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# ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A $2 \times 2$ REACTION-DIFFUSION SYSTEM WITH A CROSS DIFFUSION MATRIX ON UNBOUNDED DOMAINS

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Abstract. This article concerns the behavior at $\mp\infty$ of solutions to a reaction-diffusion system with a cross diffusion matrix on unbounded domains. We show that the solutions satisfy the free diffusion system for all positive time whenever the initial distribution has limits at $\mp\infty$.

## 1. Introduction

In this paper, we investigate the system of reaction-diffusion equations

$$\begin{aligned} u_t &= a\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial u}{\partial x} + b\frac{\partial^2 v}{\partial x^2} + f(t,u,v), \quad x \in \mathbb{R},\ t > 0, \\ v_t &= c\frac{\partial^2 u}{\partial x^2} + d\frac{\partial^2 v}{\partial x^2} + \beta\frac{\partial v}{\partial x} + g(t,u,v), \quad x \in \mathbb{R},\ t > 0, \end{aligned} \tag{1.1}$$

supplemented with the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}. \tag{1.2}$$

The diffusion coefficients $a$ and $d$ are positive constants while the diffusion coefficients $b, c$ and the coefficient $\beta$ are arbitrary constants. We assume also the following three conditions:

- (H1) $(a-d)^2 + 4bc > 0$, $cd \neq 0$ and $ad > bc$.
- (H2) $u_0, v_0 \in X$.
- (H3) $f(t,u,v)$ and $g(t,u,v) \in X$, for all $t > 0$ and $u, v \in X$. Moreover $f$ and $g$ are locally Lipshitz; namely, for all $t_1 \geq 0$ and all constant $k > 0$, there exist a constant $L = L(k, t_1) > 0$ such that
$$|f(t,w_1) - f(t,w_2)| \leq L|w_1 - w_2|,$$
is verified for all $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2) \in \mathbb{R} \times \mathbb{R}$ with $|w_1| \leq k$ , $|w_2| \leq k$ and $t \in [0, t_1]$.

System (1.1) with specific functional responses has received extensive mathematical treatment since the addition of diffusive terms to the Lotka-Volterra systems. For the case of bounded regions, the questions of existence of globally bounded solutions

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and their large time behavior have been well studied; various results are presented by Rothe [13]. Some situations of unbounded regions are presented in [11].

The system with triangular diffusion matrix

$$\begin{aligned} u_t &= a\Delta u - uh(v), \quad (x,t) \in \Omega \times (0,\infty), \\ v_t &= b\Delta u + d\Delta v + uh(v), \quad (x,t) \in \Omega\times, (0,\infty), \end{aligned} \tag{1.3}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with Neumann boundary conditions, $b \geq 0$, $a > d$, $v_0 \geq \frac{b}{a-d} u_0 \geq 0$, and $h(s)$ is a differentiable nonnegative function on $\mathbb{R}$ has been studied by Kirane. In [8], He proved that if $a > d > 0$, $b \geq 0$, $b^2 < 4ad$, the solution $(u, v)$ converges uniformly in $\overline{\Omega}$ to a constant $(k_1, k_2)$ such that $k_1 \geq 0$, $k_2 \geq 0$ and $k_1 h(k_2) = 0$.

Such equations describe reaction-diffusion processus in physics, chemistry, biology and population dynamics.

Collet and Xin [5] have studied the same system (1.3) on $\mathbb{R}^n$ with a diagonal diffusion matrix ($b \neq 0$) and $h(v) = v^m$, where $m \in \mathbb{N}^\star$. They proved the existence of global solutions and showed that the $L^\infty$ norm of $v$ cannot grow faster than $O(\ln t)$. Also, the system was studied by Avrin [1] when $b = 0$, $v = \exp\{-E/v\}$, $E > 0$ and the space variable is in $\mathbb{R}$.

The system (1.3) with a triangular diffusion matrix in the case of unbounded domain and $h(v) = v^m$ is studied by Badraoui in [2, 3]. In [3] he showed the existence of global classical solution if $v_0(x) \geq \frac{b}{a-d} u_0(x)$ and $a > d$, $b > 0$, or $a < 0, b < 0$. In [3] he proved that the $L^\infty$ norm of $v$ cannot grow faster than $O(\ln t)$.

Kouachi [10] obtained a result concerning uniform boundedness of solutions to a system like (1.3) with a general full matrix of diffusion coefficients satisfying a balance law. This result is generalized after by Kouachi [9] who used the notion of invariant regions and Lyapunov functional.

Surprisingly enough, less attention has been given to the behavior of the solutions when the spatial variable $x$ approaches infinity despite the usefulness of this type of result for the numerical treatment of such problems. We are only aware of the article of Gladnov [7] which generalizes a result of behavior as $x$ approaches infinity of a semi-linear equation posed in $\mathbb{R}^+$ studied by Beberns and Fulks [4].

In this paper, we investigate the behavior of solutions to system (1.1) for large $x$. We show first that the linear operator

$$A = \begin{pmatrix} a(\cdot)_{xx} + \beta(\cdot)_x & b(\cdot)_{xx} \\ c(\cdot)_{xx} & d(\cdot)_{xx} + \beta(\cdot)_x \end{pmatrix}$$

generates an analytic semi-group over the Banach space $C_{UB}(\mathbb{R}) \times C_{UB}(\mathbb{R})$, where $C_{UB}(\mathbb{R})$ is the space of bounded uniformly continuous real-valued functions on $\mathbb{R}$, endowed with the norm of the uniform convergence. After, we show that if the initial conditions $u_0$ and $v_0$ have finite limits as $x$ approaches $\pm\infty$, the system converges when $x$ approaches $\pm\infty$ to the ordinary differential system associated to it.

We will use the following notation:

Let $X = (C_{UB}(\mathbb{R}), \|\cdot\|)$ be the space of bounded uniformly continuous real-valued functions on $\mathbb{R}$.

For $u : [0,T] \to X$ a continuous function, we use the norm

$$\|u\|_1 = \max_{t \in [0,T]} \|u(t)\|.$$

For $w = (u, v) \in X \times X$; we define

$$\|w\| = \|u\| + \|v\|.$$

Let $f(t, w) = (f(t, u, v), g(t, u, v))^t \equiv \begin{pmatrix} f(t,u,v) \\ g(t,u,v) \end{pmatrix}$.

## 2. Existence of a local solution

It is well known that for all $\lambda > 0$, the linear operator $\lambda \frac{\partial^2}{\partial x^2} + \beta \frac{\partial}{\partial x}$ generate analytic semigroup of contractions $G(t)$ on the Banach space. This semigroup is given explicitly by the expression

$$[G(t)u](x) = \frac{1}{\sqrt{4\pi\lambda t}} \int_{\mathbb{R}} \exp(-\frac{|x + \beta t - \xi|^2}{4\lambda t}) u(\xi) d\xi.$$

We recall here that Chen Caisheng [3] showed that the linear operator $\begin{pmatrix} a\Delta & b\Delta \\ c\Delta & d\Delta \end{pmatrix}$ generates an analytic semigroup of contractions on the space $L^p(\Omega) \times L^p(\Omega)$ $(1 \leq p < \infty)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$.

Inspired by this result, we show that the linear operator

$$\begin{pmatrix} a(\cdot)_{xx} + \beta(\cdot)_x & b(\cdot)_{xx} \\ c(\cdot)_{xx} & d(\cdot)_{xx} + \beta(\cdot)_x \end{pmatrix}$$

generates an analytic semigroup of contractions on the Banach space $X \times X$.

**Proposition 2.1.** *Assuming (H1)-(H2), the linear operator*

$$A = \begin{pmatrix} a(\cdot)_{xx} + \beta(\cdot)_x & b(\cdot)_{xx} \\ c(\cdot)_{xx} & d(\cdot)_{xx} + \beta(\cdot)_x \end{pmatrix}$$

*generates an analytic semigroup of contractions on the space $X \times X$, given explicitly by*

$$S(t) = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} (\lambda_2 - a)S_1(t) + (a - \lambda_1)S_2(t) & -bS_1(t) + bS_2(t) \\ -cS_1(t) + cS_2(t) & (\lambda_2 - d)S_1(t) + (d - \lambda_1)S_2(t) \end{pmatrix}, \tag{2.1}$$

*where*

$$\lambda_1 = \frac{1}{2}(a + d - \sqrt{(a-d)^2 + 4bc}), \quad \lambda_2 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc}),$$

*and $S_1(t)$ and $S_2(t)$ are the semigroups generated by the linear operators $\lambda_1 \frac{\partial^2}{\partial x^2} + \beta \frac{\partial}{\partial x}$ and $S_2(t)$ respectively.*

It should be noted that $\lambda_1, \lambda_2 > 0$.

*Proof.* It is clear that $S(0) = I$. It is suffices to prove (2.1) for any $w = (u, v)$ in

$$D(A) = \{(u, v) : u, v, u_{xx}, v_{xx} \in C_{UB}(\mathbb{R})\}.$$

We have

(i) $\lim_{t \searrow 0} \frac{S(t)w - w}{t} = Aw$, in $X$,
(ii) $S(t + \tau)w = S(t)S(\tau)w$, for any $t, \tau \geq 0$.

In fact, we have

$$
\begin{aligned}
&\lim_{t\searrow 0}\frac{1}{t}\{S(t)w-w\}\\
&=\frac{1}{\lambda_2-\lambda_1}\\
&\quad\times\lim_{t\searrow 0}\begin{pmatrix}\frac{1}{t}\{(\lambda_2-a)S_1(t)u+(a-\lambda_1)S_2(t)u-u-bS_1(t)v+(\lambda_1-a)S_2(t)v\}\\ \frac{1}{t}\{-cS_1(t)u+cS_2(t)u+(\lambda_2-d)S_1(t)v+(d-\lambda_1)S_2(t)v-v\}\end{pmatrix}.
\end{aligned}
$$

For the first component, we have

$$
\begin{aligned}
&\frac{1}{\lambda_2-\lambda_1}\lim_{t\searrow 0}\frac{1}{t}\{(\lambda_2-a)S_1(t)u+(a-\lambda_1)S_2(t)u-u-bS_1(t)v+(\lambda_1-a)S_2(t)v\}\\
&=\frac{1}{\lambda_2-\lambda_1}\lim_{t\searrow 0}\{(\lambda_2-a)\frac{S_1(t)u-u}{t}+(a-\lambda_1)\frac{S_2(t)u-u}{t}\\
&\quad-b\frac{S_1(t)v-v}{t}+b\frac{S_2(t)v-v}{t}\}\\
&=\frac{1}{\lambda_2-\lambda_1}\{(\lambda_2-a)(\lambda_1u_{xx}+\beta u_x)+(a-\lambda_1)(\lambda_2u_{xx}+\beta u_x)-b(\lambda_1v_{xx}+\beta v_x)\}\\
&\quad+\frac{1}{\lambda_2-\lambda_1}\{b(\lambda_2v_{xx}+\beta v_x)\}\\
&=au_{xx}+\beta u_x+bv_{xx},
\end{aligned}
$$

in $C_{UB}(\mathbb{R})$. Similarly, we obtain

$$
\begin{aligned}
&\frac{1}{\lambda_2-\lambda_1}\lim_{t\searrow 0}\frac{1}{t}\{-cS_1(t)u+cS_2(t)u+(\lambda_2-d)S_1(t)v+(d-\lambda_1)S_2(t)v-v\}\\
&=cu_{xx}+dv_{xx}+\beta v_x,
\end{aligned}
$$

in $C_{UB}(\mathbb{R})$. Therefore (i) is true. Also, by direct computation, we see that (ii) holds. □

As a consequence of this result we have the following proposition.

**Proposition 2.2.** *Let (H1)-(H3) be satisfied. Then, the system* (1.1)-(1.2) *has a unique local solution* $(u,v)\in(C[0,T_0[,X\times X)$ *for some* $0<T_0<\infty$.

*Proof.* It suffices to set

$$
A=\begin{pmatrix}a(\cdot)_{xx}+\beta(\cdot)_x & b(\cdot)_{xx}\\ c(\cdot)_{xx} & d(\cdot)_{xx}+\beta(\cdot)_x\end{pmatrix},
$$
$$
w_0=(u_0,v_0)^t.
$$

Then, the system (1.1), (1.2) is written as

$$
w_t=Aw+F(t,w), \tag{2.2}
$$
$$
w(0)=w_0. \tag{2.3}
$$

Taking into account [12, proposition 5.1, theorem 6.1.4], the proof is complete. □

Let

$$
C_\pm:=\{u\in X:\lim_{x\to\pm\infty}u(x)\text{ exist}\}.
$$

## 3. Behavior of solutions as $x \to \infty$

It turns out that if $u_0, v_0 \in C_\pm$ then the diffusive system, for $x$ large, will behave like the system of ordinary differential equations associated to it, and hence, for $x$ large, it can be replaced by the latter which is simpler to analyze.

For instance, for the numerical treatment of system (1.1)-(1.2), one can develop a numerical scheme for an approximated problem through a truncated domain $[-R, R]$ and use the system of ordinary differential equations in $\mathbb{R}\backslash[-R, R]$.

**Theorem 3.1.** *Under the assumptions (H1)-(H3), if $u_0, v_0 \in C_+$ , then $u(t), v(t) \in C_+$, for all $t \in [0, t[$ where $t < t_{\max}$. Moreover, $U(t) \equiv \lim_{x\to+\infty} u(x,t)$ and $V(t) \equiv \lim_{x\to+\infty} v(x,t)$ satisfy the system of ordinary differential equations*

$$\begin{aligned} U'(t) &= f(t, U(t), V(t)), \\ V'(t) &= g(t, U(t), V(t)), \end{aligned} \tag{3.1}$$

*for any $t < t_{\max}$, with the initial data*

$$U(0) = \lim_{x\to+\infty} u_0(x), \quad V(0) = \lim_{x\to+\infty} v_0(x). \tag{3.2}$$

*Proof.* The solution $(u, v)$ satisfies the system of integral forms

$$\begin{aligned} (\lambda_2 - \lambda_1)u(t) &= S_1(t)((\lambda_2 - a)u_0 - bv_0) + S_2(t)((a - \lambda_1)u_0 + bv_0) \\ &\quad + \int_0^t S_1(t-\tau)((\lambda_2 - a)f(\tau, u, v) - bg(\tau, u, v))d\tau \\ &\quad + \int_0^t S_2(t-\tau)((a - \lambda_1)f(\tau, u, v) + bg(\tau, u, v))d\tau, \end{aligned} \tag{3.3}$$

$$\begin{aligned} (\lambda_2 - \lambda_1)v(t) &= S_1(t)(-cu_0 + (\lambda_2 - d)v_0) + S_2(t)(cu_0 + (d - \lambda_1)v_0) \\ &\quad + \int_0^t S_1(t-\tau)(-cf(\tau, u, v) + (\lambda_2 - d)g(\tau, u, v))d\tau \\ &\quad + \int_0^t S_2(t-\tau)(cf(\tau, u, v) + (d - \lambda_1)g(\tau, u, v))d\tau. \end{aligned} \tag{3.4}$$

Changing the spatial variable, $u$ and $v$ can be written as

$$\begin{aligned} &(\lambda_2 - \lambda_1)u(x,t) \\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}((\lambda_2 - a)u_0 - bv_0)(y,t)d\eta + \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}((a - \lambda_1)u_0 + bv_0)(z,t)d\eta \\ &\quad + \frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_1(y_\tau, \tau)d\eta d\tau + \frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_2(z_\tau, \tau)d\eta d\tau, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &(\lambda_2 - \lambda_1)v(x,t) \\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}(-cu_0 + (\lambda_2 - d)v_0)(y,t)d\eta + \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}(cu_0 + (d - \lambda_1)v_0)(z,t)d\eta \\ &\quad + \frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_3(y_\tau, \tau)d\eta d\tau + \frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_4(z_\tau, \tau)d\eta d\tau, \end{aligned} \tag{3.6}$$

where

$$
\begin{gathered}
y = x + \beta t + 2\eta\sqrt{\lambda_1 t}, \\
z = x + \beta t + 2\eta\sqrt{\lambda_2 t}, \\
y_\tau = x + \beta(t-\tau) + 2\eta\sqrt{\lambda_1 t}, \\
z_\tau = x + \beta(t-\tau) + 2\eta\sqrt{\lambda_2 (t-\tau)},
\end{gathered}
$$

and

$$
\begin{gathered}
h_1(y_\tau,\tau) = ((\lambda_2 - a) f(.,u,v) - bg(.,u,v))(y_\tau,\tau), \\
h_2(z_\tau,\tau) = ((a-\lambda_1) f(.,u,v) + bg(.,u,v))(z_\tau,\tau), \\
h_3(y_\tau,\tau) = (-cf(.,u,v) + (\lambda_2 - d) g(.,u,v))(y_\tau,\tau), \\
h_4(z_\tau,\tau) = (cf(.,u,v) + (d-\lambda_1) g(.,u,v))(z_\tau,\tau).
\end{gathered}
$$

To show that $u$ and $v$ have limits when $x \to +\infty$, for any positive $t < t_{\max}$, it suffices to verify that for any sequence of real numbers $(x_n)_n$ satisfying $\lim_{n\to\infty} x_n = +\infty$, the sequences $(u(x_n,t))_{n\geq 1}$ and $(v(x_n,t))_{n\geq 1}$ are Cauchy sequences in $\mathbb{R}$. To do so, let $t < t_{\max}$, and set

$$
\begin{gathered}
y_n = x_n + \beta t + 2\eta\sqrt{\lambda_1 t}, \quad y_{\tau,n} = x_n + \beta(t-\tau) + 2\eta\sqrt{\lambda_1(t-\tau)}, \\
z_n = x_n + \beta t + 2\eta\sqrt{\lambda_2 t}, \quad z_{\tau,n} = x_n + \beta(t-\tau) + 2\eta\sqrt{\lambda_2(t-\tau)}.
\end{gathered}
$$

Then from (3.5)–(3.6), we get

$$
\begin{aligned}
&|\lambda_2 - \lambda_1||u(x_m,t) - u(x_n,t)| \\
&\leq \frac{|\lambda_2 - a|}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} |u_0(y_m) - u_0(y_n)| d\eta + \frac{|b|}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} |v_0(y_m) - v_0(y_n)| d\eta \\
&\quad + \frac{|a - \lambda_1|}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} |u_0(z_m) - u_0(z_n)| d\eta + \frac{|b|}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} |v_0(z_m) - v_0(z_n)| d\eta \\
&\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{\mathbb{R}} e^{-\eta^2} |h_3(y_{\tau,m},\tau) - h_3(y_{\tau,n},\tau)| d\eta d\tau \\
&\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{\mathbb{R}} e^{-\eta^2} |h_4(z_{\tau,m},\tau) - h_4(z_{\tau,n},\tau)| d\eta d\tau.
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
&|\lambda_2 - \lambda_1||v(x_m,t) - v(x_n,t)| \\
&\leq \frac{|c|}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} |u_0(y_m) - u_0(y_n)| d\eta + \frac{|\lambda_2 - d|}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} |v_0(y_m) - v_0(y_n)| d\eta \\
&\quad + \frac{|c|}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} |u_0(z_m) - u_0(z_n)| d\eta + \frac{|d - \lambda_1|}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} |v_0(z_m) - v_0(z_n)| d\eta \\
&\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{\mathbb{R}} e^{-\eta^2} |h_3(y_{\tau,m},\tau) - h_3(y_{\tau,n},\tau)| d\eta d\tau \\
&\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{\mathbb{R}} e^{-\eta^2} |h_4(z_{\tau,m},\tau) - h_4(z_{\tau,n},\tau)| d\eta d\tau.
\end{aligned} \tag{3.8}
$$

Since $u_0, v_0 \in C_+$, for any positive $\varepsilon > 0$, there is a natural number $n_0$ such that for any $m$, $n > n_0$

$$
\begin{aligned}
|u_0(y_m) - u_0(y_n)| &< \frac{\varepsilon|\lambda_2 - \lambda_1|}{D}, \\
|u_0(z_m) - v_0(z_n)| &< \frac{\varepsilon|\lambda_2 - \lambda_1|}{D}, \\
|v_0(y_m) - v_0(y_n)| &< \frac{\varepsilon|\lambda_2 - \lambda_1|}{D}, \\
|v_0(z_m) - v_0(z_n)| &< \frac{\varepsilon|\lambda_2 - \lambda_1|}{D},
\end{aligned}
\tag{3.9}
$$

where $D = 4\max\{|b|, |c|, |\lambda_2 - a|, |a - \lambda_1|, |\lambda_2 - d|, |d - \lambda_1|\}$. On the other hand, it is easy to show that for any $\varphi \in X$, we have the estimate

$$
\|\frac{d}{dx}G(t)\varphi\| \le \frac{\|\varphi\|}{\sqrt{\lambda\pi}}t^{-1/2}, \tag{3.10}
$$

for all $t < t_{\max}$ (see Appendix). Hence, for all continuous function $\Psi : [0, T] \to X$, we have

$$
\|\frac{d}{dx}\int_0^t G(t-\tau)\Psi(\tau)d\tau\| \le 2\frac{\|\Psi\|_1}{\sqrt{\lambda\pi}}t^{-1/2}, \tag{3.11}
$$

for all $t \in [0, T]$, where $T < t_{\max}$.

Here, $G(t)$ is the semigroup generated by the operator $\lambda\Delta$ ($\lambda > 0$) on $X$, and $\|\Psi\|_1 = \max_{t\in[0,T]} \|\Psi(t)\|$. Also, from (3.10), (3.11), (3.3), (3.4) we get

$$
\begin{aligned}
&\|\frac{du(t)}{dx}\| \\
&\le \frac{1}{|\lambda_2 - \lambda_1|}\{\frac{|\lambda_2 - a|}{\sqrt{\lambda_1\pi}}\|u_0\| + \frac{|b|}{\sqrt{\lambda_1\pi}}\|v_0\| + \frac{|a - \lambda_1|}{\sqrt{\lambda_2\pi}}\|u_0\| + \frac{|b|}{\sqrt{\lambda_2\pi}}\|v_0\|\}t^{-1/2} \\
&\quad + \frac{2}{|\lambda_2 - \lambda_1|}\{\frac{|\lambda_2 - a|}{\sqrt{\lambda_1\pi}}\|f\|_1 + \frac{|b|}{\sqrt{\lambda_1\pi}}\|g\|_1 + \frac{|a - \lambda_1|}{\sqrt{\lambda_2\pi}}\|f\|_1 + \frac{|b|}{\sqrt{\lambda_2\pi}}\|g\|_1\}t^{1/2},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&\|\frac{dv(t)}{dx}\| \\
&\le \frac{1}{|\lambda_2 - \lambda_1|}\{\frac{|c|}{\sqrt{\lambda_1\pi}}\|u_0\| + \frac{|\lambda_2 - d|}{\sqrt{\lambda_1\pi}}\|v_0\| + \frac{|c|}{\sqrt{\lambda_2\pi}}\|u_0\| + \frac{|d - \lambda_1|}{\sqrt{\lambda_2\pi}}\|v_0\|\}t^{-1/2} \\
&\quad + \frac{2}{|\lambda_2 - \lambda_1|}\{\frac{|c|}{\sqrt{\lambda_1\pi}}\|f\|_1 + \frac{|\lambda_2 - d|}{\sqrt{\lambda_1\pi}}\|g\|_1 + \frac{|c|}{\sqrt{\lambda_2\pi}}\|f\|_1 + \frac{|d - \lambda_1|}{\sqrt{\lambda_2\pi}}\|g\|_1\}t^{1/2}.
\end{aligned}
\tag{3.13}
$$

When we set

$$
\begin{aligned}
A = \max\Big\{&\frac{1}{|\lambda_2 - \lambda_1|}\{\frac{|\lambda_2 - a|}{\sqrt{\lambda_1\pi}}\|u_0\| + \frac{|b|}{\sqrt{\lambda_1\pi}}\|v_0\| + \frac{|a - \lambda_1|}{\sqrt{\lambda_2\pi}}\|u_0\| + \frac{|b|}{\sqrt{\lambda_2\pi}}\|v_0\|\}, \\
&\frac{1}{|\lambda_2 - \lambda_1|}\{\frac{|c|}{\sqrt{\lambda_1\pi}}\|u_0\| + \frac{|\lambda_2 - d|}{\sqrt{\lambda_1\pi}}\|v_0\| + \frac{|c|}{\sqrt{\lambda_2\pi}}\|u_0\| + \frac{|d - \lambda_1|}{\sqrt{\lambda_2\pi}}\|v_0\|\}\Big\}
\end{aligned}
$$

and

$$B=\max\Big\{\frac{2}{|\lambda_2-\lambda_1|}\{\frac{|\lambda_2-a|}{\sqrt{\lambda_1\pi}}\|f\|_1+\frac{|b|}{\sqrt{\lambda_1\pi}}\|g\|_1+\frac{|a-\lambda_1|}{\sqrt{\lambda_2\pi}}\|f\|_1+\frac{|b|}{\sqrt{\lambda_2\pi}}\|g\|_1\},\\ \frac{2}{|\lambda_2-\lambda_1|}\{\frac{|c|}{\sqrt{\lambda_1\pi}}\|f\|_1+\frac{|\lambda_2-d|}{\sqrt{\lambda_1\pi}}\|g\|_1+\frac{|c|}{\sqrt{\lambda_2\pi}}\|f\|_1+\frac{|d-\lambda_1|}{\sqrt{\lambda_2\pi}}\|g\|_1\}\Big\},$$

we get from (3.12)-(3.13),

$$\|\frac{d}{dx}u(t)\|\le At^{-1/2}+Bt^{1/2},\quad \|\frac{d}{dx}v(t)\|\le At^{-1/2}+Bt^{1/2}, \tag{3.14}$$

for all $t\in[0,T]$.
Let $k>0$ be a constant such that $\|u\|_1\le k$ and $\|v\|_1\le k$. Using the Lagrange theorem and the estimates (3.14) we obtain

$$\begin{aligned} |u(x_m,t)-u(x_n,t)|&\le|x_m-x_n|\|\frac{\partial u}{\partial x}(x',t)\|\le|x_m-x_n|\big(At^{-1/2}+Bt^{1/2}\big),\\ |v(x_m,t)-v(x_n,t)|&\le|x_m-x_n|\|\frac{\partial v}{\partial x}(x'',t)\|\le|x_m-x_n|\big(At^{-1/2}+Bt^{1/2}\big)\end{aligned} \tag{3.15}$$

for all $t\in[0,T]$. Here, $x',x''$ are points between $x_m$ and $x_n$, and $L=L(k,T)>0$ is a constant. On the other hand, we have from (H3) and (3.15),

$$\begin{aligned} &|h_1(y_{\tau,m},\tau)-h_1(y_{\tau,n},\tau)|\\ &\le|\lambda_2-a||f(\tau,u(y_{\tau,m},\tau),v(y_{\tau,m},\tau))-f(\tau,u(y_{\tau,n},\tau),v(y_{\tau,n},\tau))|\\ &\quad+|b||g(\tau,u(y_{\tau,m},\tau),v(y_{\tau,m},\tau))-g(\tau,u(y_{\tau,n},\tau),v(y_{\tau,n},\tau))|\\ &\le L\max\{|\lambda_2-a|,|b|\}\{|u(y_{\tau,m},\tau)-u(y_{\tau,n},\tau)|+|v(y_{\tau,m},\tau)-v(y_{\tau,n},\tau)|\}\\ &\le 2L\max\{|\lambda_2-a|,|b|\}|x_m-x_n|(A\tau^{-1/2}+B\tau^{1/2}),\end{aligned}$$

$$\begin{aligned} &|h_2(z_{\tau,m},\tau)-h_2(z_{\tau,n},\tau)|\\ &\le|a-\lambda_1||f(\tau,u(z_{\tau:m},\tau),v(z_{\tau:m},\tau))-f(\tau,u(z_{\tau:n},\tau),v(z_{\tau:n},\tau))|\\ &\quad+|b||g(\tau,u(z_{\tau:m},\tau),v(_{\tau,m},\tau))-g(\tau,u(y_{\tau,n},\tau),v(y_{\tau,n},\tau))|\\ &\le L\max\{|a-\lambda_1|,|b|\}\{|u(z_{\tau,m},\tau)-u(z_{\tau,n},\tau)|+|v(z_{\tau,m},\tau)-v(z_{\tau,n},\tau)|\}\\ &\le 2L\max\{|a-\lambda_1|,|b|\}|x_m-x_n|(A\tau^{-1/2}+B\tau^{\frac12}),\end{aligned}$$

$$\begin{aligned} &|h_3(y_{\tau,m},\tau)-h_3(y_{\tau,n},\tau)|\\ &\le|c||f(\tau,u(y_{\tau,m},\tau),v(y_{\tau,m},\tau))-f(\tau,u(y_{\tau,n},\tau),v(y_{\tau,n},\tau))|\\ &\quad+|\lambda_2-d||g(\tau,u(y_{\tau,m},\tau),v(y_{\tau,m},\tau))-g(\tau,u(y_{\tau,n},\tau),v(y_{\tau,n},\tau))|\\ &\le L\max\{|c|,|\lambda_2-d|\}\{|u(y_{\tau,m},\tau)-u(y_{\tau,n},\tau)|+|v(y_{\tau,m},\tau)-v(y_{\tau,n},\tau)|\}\\ &\le 2L\max\{|c|,|\lambda_2-d|\}|x_m-x_n|(A\tau^{-1/2}+B\tau^{1/2}),\end{aligned}$$

and

$$\begin{aligned} &|h_4(z_{\tau,m},\tau)-h_4(z_{\tau,n},\tau)|\\ &\le|c||f(\tau,u(z_{\tau,m},\tau),v(z_{\tau,m},\tau))-f(\tau,u(z_{\tau,n},\tau),v(z_{\tau,n},\tau))|\\ &\quad+|d-\lambda_1||g(\tau,u(z_{\tau,m},\tau),v(_{\tau,m},\tau))-g(\tau,u(y_{\tau,n},\tau),v(y_{\tau,n},\tau))|\\ &\le L\max\{|c|,|d-\lambda_1|\}\{|u(z_{\tau,m},\tau)-u(z_{\tau,n},\tau)|+|v(z_{\tau,m},\tau)-v(z_{\tau,n},\tau)|\}\\ &\le 2L\max\{|c|,|d-\lambda_1|\}|x_m-x_n|(A\tau^{-1/2}+B\tau^{1/2}).\end{aligned}$$

Let

$$M|\lambda_2-\lambda_1| = 2L\max\{|b|,|c|,|\lambda_2-a|,|a-\lambda_1|,|\lambda_2-d|,|d-\lambda_1|\}\ .$$

Then

$$\begin{aligned}
|h_1(y_{\tau,m},\tau)-h_1(y_{\tau,n},\tau)| &\le M|\lambda_2-\lambda_1||x_m-x_n|(A\tau^{-1/2}+B\tau^{1/2}),\\
|h_2(z_{\tau,m},\tau)-h_2(z_{\tau,n},\tau)| &\le M|\lambda_2-\lambda_1||x_m-x_n|(A\tau^{-1/2}+B\tau^{1/2}),\\
|h_3(y_{\tau,m},\tau)-h_3(y_{\tau,n},\tau)| &\le M|\lambda_2-\lambda_1||x_m-x_n|(A\tau^{-1/2}+B\tau^{1/2})\\
|h_4(z_{\tau,m},\tau)-h_4(z_{\tau,n},\tau)| &\le M|\lambda_2-\lambda_1||x_m-x_n|(A\tau^{-1/2}+B\tau^{1/2}).
\end{aligned}\tag{3.16}$$

Inserting (3.9) and (3.16) in (3.7)-(3.8), we get for any $m$, $n>n_0$

$$\begin{aligned}
|u(x_m,t)-u(x_n,t)| &\le \varepsilon + M|x_m-x_n|(2At^{1/2}+\frac{2}{3}Bt^{3/2}),\\
|v(x_m,t)-v(x_n,t)| &\le \varepsilon + M|x_m-x_n|(2At^{1/2}+\frac{2}{3}Bt^{\frac{3}{2}}),
\end{aligned}\tag{3.17}$$

for all $t\in[0,T]$. Setting

$$\begin{aligned}
y'_n &= y_{\tau,n}+\beta\tau+2\eta\sqrt{\lambda_1\tau}, \quad y'_{\sigma,n} = y_{\tau,n}+\beta(\tau-\sigma)+2\eta\sqrt{\lambda_1(\tau-\sigma)},\\
z'_n &= z_{\tau,n}+\beta\tau+2\eta\sqrt{\lambda_2\tau}, \quad z'_{\sigma,n} = z_{n,\tau}+\beta(\tau-\sigma)+2\eta\sqrt{\lambda_2(\tau-\sigma)}.
\end{aligned}$$

Then, from (3.9) and (3.16) into (3.7)-(3.8), we obtain

$$\begin{aligned}
&|\lambda_2-\lambda_1||u(y_{\tau,m},\tau)-u(y_{\tau,n},\tau)|\\
&\le \frac{|\lambda_2-a|}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}|u_0(y'_m)-u_0(y'_n)|d\eta + \frac{|b|}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}|v_0(y'_m)-v_0(y'_n)|d\eta\\
&\quad + \frac{|a-\lambda_1|}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}|u_0(z'_m)-u_0(z'_n)|d\eta + \frac{|b|}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}|v_0(z'_m)-v_0(z'_n)|d\eta\\
&\quad + \frac{1}{\sqrt{\pi}}\int_0^\tau\int_{\mathbb{R}} e^{-\eta^2}|h_1(y'_{\sigma,m},\sigma)-h_1(y'_{\sigma,n},\sigma)|d\eta d\sigma\\
&\quad + \frac{1}{\sqrt{\pi}}\int_0^\tau\int_{\mathbb{R}} e^{-\eta^2}|h_2(z'_{\sigma,m},\sigma)-h_2(z'_{\sigma,n},\sigma)|d\eta d\sigma\\
&\le \varepsilon|\lambda_2-\lambda_1| + M|\lambda_2-\lambda_1||x_m-x_n|(2A\tau^{\frac{1}{2}}+\frac{2}{3}B\tau^{\frac{3}{2}})
\end{aligned}$$

and

$$\begin{aligned}
&|\lambda_2-\lambda_1||v(z_{\tau,m},\tau)-v(z_{\tau,n},\tau)|\\
&\le \frac{|c|}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}|u_0(y'_m)-u_0(y'_n)|d\eta + \frac{|\lambda_2-d|}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}|v_0(y'_m)-v_0(y'_{\sigma,n})|d\eta\\
&\quad + \frac{|c|}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}|u_0(z'_m)-u_0(z'_n)|d\eta + \frac{|d-\lambda_1|}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}|v_0(z'_m)-v_0(z'_n)|d\eta\\
&\quad + \frac{1}{\sqrt{\pi}}\int_0^\tau\int_{\mathbb{R}} e^{-\eta^2}|h_3(y'_{\sigma,m},\sigma)-h_3(y'_{\sigma,n},\sigma)|d\eta d\sigma\\
&\quad + \frac{1}{\sqrt{\pi}}\int_0^{t\tau}\int_{\mathbb{R}} e^{-\eta^2}|h_4(z'_{\sigma,m},\sigma)-h_4(z'_{\sigma,n},\sigma)|d\eta d\sigma\\
&\le \varepsilon|\lambda_2-\lambda_1| + M|\lambda_2-\lambda_1||x_m-x_n|(2A\tau^{\frac{1}{2}}+\frac{2}{3}B\tau^{\frac{3}{2}}).
\end{aligned}$$

Whence

$$|u(y_{\tau,m},\tau)-u(y_{\tau,n},\tau)|\le \varepsilon+M|x_m-x_n|(2A\tau^{1/2}+\frac{2}{3}B\tau^{\frac{3}{2}}) \tag{3.18}$$

$$|v(z_{\tau,m},\tau)-v(z_{\tau,n},\tau)|\le \varepsilon+M|x_m-x_n|(2A\tau^{1/2}+\frac{2}{3}B\tau^{\frac{3}{2}}), \tag{3.19}$$

and from (3.18)-(3.19) in (3.7)-(3.8) we get

$$\begin{gathered}|u(y_m,t)-u(y_n,t)|\le \varepsilon(1+Mt)+M^2|x_m-x_n|(\frac{2^2}{3}At^{\frac{3}{2}}+\frac{2^2}{3\times 5}Bt^{\frac{5}{2}})\\ |v(z_m,t)-u(z_n,t)|\le \varepsilon(1+Mt)+M^2|x_m-x_n|(\frac{2^2}{3}At^{\frac{3}{2}}+\frac{2^2}{3\times 5}Bt^{\frac{5}{2}}),\end{gathered} \tag{3.20}$$

for all $t\in[0,T]$. Iterating this operation $N$ times we obtain

$$\begin{aligned}|u(x_m,t)-u(x_n,t)|\le \varepsilon\big(1+Mt+\frac{(Mt)^2}{2!}\dots\frac{(Mt)^{n-1}}{(N-1)!}\big)\\ +|x_m-x_n|\Big(\frac{(2M)^N}{1\times 3\times 5\times\dots\times(2N-1)}At^{N-\frac{1}{2}}\\ +\frac{(2M)^N}{1\times 3\times 5\times\dots\times(2N+1)}Bt^{N+\frac{1}{2}}\Big),\end{aligned}$$

and

$$\begin{aligned}|v(x_m,t)-v(x_n,t)|\le \varepsilon\big(1+Mt+\frac{(Mt)^2}{2!}\dots\frac{(Mt)^{n-1}}{(N-1)!}\big)\\ +|x_m-x_n|\big(\frac{(2M)^N}{1\times 3\times 5\times\dots\times(2N-1)}\\ \times At^{N-\frac{1}{2}}\frac{(2M)^N}{1\times 3\times 5\times\dots\times(2N+1)}Bt^{N+\frac{1}{2}}\big).\end{aligned}$$

Passing to the limit when $N$ approaches infinity, we obtain

$$|u(x_m,t)-u(x_n,t)|\le \varepsilon e^{Mt},\quad |v(x_m,t)-v(x_n,t)|\le \varepsilon e^{Mt}, \tag{3.21}$$

for all $t\in[0,T]$. From these inequalities, we deduce that the sequences $(u(x_n,t))_n$ and $(v(x_n,t))_n$ are Cauchy sequences of continuous functions from $[0,T]$ into $X$, hence they converge uniformly on $[0,T]$ to some continuous functions $U$ and $V$, respectively.

The solution $(u,v)$ satisfies the system of integral equation

$$\begin{aligned}&(\lambda_2-\lambda_1)u(x,t)\\ &=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-\eta^2}[(\lambda_2-a)u_0-bv_0](y,t)d\eta+\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-\eta^2}[(a-\lambda_1)u_0+bv_0](z,t)d\eta\\ &\quad+\frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}}e^{-\eta^2}h_1(y_\tau,\tau)d\eta d\tau+\frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}}e^{-\eta^2}h_2(z_\tau,\tau)d\eta d\tau,\end{aligned}$$

$$\begin{aligned}&(\lambda_2-\lambda_1)v(x,t)\\ &=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-\eta^2}[-cu_0+(\lambda_2-d)v_0](y,t)d\eta+\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-\eta^2}[cu_0+(d-\lambda_1)v_0](z,t)d\eta\\ &\quad+\frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}}e^{-\eta^2}h_3(y_\tau,\tau)d\eta d\tau+\frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}}e^{-\eta^2}h_4(z_\tau,\tau)d\eta d\tau.\end{aligned}$$

With the previous substitution of the spatial variable, and for any sequence $(x_n)_n$ tending to $+\infty$, we have

$$\begin{aligned}
&(\lambda_2-\lambda_1)u(x_n,t)\\
&=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}[(\lambda_2-a)u_0-bv_0](y_n,t)d\eta+\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}[(a-\lambda_1)u_0+bv_0](z_n,t)d\eta\\
&\quad+\frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_1(y_{\tau,n},\tau)d\eta d\tau+\frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_2(z_{\tau,n},\tau)d\eta d\tau,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&(\lambda_2-\lambda_1)v(x_n,t)\\
&=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}[-cu_0+(\lambda_2-d)v_0](y_n,t)d\eta\\
&\quad+\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}[cu_0+(d-\lambda_1)v_0](z_n,t)d\eta\\
&\quad+\frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_3(y_{\tau,n},\tau)d\eta d\tau+\frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_4(z_{\tau,n},\tau)d\eta d\tau.
\end{aligned} \tag{3.23}$$

By the dominated convergence theorem we have

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}} e^{-\eta^2}[(\lambda_2-a)u_0-bv_0](y_n,t)d\eta&=\sqrt{\pi}\{(\lambda_2-a)U_0-bV_0\},\\
\lim_{n\to\infty}\int_{\mathbb{R}} e^{-\eta^2}[(a-\lambda_1)u_0+bv_0](z_n,t)d\eta&=\sqrt{\pi}\{(a-\lambda_1)U_0+bV_0\},\\
\lim_{n\to\infty}\int_{\mathbb{R}} e^{-\eta^2}[-cu_0+(\lambda_2-d)v_0](y_n,t)d\eta&=\sqrt{\pi}\{-cU_0+(\lambda_2-d)V_0\},\\
\lim_{n\to\infty}\int_{\mathbb{R}} e^{-\eta^2}[cu_0+(d-\lambda_1)v_0](z_n,t)d\eta&=\sqrt{\pi}\{cU_0+(d-\lambda_1)V_0\},
\end{aligned} \tag{3.24}$$

where $U_0=\lim_{n\to\infty}u_0(x_n)$ and $V_0=\lim_{n\to\infty}v_0(x_n)$. We also have

$$|e^{-\eta^2}h_i(y_{\tau,n},\tau)|\le C(T)e^{-\eta^2},$$

for $i=1,2,3,4$ and all $0\le\tau\le t\le T$, where

$$C(T)=\max\big\{|\lambda_2-a|,|b|,|a-\lambda_1|,|c|,|d-\lambda_1|\big\}(\|f\|_1+\|g\|_1)$$

Using again the dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_1(y_{\tau,n},\tau)\\
&=\sqrt{\pi}\int_0^t\{(\lambda_2-a)f(\tau,U(\tau),v(\tau))-bg(\tau,U(\tau),v(\tau))\}d\tau,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t\int_{\mathbb{R}} e^{-\eta^2}h_2(y_{\tau,n},\tau)\\
&=\sqrt{\pi}\int_0^t\{(a-\lambda_1)f(\tau,U(\tau),v(\tau))+bg(\tau,U(\tau),v(\tau))\}d\tau.
\end{aligned} \tag{3.26}$$

We have also

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t\int_{\mathbb{R}}e^{-\eta^2}h_3(y_{\tau,n},\tau)\\&=\sqrt{\pi}\int_0^t\{-cf(\tau,U(\tau),v(\tau))+(\lambda_2-d)g(\tau,U(\tau),v(\tau))\}d\tau,\end{aligned}\tag{3.27}$$

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t\int_{\mathbb{R}}e^{-\eta^2}h_4(y_{\tau,n},\tau)\\&=\sqrt{\pi}\int_0^t\{cf(\tau,U(\tau),v(\tau))+(d-\lambda_1)g(\tau,U(\tau),v(\tau))\}d\tau.\end{aligned}\tag{3.28}$$

Thanks to (3.24) and (3.25)-(3.28), if we pass to the limit in (3.22)-(3.23), we obtain

$$U(t)=U_0+\int_0^t f(\tau,U(\tau),V(\tau))d\tau,$$
$$V(t)=V_0+\int_0^t g(\tau,U(\tau),V(\tau))d\tau,$$

for all $0\le t\le T$. The ordinary differential system then follows. □

We remark remark that the same analysis holds for

$$u_0,v_0\in C_-\equiv\{u\in X:\lim_{x\to-\infty}u(x)\text{ exist}\}.$$

**Conclusions.** We have proved the result of asymptotic behavior when $x\to\infty$ thanks to the explicit expression of the semigroup generated by the linear operator

$$A=\begin{pmatrix}a(.)_{xx}+\beta(.)_x & b(.)_{xx}\\ c(.)_{xx} & d(.)_{xx}+\lambda(.)_x\end{pmatrix},$$

where $\lambda=\beta$ in the space $X^2$, where $X=(C_{UB}(\mathbb{R}),\|.\|)$ under some conditions over the coefficients $a,b,c$ and $d$. The analytic expression of the semigroup generated by the operator $A$ if $\lambda\neq\beta$ still an open problem.

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