epsilon. \tag{3.1}$$

Let $\psi \in C(K,\mathbb{R})$ be such that

$$\|\varphi - \psi\|_\infty < \frac{\alpha(\epsilon)}{2}. \tag{3.2}$$

We denote:

$$\begin{aligned} K_{-,-} &= \{x \in K : \varphi(x) \le e\} \cap \{x \in K : \psi(x) \le e\} \\ K_{-,+} &= \{x \in K : \varphi(x) \le e\} \cap \{x \in K : \psi(x) > e\} \\ K_{+,-} &= \{x \in K : \varphi(x) > e\} \cap \{x \in K : \psi(x) \le e\}. \end{aligned}$$

We have:

$$\begin{aligned} \mathcal{H}_e(\varphi) &= \mathrm{Vol}(K_{-,-}) + \mathrm{Vol}(K_{-,+}) \\ \mathcal{H}_e(\psi) &= \mathrm{Vol}(K_{-,-}) + \mathrm{Vol}(K_{+,-}). \end{aligned}$$

Then

$$\mathcal{H}_e(\varphi) - \mathcal{H}_e(\psi) = \mathrm{Vol}(K_{-,+}) - \mathrm{Vol}(K_{+,-}).$$

By (3.1) and (3.2), we have

$$K_{-,+} \cap K_\epsilon = \emptyset \quad \text{and} \quad K_{+,-} \cap K_\epsilon = \emptyset,$$

hence

$$K_{-,+} \subset B(x_0,\epsilon) \quad \text{and} \quad K_{+,-} \subset B(x_0,\epsilon).$$

Then

$$\mathrm{Vol}(K_{-,+}) \le 2\epsilon \quad \text{and} \quad \mathrm{Vol}(K_{+,-}) \le 2\epsilon.$$

Finally we get $|\mathcal{H}_e(\varphi) - \mathcal{H}_e(\psi)| \le 4\epsilon$. □

Definition 3.2. Let O be an open bounded set in $\mathbb{R}^n$, and let φ a function in $C^\infty(O,\mathbb{R})$. We say that φ has an isolated local minimum (resp. maximum) of order $p \in \mathbb{N}^*$ at $x_0 \in O$, if the Taylor expansion of φ near x_0 is as follows

$$\varphi(x + x_0) = \varphi(x_0) + \sum_{i=1}^{n} \alpha_i x_i^{2p} + \sum_{\sigma\in(\mathbb{N})^n;|\sigma|=p} a_\sigma x^{2\sigma} + \mathcal{O}(|x|^{2p+1})$$

with $\alpha_i > 0$, $a_\sigma \ge 0$ (resp. $\alpha_i < 0$, $a_\sigma \le 0$). $\sigma = (\sigma_1,\dots,\sigma_n) \in (\mathbb{N})^n$, $x^{2\sigma}$ denotes $x_1^{2\sigma_1}\dots x_n^{2\sigma_n}$ and $|\sigma| = \sigma_1 + \dots + \sigma_n$.

We now return to the real valued function $I(e) := \int_{\{x\in O:\varphi(x)\le e\}} dx$ introduced in section 2. Let H denote the Heaviside function.

Lemma 3.3. *Suppose that φ has an isolated local extremum of order $p \in \mathbb{N}^*$ at x_0. If $\nabla\varphi(x) \ne 0$ for all $x \in \Sigma_{e_0}\backslash\{x_0\}$, then*

(i) *If e_0 is a minimum,*

$$I(e) = g(e - e_0) + H(e - e_0)(e - e_0)^{\frac{n}{2p}}(C + R(e)) \tag{3.3}$$

with $C > 0$, $\lim_{e\to e_0} R(e) = 0$ and g analytic function.

(ii) *If e_0 is a maximum,*

$$I(e) = g(e - e_0) + H(e_0 - e)(e_0 - e)^{\frac{n}{2p}}(C + R(e)) \tag{3.4}$$

with $C > 0$, $\lim_{e\to e_0} R(e) = 0$ and g analytic function.

Proof. (i) We note that if e_0 is a minimum for φ then there exists $\epsilon > 0$ such that $\varphi(x + x_0) \geq e_0$ for all $x \in B(0, \epsilon)$. We write

$$I(e) = \int_{\{x\in B(0,\epsilon),\, \varphi(x)\leq e\}} dx + \int_{\{x\in O\setminus B(0,\epsilon),\, \varphi(x)\leq e\}} dx.$$

By Lemma 2.1, the second term in the right-hand side is analytic near e_0. Let:

$$I_\epsilon(e) := \int_{\{x\in B(0,\epsilon),\, \varphi(x)\leq e\}} dx.$$

For $e < e_0$, $I_\epsilon(e) = 0$. For $e > e_0$, we can write

$$\varphi(x_0 + x) = e_0 + D_{2p}(x) + \mathcal{O}(|x|^{2p+1})$$

with ,

$$D_{2p}(x) = \sum_{i=1}^{n} \alpha_i x_i^{2p} + \sum_{\sigma\in(\mathbb{N})^n;|\sigma|=p} a_\sigma x^{2\sigma}.$$

Up to $\epsilon > 0$, we have for all $x \in B(0, \epsilon)$,

$$|\mathcal{O}(|x|^{2p+1})| \leq \frac{1}{2} D_{2p}(x).$$

Hence

$$J_e := \{x \in B(0, \epsilon) : \varphi(x + x_0) \leq e\} \subset \{x \in B(0, \epsilon) : D_{2p}(x) \leq 2(e - e_0)\}$$

Since $a_\sigma \geq 0$ for all σ, we have

$$J_e \subset \{x \in B(0, \epsilon) : \sum_{i=1}^{n} \alpha_i x_i^{2p} \leq 2(e - e_0)\} \subset B(0, c(e - e_0)^{\frac{1}{2p}})$$

with $c > 0$. Therefore,

$$I_\epsilon(e) = \int_{\{x\in B(0,\epsilon)\cap B(0,c(e-e_0)^{\frac{1}{2p}}):\varphi(x_0+x)\leq e\}} dx.$$

Up to reduce $e - e_0$, we can suppose that $c(e - e_0)^{\frac{1}{2p}} < \epsilon$. Then we get

$$I_\epsilon(e) = \int_{\{x\in B(0,c(e-e_0)^{\frac{1}{2p}}):\varphi(x_0+x)\leq e\}} dx.$$

By the scaling $x = (e - e_0)^{\frac{1}{2p}} y$, we get

$$I_\epsilon(e) = (e - e_0)^{\frac{n}{2p}} \int_{\{y\in B(0,c):D_{2p}(y)+(e-e_0)^{\frac{1}{2p}}\Psi_e(y)\leq 1\}} dy,$$

with Ψ_e bounded on $B(0, c)$ uniformly on e near e_0. By Lemma 3.1, we get, for $e > e_0$,

$$I_\epsilon(e) = (e - e_0)^{\frac{n}{2p}} \Big(\int_{\{y\in B(0,c):D_{2p}(y)\leq 1\}} dy + R(e)\Big)$$

with $\lim_{e\to e_0} R(e) = 0$. □

By Lemma 3.3 and the representation (1.2) of ρ we obtain the following result.

Lemma 3.4. *Let e_0 be a simple eigenvalue of P_0. We assume that*

(i) *There exist i_0 and k_0 such that $\lambda_{i_0}(k_0) = e_0$.*
(ii) *e_0 is an isolated local extremum of order p for λ_{i_0}.*

(iii) $\nabla\lambda_{i_0}(k) \neq 0$, *for all* $k \in \lambda_{i_0}^{-1}(e_0)$, $k \neq k_0$. *Moreover* $\nabla\lambda_i(k) \neq 0$, *for all* $k \in \lambda_i^{-1}(\{e_0\})$, $i \neq i_0$.

Then there exists an open interval J such that the density of states measures has the representation (3.3), (3.4) *in lemma 3.3.*

Therefore, by [2, Theorem 1.6], we have the following result.

Theorem 3.5. *Let e_0 and J be as in Lemma 3.4, I satisfying (H2) and let* $E \in (e_0 + \operatorname{sing}\operatorname{supp}_{\mathrm{a}}(\mu)) \cap I$ *be such that* $(E - \operatorname{supp}(\mu)) \subset J$. *Then for all h-independent complex neighborhood Ω of E, there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that for $h \in]0, h_0[$,*

$$\#\{z \in \Omega; z \in \mathrm{Res}P(h)\} \geq C_\Omega h^{-n}.$$

Remark 3.6. *The hypothesis (iii) in Lemma 3.4, implies that the (λ_i) have no more critical point at the e_0 level other than λ_{i_0}'s one at k_0. In the following lemmas we consider the case of finite number of extrema at the same level. For simplicity we state these lemmas for only two extrema.*

By Lemma 3.3 and the representation (2) of ρ, we have the following result.

Lemma 3.7. *Let e_0 be a simple eigenvalue of P_0. We assume that:*

(i) *There exist i_1 and k_1, i_2 and k_2 such that* $\lambda_{i_1}(k_1) = \lambda_{i_2}(k_2) = e_0$.

(ii) λ_{i_1} *(resp. λ_{i_2}) has an isolated local minimum at the e_0 level of order p_1 (resp. p_2) at k_1 (resp. k_2).*

(iii) *The λ_i have no more critical points at the e_0 level other than λ_{i_1}'s one at k_1 and λ_{i_2}'s one at k_2.*

Then there exists an open interval J such that the density of states measures has the representation

$$\rho(e) = g(e - e_0) + H(e - e_0)(e - e_0)^{\frac{n}{2p}}(C + R(e)),$$

with $C > 0$, $\lim_{e \to e_0} R(e) = 0$, *$g$ analytic function and* $p = \max(p_1, p_2)$.

Lemma 3.8. *Let e_0 be a simple eigenvalue of P_0. We assume that:*

(i) *There exist i_1 and k_1, i_2 and k_2 such that* $\lambda_{i_1}(k_1) = \lambda_{i_2}(k_2) = e_0$.

(ii) λ_{i_1} *(resp. λ_{i_2}) has an isolated local minimum (resp. maximum) at the e_0 level of order p_1 (resp. p_2) at k_1 (resp. k_2). Moreover if $p_1 = p_2$ then we assume that* $\frac{n}{2p_1} \notin \mathbb{N}$.

(iii) *The λ_i have no more critical points in the e_0 level other than λ_{i_1}'s one at k_1 and λ_{i_2}'s one at k_2.*

Then there exists an open interval J such that the density of states measures has the representation

$$\rho(e) = g(e - e_0) + H(e - e_0)(e - e_0)^{\frac{n}{2p_1}}(C_1 + R_1(e)) + H(e_0 - e)(e_0 - e)^{\frac{n}{2p_2}}(C_2 + R_2(e))$$

with $C_1 > 0$, $C_2 > 0$, $\lim_{e \to e_0} R_1(e) = \lim_{e \to e_0} R_2(e) = 0$ *and g analytic function.*

Therefore, by [2, Theorem 1.6], we have the following theorem.

Theorem 3.9. *Let e_0 and J be as in Lemma 3.7 or Lemma 3.8, I satisfying (H2) and let* $E \in (e_0 + \operatorname{sing}\operatorname{upp}_{\mathrm{a}}(\mu)) \cap I$ *be such that* $(E - \operatorname{supp}(\mu)) \subset J$. *Then for all h-independent complex neighborhood Ω of E, there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that for $h \in]0, h_0[$,*

$$\#\{z \in \Omega; z \in \mathrm{Res}P(h)\} \geq C_\Omega h^{-n}.$$

4. Lower bound of the number of resonances near a conic singularity of the density of states

In this section we study resonances near a singularity of $\rho(\lambda)$ generated by a bands crossing. We assume that λ_j is a double eigenvalues

$$\lambda_{j-1}(k_0) < \lambda_j(k_0) = e_0 = \lambda_{j+1}(k_0) < \lambda_{j+2}(k_0)$$

and that for all $k \neq k_0$ such that $\lambda_i(k) = e_0$, $\lambda_i(k)$ is simple and $\nabla\lambda_i(k) \neq 0$.

Since P_k is analytic in k, this implies that for $|k-k_0| \leq \delta$ (with δ small enough), the span $V(k)$, of the eigenvectors of P_k corresponding to eigenvalues in the set $\{e : |e-e_0| \leq \delta\}$ has a basis $\psi_j(x,k), \psi_{j+1}(x,k)$, which is orthonormal and real analytic in k. The restriction of P_k to $V(k)$ has the matrix

$$\begin{pmatrix} \alpha(k) & \overline{b(k)} \\ b(k) & \beta(k) \end{pmatrix},$$

which can be written as

$$\begin{pmatrix} a(k)+c(k) & b_1(k)-ib_2(k) \\ b_1(k)+ib_2(k) & a(k)-c(k) \end{pmatrix},$$

where $a(k) = (\alpha(k)+\beta(k))/2$, $c(k) = (\alpha(k)-\beta(k))/2$, $b_1(k)$ and $b_2(k)$ are real valued. Next the periodic potential is assumed to have the symmetry $V(x) = V(-x)$. This symmetry is typical of metals. This symmetry forces $b(k)$ to be real valued (i.e., $b_2(k) = 0$). Consequently, near k_0 we have

$$E_j(k) = a(k) - \sqrt{c^2(k)+b^2(k)}, \quad E_{j+1}(k) = a(k) + \sqrt{c^2(k)+b^2(k)}.$$

The case $n = 2$ is treated in [1]. We consider here the case $n = 3$. We assume that $\nabla b(k_0), \nabla c(k_0)$ are independent and

$$\|\nabla_{b,c}a(k_0)\| < 1 \tag{4.1}$$

Nedelec in [2] section 6 studied singularity of volumes of matrix problem in some equivalent situations. She gets C^∞ singularities. Following the same method we get a more precise result.

Lemma 4.1. *We assume that $a/_{\{b=c=0\}}$ is non-degenerate at e_0. Then, there exist f and g, analytic near e_0, such that*

$$\rho(e) = f(e-e_0) + H(e-e_0)g(\sqrt{e-e_0}), \tag{4.2}$$

with $g(.) \neq 0$.

Proof. Without loss of generality we may assume that $e_0 = 0$ and $k_0 = 0$. Let $S = \{k \in \mathbb{R}^3; b(k) = c(k) = 0\}$. Since $\nabla b(k_0)$, $\nabla c(k_0)$ are independent then the system $(\nabla b(k_0), \nabla c(k_0), v)$ is a basis of $\mathbb{R}^3$ for all $v \neq 0$ in $T_{k_0}S$, (where $T_{k_0}S$ denotes the tangent space of S at k_0). Therefore, we can choose as coordinates

$$y_1 = b(k), \quad y_2 = c(k), \quad z = v.k$$

With this change of variables we get

$$\rho(e) = \int_{\{G(y,z)-|y|\leq e,\,(y,z)\in W\}} J(y,z)dy\,dz + \int_{\{G(y,z)+|y|\leq e,\,(y,z)\in W\}} J(y,z)dy\,dz + h(e)$$

where J is analytic in W a complex neighborhood of $(0,0)$, $G(y,z)=a(k)$ and h is analytic near 0.

By polar coordinates $y\to r(\cos(\theta),\sin(\theta)):=r\omega$, W moves into W_1 and we obtain

$$\begin{aligned}\rho(e) &= \int_{\{G(r\omega,z)-r\le e,\,(r,\omega,z)\in W_1\}} J(r\omega,z)rdrd\omega dz\\ &\quad+\int_{\{G(r\omega,z)+r\le e,\,(r,\omega,z)\in W_1\}} J(r\omega,z)r\,dr\,d\omega\,dz+h(e)\\ &= -\int_{\{G(r\omega,z)+r\le e,\,(-r,-\omega,z)\in W_1\}} J(r\omega,z)r\,dr\,d\omega\,dz\\ &\quad+\int_{\{G(r\omega,z)+r\le e,\,(r,\omega,z)\in W_1\}} J(r\omega,z)r\,dr\,d\omega\,dz+h(e)\end{aligned}$$

In the first integral of the last equation we have use the change $(r,\omega)\to(-r,-\omega)$. The assumption that $a/_{\{b=c=0\}}$ is non-degenerate implies $G(0,0)=0$, $\partial_z G(0,0)=0$ and $\nabla_z^2 G(0,0)\neq 0$. We may assume that $\nabla_z^2 G(0,0)>0$. Applying Taylor's formula to the function $y\to a(y,z)$, we get

$$G(r\omega,z)=G(0,z)+rG_1(r,\omega,z),$$

The condition (4.1) yields $|G_1|<1$.

$$G(r\omega,z)+r=G(0,z)+r(G_1(r,\omega,z)+1).$$

The change of variable $\tilde r=r(G_1(r,\omega,z)+1)$ leads to

$$\begin{aligned}\rho(e) &= -\int_{\{G(0,z)+\tilde r\le e,\,\tilde r<0,W_1\}} J_1(\tilde r,\omega,z)d\tilde r d\omega dz\\ &\quad+\int_{\{G(0,z)+\tilde r\le e,\,\tilde r>0,W_1\}} J_1(\tilde r,\omega,z)d\tilde r d\omega dz+h(e).\end{aligned}$$

Since $G(0,0)=0$, $\partial_z G(0,0)=0$ and $\nabla_z^2 G(0,0)>0$, there exists $\alpha(z)$ such that $G(0,z)=\alpha(z)z^2$, with $\alpha(0)>0$. Hence,

$$\begin{aligned}\rho(e) &= -\int_{\{z^2+\tilde r\le e,\,\tilde r<0,W_2\}} J_2(\tilde r,\omega,z)d\tilde r d\omega dz\\ &\quad+\int_{\{z^2+\tilde r\le e,\,\tilde r>0,W_2\}} J_2(\tilde r,\omega,z)d\tilde r d\omega dz+h(e)\\ &= -\int_{\{z^2+\tilde r\le e,\,W_2\}} J_2(\tilde r,\omega,z)d\tilde r d\omega dz\\ &\quad+2\int_{\{z^2+\tilde r\le e,\,\tilde r>0,W_2\}} J_2(\tilde r,\omega,z)d\tilde r d\omega dz+h(e)\end{aligned}$$

The first integral in the last equation is an analytic function in e near 0. If $e<0$ the set $\{z^2+\tilde r\le e:\tilde r>0,W_2\}$ is empty, then $\rho(e)$ is reduced to the first integral. If $e>0$ the second integral is a non vanishing function near 0. Moreover this function is analytic in term of $\sqrt{e}$. This yields analytic singularity for ρ. □

This lemma and [2, Theorem 1.6] lead to the following theorem.

Theorem 4.2. *Let J be an open interval in which (4.2) is valid. Let I satisfying (H2) and let $E\in I\cap(e_0+\operatorname{sing}\operatorname{supp}_{\mathrm{a}}(\mu))$ be such that $(E-\operatorname{supp}(\mu))\subset J$. Then* *for all h-independent complex neighborhood Ω of E, there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that for $h \in]0, h_0[$,*

$$\#\{z \in \Omega; z \in \mathrm{Res} P(h)\} \geq C_\Omega h^{-n}.$$

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