Electronic Journal of Differential Equations, Vol. 2006(2006), No. 66, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

IMPULSIVE DIFFERENTIAL INCLUSIONS WITH CONSTRAINS

TZANKO DONCHEV

Abstract. In the paper, we study weak invariance of differential inclusions with non-fixed time impulses under compactness type assumptions. When the right-hand side is one sided Lipschitz an extension of the well known relaxation theorem is proved. In this case also necessary and sufficient condition for strong invariance of upper semi continuous systems are obtained. Some properties of the solution set of impulsive system (without constrains) in appropriate topology are investigated.

1. Preliminaries

This paper is concerned with the impulsive differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad x(0) = x_0 \in D, \text{ a.e. } t \in I = [0,1], \ t \neq \tau_i(x), \tag{1.1}$$

$$\Delta x|_{t=\tau_i(x)} = S_i(x), \quad i = 1, \dots, p, \ x(t) \in D, \tag{1.2}$$

Here D is a closed subset of a Banach space E and $F : I \times D \to E$ is multifunction with nonempty compact values. Every absolutely continuous on (τ_i, τ_{i+1}) function for $i = 0, 1, \dots, p, p+1$ ($\tau_0 = 0$ and $\tau_{p+1} = 1$) with (possible) jumps $\Delta x|_{t=\tau_i(x)} = S_i(x(\tau_i(x) - 0))$ called impulses, i.e. $x(\tau_i(x) + 0) = x(\tau_i(x) - 0) + S_i(x(\tau_i(x) - 0))$, is said to be a solution of (1.1)–(1.2).

Further we assume that $x + S_i(x) \in D$ for every $x \in D$.

Differential inclusions without impulses have been studied extensively; see [2, 3, 9, 12, 14, 16, 20, 29] and references therein. We refer to [5, 18, 22, 23, 28] for the theory of impulsive differential equations. The existence of solutions of impulsive differential inclusions in infinite dimensional spaces is very comprehensively studied in [7], see also [6, 8, 30]. In these works the authors use mainly fixed point arguments. Method of averaging and some other qualitative properties of impulsive differential inclusions are studied in [25, 26, 27]. We refer to [1, 4, 11] for impulsive differential inclusions with constrains in finite dimensional space.

Our first purpose is to present sufficient (and necessary) conditions for the existence of solutions in arbitrary (not necessarily separable) Banach space when the right-hand side is almost USC. We also prove sufficient conditions when the right-hand side is USC at some points and LSC in others. This is done in the second section. Notice that our compactness conditions are weaker that those in [7, 6, 8].

2000 *Mathematics Subject Classification.* 34A60, 34A37, 34B15, 49K24.
Key words and phrases. Impulsive differential inclusions; constrains; invariant solutions.

Submitted March 27, 2006. Published May 26, 2006.
Supported by grant DP0346099 from the Australian Research Council.

In the present paper, we are not able to use fixed points approach. We follow the method used in [14, 9] with some modifications.

Our second purpose is to describe the structure of solution set (in appropriate metric). We extend the well-known relaxation theorem. We use a modification of the very short proof of this theorem presented in [15] for systems without impulses. Namely we prove that the solution set of

$$\dot{x}(t) \in \overline{\text{ext}} F(t, x(t)), \quad x(0) = x_0 \text{ a.e. } t \in I = [0, 1], \; t \neq \tau_i(x), \tag{1.3}$$

$$\Delta x|_{t=\tau_i(x)} = S_i(x(\tau_i(x) - 0)), \quad i = 1, \dots, p, \; x(t) \in D, \tag{1.4}$$

is dense in the solution set of (1.1)–(1.2). We do not know any related results in this case (impulsive system with non-fixed time of impulses).

For problems without constrains in finite dimension, i.e. $D \equiv \mathbb{R}^n$, we show that the solution set of (1.1)–(1.2) is R_δ.

Notation and terminology. The multifunction $G : E \to E$ with nonempty closed bounded values is said to be upper semi-continuous (USC) at x_0, when for every $\varepsilon > 0$ there exists $\delta > 0$ with $G(x_0) + \varepsilon\mathbb{B} \supset G(x_0 + \delta\mathbb{B})$. Here $\mathbb{B}$ is the open unit ball. The multifunction $G(\cdot)$ is said to be lower semi-continuous (LSC) at x_0 when for every $f \in G(x_0)$ and every sequence $\{x_i\}_{i=1}^{\infty}$ converging to x_0 there exist $f_i \in G(x_i)$ such that $f_i \to f_0$. When $G(\cdot)$ is USC (LSC) at every $x \in D$ it is called USC (LSC). The multifunction $F(\cdot, \cdot)$ is said to be almost USC when for every $\varepsilon > 0$ there exists a compact set $I_\varepsilon \subset I$ with Lebesgue measure $\text{meas}(I_\varepsilon) > 1 - \varepsilon$ such that $F(\cdot, \cdot)$ is USC on $I_\varepsilon \times D$. The almost LSC maps are defined analogously.

Given $M > 0$ we define the cone $\Gamma^M = \{(t, x) \in \mathbb{R}^+ \times E : \|x\|_E \le Mt\}$. If $\Omega \subset \mathbb{R} \times E$ is nonempty and $f : \Omega \to E$ we say that f is Γ^M-continuous at the point (t_0, x_0) when, given $\varepsilon > 0$ one can find $\delta > 0$ such that $(t, x) \in \Omega$, $t_0 < t < t_0 + \delta$ and $(t, x) - (t_0, x_0) \in \Gamma^M$ imply $|f(t, x) - f(t_0, x_0)| < \varepsilon$. The function$f$ is said to be Γ^M-continuous if it is Γ^M-continuous at each point of Ω.

For $A, B \subset E$ recall that $\text{dist}(a; B) = \inf_{b \in B} |a - b|$; $\text{ex}(A, B) = \sup_{a \in A} \text{dist}(a, B)$ and $D_H(A, B) = \max\{\text{ex}(A, B), \text{ex}(B, A)\}$ is the Hausdorff distance. Let $D \subset \mathbb{R}^n$ be a closed set.

We denote by $\text{ext}\, A$ the set of all extreme points of A.

$T_D(x) := \left\{v : \liminf_{h \downarrow 0} \frac{\text{dist}(x + hv; D)}{h} = 0\right\}$ is the Bouligand contingent cone of D at x.

The following definition is taken from [13] (see also [17]).

Let $D \subset \mathbb{R}^n$ be (locally) closed. A proximal normal to D at a point $x \in D$ is a vector $\xi \in \mathbb{R}^n$ such that there exists $\alpha > 0$ with $\langle \xi, x' - x \rangle \le \alpha |x' - x|^2$, $\forall x' \in D$. The set of all such vectors is a cone denoted by $N_D^P(x)$ and it is called proximal normal cone to D at x. If no such a vector exists we set $N_D^P(x) = 0$.

Denote $\mathbb{R}^+ = [0, \infty)$. A Caratheodory function $w : I \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be Kamke function if it is integrally bounded on the bounded sets, $w(t, 0) \equiv 0$ and the only solution of the differential equation $\dot{s}(t) = w(t, s(t))$, $s(0) = 0$ is $s(t) \equiv 0$.

In the next section we study the existence of solutions under compactness type conditions. Notice that our assumptions are more general that those in [14, 9]. It is also impossible to use fixed point arguments (because the constrain sets are non-convex). To our knowledge there are no existence results in the existing literature when the right-hand side changes its kind of semicontinuity as Theorem 2.8 below.

In the last section we consider differential inclusion (1.1)–(1.2) in $\mathbb{R}^n$. We prove that the solution set is R_δ in appropriate metric.

The proof of the relaxation theorem is a modification of the author's proof ([15]). It will be very difficult (if possible at all) to prove such a theorem using the approach of Pianigiani and Tolstonogov ([24, 29]). We also extend a very recent strong invariance result of [17].

2. Existence of solutions

In this section we study the existence of solutions for the Cauchy problem (1.1)–(1.2). First we need a result for a problem without impulses. We consider

$$\dot{x}(t) \in F(t, x(t)), \quad x(0) = x_0 \in D, \ t \in I.$$

In this case we need the following hypotheses:

(H1) There exists a Kamke function $\omega(\cdot,\cdot)$ such that $\chi(F(t, A)) \le \omega(t, \chi(A))$ for every bounded $A \subset D$ and a.e. $t \in I$. Here

$$\chi(A) = \inf\{r > 0 : A \text{ can be covered by finitely many balls of radius } \le r\}$$

is the Hausdorff measure of non-compactness.

(H2) There exists a null set $\mathcal{N} \subset I$ such that $F(t, x) \cap T_D(x) \neq \emptyset$ for every $t \in I \setminus \mathcal{N}$ and every $x \in D$.

The following theorem has been proved under condition stronger than (H1); see for example [9, 14]. We present a complete proof because this theorem will be essential in this paper.

Theorem 2.1. *Let $F(\cdot,\cdot)$ be almost USC with nonempty convex compact values satisfying (H1). Assume there exist an $L_1(I, \mathbb{R}^+)$ function $\lambda(\cdot)$ such that $|F(t, x)| \le \lambda(t)(1+|x|)$ (linear growth). The system* (2) *admits a solution defined on the whole interval I for every $x_0 \in D$ if and only if (H2) holds.*

Proof. As it is shown in [14], one can reduce the growth condition to the case $|F(t, x)| \le C$ for some constant $C > 0$ without destroying the other hypotheses.

For $\varepsilon > 0$ we will prove that there exists ε –solution $x(\cdot)$ on $[0, 1]$, i.e.

(1) $\operatorname{dist}(x(t), D) < \varepsilon$ for every $t \in I$,
(2) $\dot{x}(t) \in F(t, x(t) + \varepsilon\mathbb{B} \cap D)$ on a set I_ε with measure greater than $1 - \varepsilon$
(3) $\dot{x}(t) \in F(t, x(t)) + 2C\mathbb{B}$ otherwise.

Fix $\varepsilon > 0$. There exists a set $I_\varepsilon \subset I$ wit Lebesgue measure $\operatorname{meas}(I_\varepsilon) > 1 - \varepsilon$ such that $F(\cdot,\cdot)$ is USC on $I_\varepsilon \times E$ and $F(t, x) \cap T_D(x) \neq \emptyset$ for every $x \in D$ and every $t \in I_\varepsilon$. One can suppose also without loss of generality that $\omega(\cdot,\cdot)$ is (uniformly) continuous on $I_\varepsilon \times [0, 2M]$.

Since $x_0 \in D$, there exists a maximal number $0 \le \tau \le 1$ such that there exists a ε –solution $x(\cdot)$ on $[0, \tau)$ and $x(\tau) := \lim_{t\uparrow\tau} x(\tau) \in D$ ($x(\tau)$ exists, because $x(\cdot)$ is C–Lipschitz). We are done if $\tau = 1$. Let $\tau < 1$. Two cases are possible:

Case a: $\tau \in I_\varepsilon$. Since $F(\tau, x(\tau)) \cap T_D(x(\tau)) \neq \emptyset$, one has that there exist $f \in F(\tau, x(\tau)) \cap T_D(x(\tau))$ and sequences $h_n \downarrow 0$ and $y_n \to 0$ such that $x(\tau) + h_n(f - y_n) \in D$. Let $\delta > 0$ be such that be such that $\omega(t, s) - \omega(\tau, \xi)| < \varepsilon$ and $F(\tau, x(\tau)) \supset F(t, y)$ when $|t - \tau| < \delta$, $|s - \xi| < C\delta$ and $|x(\tau) - y| < C\delta$ $(t, \tau \in I_\varepsilon)$. If $h_n < \delta$, then we let $x(t) = x(\tau) + (t - \tau)(f - y_n)$. Thus $x(\tau + h_n) \in D$ and it is easy to see that $x(\cdot)$ satisfies (1), (2) (3) on $[0, \tau + h_n]$.

Case b: $\tau \notin I_\varepsilon$. Since $I \setminus I_\varepsilon$ is open, one has that it is an union of countable many pairwise disjoint open intervals. That means that there exists $T > \tau$ such that $(\tau, T) \subset I \setminus I_\varepsilon$ and $T \in I_\varepsilon$. We let $x(t) \equiv x(\tau)$ on $[\tau, T]$. Evidently $x(\cdot)$ satisfies (1), (2) and (3) on $[0, T]$.

Applying Zorn' lemma one obtains that $x(\cdot)$ is extendable on $[0, 1]$.

Let $\varepsilon_{i+1} = \frac{\varepsilon_i}{3}$ $(\varepsilon_0 = \varepsilon)$ and let $\{x^i(\cdot)\}_{i=1}^{\infty}$ be a sequence of ε_i– solutions. Their derivatives $\dot{x}_n(\cdot)$ are strongly measurable and hence almost separable valued. Therefore there exist a separable space $X_0 \subset E$ and a null set A such that which $\dot{x}_n(t) \in X_0$ for every n and every $t \in I \setminus A$. We can assume without loss of generality that $x_n(t)$ are in X_0. Define $B(t) = \chi\big(\cup_{n=1}^{\infty} \{x_n(t)\}\big)$. From [14, Proposition 9.3], we know that

$$\chi\Big(\Big\{ \int_t^{t+h} \dot{x}_k(t) : k \geq 1\Big\}\Big)\, dt \leq \int_t^{t+h} \chi\big(\{\dot{x}_k(t) :\ k \geq 1\}\big)\, dt.$$

Taking into account the definition of $x_n(\cdot)$ one has that $B(\cdot)$ is absolutely continuous and for every $\varepsilon > 0$ there exists a compact set I_ε with $\operatorname{meas}(I_\varepsilon) > 1 - \varepsilon$ such that $\dot{B}(t) \leq \omega(t, B(t)) + \varepsilon$ on I_ε and $\dot{B}(t) \leq \omega(t, B(t)) + 2C$ on $I \setminus I_\varepsilon$. Since $\varepsilon > 0$ is arbitrary one has that $\dot{B}(t) \leq \omega(t, B(t))$ for a.a. $t \in I$. However, $B(0) = 0$ and hence $B(t) \equiv 0$.

Due to Arzela's theorem the sequence $\{x_n(\cdot)\}_{n=1}^{\infty}$ is $C(I, E)$ precompact. Hence passing to subsequences $v_n(t) \to x(t)$ uniformly on I. The proof that $x(\cdot)$ is a solution of (1.1)–(1.2) is standard. □

We will use the following assumptions in this article:

- (A1) $\tau_i(\cdot)$ are Lipschitz function with a constant N, and $\tau_i(x) \geq \tau_i(x + S_i(x))$.
- (A2) $\tau_i(x) < \tau_{i+1}(x)$ for every $x \in D$.
- (A3) There exists a constant C such that $|F(t, x)| \leq C$ for every $x \in D$ and a.e. $t \in I$ and $NC < 1$.

These assumptions prevent the beating phenomena (see the following lemma, which proof follows [26, 27]).

Lemma 2.2. *Under (A1)–(A3), every solution of* (1.1)–(1.2) *(if it exists) intersects every surface $t = \tau_i(x)$ at most once.*

Proof. Suppose the contrary, i.e. there exists a solution $x(t)$ which pass through the surface $t = \tau_i(x)$ at the time $t' = \tau_i(x') + 0$ and on the time t" the same surface $(t", x")$, $t" = \tau_i(x")$.

Due to (A2), $x(\cdot)$ is continuous on the interval $(t', t")$. and $\dot{x}(t) \in F(t, x(t))$. Denote $h_i = \int_{t'}^{t"} \dot{x}(t)\, dt$. By (A2) and (A3), we have

$$\begin{aligned} t" - t' &= \tau_i(x") - \tau_i(x') \\ &= \tau_i(x' + S_i(x') + h_i) - \tau_i(x' + S_i(x')) + \tau_i(x' + S_i(x')) - \tau_i(x') \\ &\leq N \int_{t'}^{t"} |\dot{x}(t)|\, dt + \tau_i(x' + S_i(x')) - \tau_i(x') \\ &\leq NC(t" - t') + \tau_i(x' + S_i(x')) - \tau_i(x') \end{aligned}$$

i.e.

$$(1 - NC)(t" - t') \leq \tau_i(x' + S_i(x')) - \tau_i(x') \leq 0$$

which is a contradiction. □

Theorem 2.3. *Let F be almost USC with convex (and compact) values. If (A1)–(A3), (H1)–(H2) hold, then the system* (1.1)–(1.2) *has a solution.*

Remark 2.4. Obviously the conclusion of Lemma 2.2 holds for the system $\dot{x} \in G(t,x)$ when $|F(t,x)| \le C$ is replaced by $G(t,x) := \overline{F(t,x) \cap C\mathbb{B}} \neq \emptyset$. Moreover, the conclusion of Theorem 2.3 holds when (H2), (A1) are replaced by *There exists a constant $C > 0$ such that $NC < 1$ and $\overline{F(t,x) \cap C\mathbb{B} \cap T_D(x)} \neq \emptyset$ for every $x \in D$ and a.a. $t \in I$.*

Proof of Theorem 2.3. Note first that due to (A3) if $G_\varepsilon(t,x) = \overline{\text{co}}\, F(([t-\varepsilon, t+\varepsilon] \cap I) \setminus A, x + \varepsilon\mathbb{B} \cap D)$ then $|G_\varepsilon(t,x)| \le C$, where A is a null set and $\mathbb{B}$ is the unit ball in E.

Let 0 be a point of impulse. Then we consider (1.1)–(1.2) with an initial condition $x_0 + S_1(0) \in D$. Consequently one can suppose without loss of generality that 0 is not impulsive point. Due to Theorem 2.1 the problem (2) admits a nonempty $C(I,E)$ compact solution set. Therefore there exists $s := \max\{\tau > 0 :$ every solution of (2) is continuous on $(0,s)\}$. If $s = 1$ then the proof is complete. Otherwise $s < 1$ is an impulsive point for some solution $x(\cdot)$, i.e. $s = \tau_1(x(s))$. Since $x(s) \in D$, one has that $x' := x(s) + S_1(x(s-0)) \in D$. We study the problem (1.1)–(1.2) on $[s,1]$ with an initial condition $x(s) = x'$. Applying Theorem 2.1 and Lemma 2.2 one can derive the existence of solution $x(\cdot)$ on $[s = \tau_1(x), T = \tau_2(x))$, where $T > s$. Since $x(\cdot)$ is C-Lipschitz, one has that $\lim_{t \uparrow T} x(t)$ exists. If $T < 1$, then we study (1.1)–(1.2) on $[T,1]$. One can extend the solution $x(\cdot)$ on the whole interval I using the same method.

Due to (A1) and (A2), there exists an interval $[0,s]$ (with $s > 0$) such that every $x^i(\cdot)$ is continuous on $[0,s]$. □

Remark 2.5. It is possible to prove local existence of solutions, when (A3) is replaced by the linear growth assumption as in Theorem 2.1. However, in this case it is possible the solution to exists only on some neighborhood of 0 (not on the whole I).

Corollary 2.6. *Under the conditions of Theorem 2.3 there exists a constant $\lambda > 0$ such that for every solution $y(\cdot)$ of (1.1)–(1.2) $\tau_{i+1}(y(t)) - \tau_i(y(t)) \ge \lambda$, $i = 1,2,\dots,p-1$.*

Proof. Suppose the contrary, i.e. there exist a sequence $\{y^k(\cdot)\}_{k=1}^\infty$ such that

$$\min_i \big(\tau_{i+1}(y^k(t)) - \tau_i(y^k(t))\big) \to 0 \quad \text{as } k \to \infty. \tag{2.1}$$

Denote by τ_i^k the i-th impulse of $y^k(\cdot)$. Passing to subsequences if necessary, we may assume that $\lim_{k\to\infty} \tau_i^k = \tau_i$. Since $y^k(\cdot)$ are C Lipschitz on every $[\tau_i^k, \tau_{i+1}^k]$ there exists a subsequence converging to a solution $y(\cdot)$ of (1.1)–(1.2) with impulsive points τ_i for $i = 1,2,\dots,p$. Due do (2.1) either $\tau_i(x(\tau_i) + S_i(x(\tau_i))) = \tau_{i+1}$ for some i-contradiction with (A1), or $\tau_i = \tau_{i+1}$-contradiction with (A2). □

Now we study the mixed semicontinuous case, namely we assume that $F(\cdot,\cdot)$ is almost USC with convex compact values in some points and it is almost LSC with compact values in others.

Note that the most papers consider the case when $F(\cdot,\cdot)$ is either almost USC or it is almost LSC. The tedious proofs of the existence result in mixed semicontinuous case were simplified in separable Banach spaces in the very recent papers [16] and [19]. Here we extend the results presented in these papers to the case of differential inclusions with impulses.

Assume that E is a separable Banach space. Let $\mathcal{A} \subset I \times D$ and let $\mathcal{A} \in \mathcal{L} \otimes \mathcal{B}$, where $\mathcal{L}$ is the class of Lebesgue measurable subsets of I and $\mathcal{B}$ – the class of Borel subsets of E. We require also that for every t with $(t,x) \in A$ the projections $\{z \in D : (t,z) \in \mathcal{A}\}$ be relatively open (in D).

(A4) $F(\cdot,\cdot)$ is almost LSC on $\mathcal{A}$. $F(\cdot,x)$ is measurable for every $x \in D$, $F(t,\cdot)$ is USC with convex values on $(I \times D) \setminus \mathcal{A}$. Moreover, there exists a constant C such that $|F(t,x)| \le C$ for every $x \in D$ and a.e. $t \in I$, and $NC < 1$.

(A5) There exists a null set $\mathcal{N} \subset I$ such that $F(t,x) \subset T_D(x)$ for every $(t,x) \in \mathcal{A}$ when $t \notin \mathcal{N}$.

The following lemma is used in the proof of Theorem 2.8 below.

Lemma 2.7 ([10, Theorem 2])**.** *Let X, Z be two Banach spaces, let $\Omega \subset I \times X$ be nonempty and let $M > 0$. Then any closed valued LSC multifunction from Ω into Z admits a Γ^M-continuous selection.*

Theorem 2.8. *Under assumptions (A1)–(A5) and (H1)–(H2), the system* (1.1)–(1.2) *has a solution.*

Proof. Since $F(\cdot,\cdot)$ is almost LSC, one has that there exists a sequence $\{J_n\}_{n=1}^{\infty}$ of pairwise disjoint compacts $J_n \subset I$ such that $F(\cdot,\cdot)$ is LSC on $(J_n \times D) \cap \mathcal{A}$ for every n. Furthermore, its union is of full measure (without loss of generality we ca assume that it is equal to $I \setminus \mathcal{N}$). Let $\Omega_n = (J_n \times D) \cap \mathcal{A}$. Then define

$$G(t,x) = \begin{cases} F(t,x) & (t,x) \in \big(J_n \times D\big) \setminus \Omega_n \\ G_n(t,x) & (t,x) \in \Omega_n,\ n = 1,2,\dots \\ 0 & t \in \mathcal{N} \end{cases}$$

Here $G_n(t,x) = \cap_{\varepsilon>0} \overline{\mathrm{co}}\, f_n(A_\varepsilon)$, $A_\varepsilon = \big([t-\varepsilon, t+\varepsilon] \times (x + \varepsilon U)\big) \cap \Omega_n$, where $f_n(\cdot,\cdot)$ are Γ^{C+1} continuous selections of $F(\cdot,\cdot)$ on Ω_n. It is easy to see that due to (H1) $G(\cdot,\cdot)$ is almost USC.

There exists a measurable selection of $T_D(x) \cap G(\cdot,x)$. From [14, proposition 5.1] we know that there exists an almost USC $G_0(t,x) \subset G(t,x)$ with convex and compact values satisfying the conditions of Theorem 2.3 such that for every measurable $u(\cdot), v(\cdot)$ with $v(t) \in G(t,(u(t))$ it follows that $v(t) \in G_0(t,(u(t))$. Let $D' = \{x_i\}_{i=1}^{\infty}$ be a dense subset of D. Fix $\varepsilon > 0$. Let $f_i(t) \in G(t,x_i) \cap T_D(x_i)$ be measurable. Therefore there exists a compact $I_\varepsilon \subset I$ with $\mathrm{meas}(I_\varepsilon) > 1 - \varepsilon$ such that $f_i(\cdot)$ are continuous on I_ε for every i. Due to (H1) for every $t \in I_\varepsilon$ every sequence $\{f_i(t)\}_{i=1}^{\infty}$ has a density point say f_t. Let $x_i \to x$, therefore $f_t \in G_0(t,x)$, because $G_0(t,\cdot)$ is USC. Thus $G_0(\cdot,\cdot)$ admits nonempty values and hence $G_0(t,x) \cap T_D(x) \neq \emptyset$. Therefore G_0 satisfies all the conditions of Theorem 2.3.

Hence the system (1.1)–(1.2) with F replaced by G_0 has a solution. $\square$

3. Properties of the solution set of impulsive differential inclusion

Let $\mathfrak{Im}_{k,L}$ be the set of all functions $x(\cdot)$ which are L-Lipschitz on $[t_i(x) + 0, t_{i+1}(x)]$ and have no more than k jump points $t_1(x) < t_2(x) < \dots < t_k(x)$. Note that in general t_i depend on x, i.e. the impulses are not fixed times.

Proposition 3.1. *The space $\mathfrak{Im}_{k,L}$ equipped with the usual $L^1(I,E)$ norm becomes a complete metric space.*

Proof. One must only show that every Cauchy sequence $\{x^k(\cdot)\}_{k=1}^{\infty}$ converges to a $\mathfrak{Im}_{k,L}$ function, because $\mathfrak{Im}_{k,L} \subset L^1$. By Egorov' theorem if a sequence $\{y^k(\cdot)\}_{k=1}^{\infty}$ of L-Lipschitz functions converges in L^1 norm to $y(\cdot)$ then the latter is also L-Lipschitz. Let $\{x^n(\cdot)\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathfrak{Im}_{k,L}$. Denote by t_j^n (as $j = 1, 2, \dots, k$) the times of (possible) jumps of $x^n(\cdot)$. Passing to subsequences if necessary one can assume that $\lim_{n\to\infty} t_j^n = t_j$ for $j = 1, 2, \dots, k$. Let $0 \le t_1 \le t_2 \le \dots \le t_k \le 1$. Given $\varepsilon > 0$ we consider $I_\varepsilon = \cup_{j=1}^{k}(t_j - \frac{\varepsilon}{k}, t_j + \frac{\varepsilon}{k})$. It is easy to see that L^1 limit of this (sub)sequence on $I \setminus I_\varepsilon$ is L-Lipschitz function. Since it is valid for every $\varepsilon > 0$ one can conclude that the limit function $x(\cdot)$ is L-Lipschitz on every (t_j, t_{j+1}). The L^1 limit is unique and hence $x(\cdot) \in \mathfrak{Im}_{k,L}$. □

We will also need the following assumption.

(A6) The functions $S_i : D \to D$ are Lipschitz continuous with constant μ such that $C\mu < 1$.

Let E be a Banach space with single valued duality map $J : E \to E^*$. Recall that the map $F : I \times D \to E$ is said to be One Sided Lipschitz (OSL) when there exists a constant L such that $h_F(t, x, J(x - y)) - h_F(t, y, J(x - y)) \le L|x - y|^2$ for every $x, y \in D$, where $h_F : I \times E \times E^* \to \mathbb{R}$ is the lower Hamiltonian defined as $h_F(t, x, p) = \inf\{\langle p, v\rangle : v \in F(t, x)\}$. We refer to [15, 16] and the references therein for theory of OSL differential inclusions.

We will use the following lemma which is a particular case of [25, Lemma 2].

Lemma 3.2. *Let $a_1, a_2, b \ge 0$ and for $i = 1, 2, \dots, p$ let*

$$\delta_i^+ \le a_1\delta_i^-, \quad \delta_i^- \le a_2\delta_{i-1}^+ + b$$

then $\delta_i^- \le b\sum_{j=0}^{i-1}(a_1a_2)^j + \delta_0(a_1a_2)^i$, where $\delta_0^+ \ge 0$.

The following result is the well known relaxation theorem. However, to our knowledge this theorem has not been studied in case of impulsive differential inclusions. We follow the proof from [15] (given there for system without impulses).

Theorem 3.3. *Let $F(\cdot,\cdot)$ be almost continuous with nonempty convex compact values. Further we assume that it is OSL and $|F(t, x)| \le C$. If $\overline{\text{ext}}\ F(t, x) \subset T_D(x)$ for every $x \in D$ and a.a. $t \in I$ then under (A1)–(A6), (H1), the solution set of* (1.3)–(1.4) *is dense in the solution set of* (1.1)–(1.2).

Proof. Let $x(\cdot)$ be a solution of (1.1)–(1.2). Denote $R(t, x) = \overline{\text{ext}} F(t, x(t))$. Since $F(\cdot,\cdot)$ is almost continuous, one has that $R(\cdot,\cdot)$ is almost LSC [29, Lemma 2.3.7]). Define:

$$G_\varepsilon(t, y) = \overline{\{v \in R(t, x(t)) : \ \langle J(x(t) - y), \dot{x}(t) - v\rangle < L|x(t) - y|^2 + \varepsilon^2/2\}}.$$

From [20, prop. 2.62 p. 55 vol. I] we know that $G_\varepsilon(\cdot,\cdot)$ is almost LSC with nonempty compact values. One can prove as in the proof of Theorem 2.8 that the system

$$\begin{gathered}\dot{y}(t) \in G_\varepsilon(t, y(t)), \quad y(0) = x_0 \text{ a.e. } t \in I = [0, 1],\ t \ne \tau_i(y), \\ \Delta y|_{t=\tau_i(y)} = S_i(y(\tau_i(y) - 0)), \quad i = 1, \dots, p,\ x(t) \in D\end{gathered} \tag{3.1}$$

has a solution. Indeed, let $g_\varepsilon(t, x) \in G_\varepsilon(t, x)$ be almost Γ^{C+1} continuous, i.e. $g_\varepsilon(\cdot,\cdot)$ is Γ^{C+1} continuous on $I_k \times D$ $(k = 1, 2, \dots)$, where $\cup_{k=1}^{\infty} I_k$ has full measure and I_k are nonempty pairwise disjoint compact sets.

We let $g_k(t,x) := \cap_{\delta>0}\overline{\mathrm{co}}\, g_\varepsilon\left((t-\delta,t+\delta)\cap I_k, x+\beta\mathbb{B}\right)$ for $t\in I_k$. Define

$$g(t,x) := \begin{cases} g_k(t,x) & t\in I_k,\ k=1,2,\dots \\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that $g(t,x)\subset F(t,x)$ is almost USC and it satisfies all the assumptions of Theorem 2.3. Hence (1.1)–(1.2) with F replaced by g admits a solution. Let $y(\cdot)$ be a solution of (3.1). On every common interval of continuity of $x(\cdot)$ and $y(\cdot)$ one has $\langle J(x(t)-y(t)), \dot{x}(t)-\dot{y}(t)\rangle \le L|x(t)-y(t)|^2+\varepsilon^2/2$. That is

$$\frac{d}{dt}|x(t)-y(t)|^2 \le 2L|x(t)-y(t)|^2+\varepsilon^2.$$

Consequently, on every such interval $[\tau,\nu]$ one has $|x(t)-y(t)|^2 \le e^{2L(t-\tau)}\delta^2 + \varepsilon^2\int_\tau^t e^{2L(t-s)}\,ds$, where $\delta=|x(\tau)-y(\tau)|$. Hence

$$|x(t)-y(t)| \le e^{L(t-\tau)}\delta + f(t)\varepsilon, \tag{3.2}$$

where $f(t)=\max\{1,e^{L(t-\tau)}\}$. Let $a_1=\frac{1+N+C\mu}{1-C\mu}$, let $a_2=\max\{e^L,1\}$ and let $b=\varepsilon e^{q(L)}$, where

$$q(L) = \begin{cases} \frac{e^L-1}{L} & L\neq 0 \\ 1 & L=0.\end{cases}$$

We are ready to apply Lemma 3.2. Let $\delta_0^+=|x_0-y_0|$ (when $y(0)=y_0\neq x_0$ in (3.1)). We will show that for fixed $\delta>0$ there exists $\varepsilon(\delta)>0$ such that: For every $0<\varepsilon<\varepsilon(\delta)$ there exists a solution $y(\cdot)$ of (3.1) such that $|x(t)-y(t)|<\delta$ for $t\in I\setminus\cup_{i=1}^p[\tau_i^-,\tau_i^+]$, where $\tau_i^-=\max\{\tau_i^x,\tau_i^y\}$ and $\tau_i^+=\min\{\tau_i^x,\tau_i^y\}$. Moreover,

$$\sum_{i=1}^p|\tau_i^x-\tau_i^y|<\delta.$$

Here τ_i^x and τ_i^y are the jump points of $x(\cdot)$ and $y(\cdot)$. First we assume that $\tau_i^x<\tau_{i+1}^y$ and $\tau_i^y<\tau_{i+1}^x$ and afterward we will see that for sufficiently small ε it is the case.

The rest of the proof is very similar to the proof of Theorem 2 of [25] and will be given, for reader convenience. Due to (3.2) one has $|x(t)-y(t)|\le a_1\delta_0^+ + b$ on $[0,\tau_1^-]$. Hence $\delta_1^-=|x(\tau_1^-0)-x(\tau_1^-0)|\le a_1\delta_0^+ + b$, because obviously $f(t)<b$ and $a_2>e^{L(t-\tau)}$ for any interval $[\tau,\nu]$. Evidently denoting $\delta_i^+=|x(\tau_i^++0)-y(\tau_i^++0)|$ one has that $\delta_{i-1}^-\le a_2\delta_i^+ + b$.

If $\tau_1^y<\tau_1^x$ then $|x(\tau_i^+-0)-y(\tau_i^--0)|\le\delta_i^- + |x(\tau_i^+-0)-x(\tau_i^-)|\le\delta_i^- + C(\tau_i^+-\tau_i^-)$. Consequently, $\tau_i^+-\tau_i^-=|\tau_i(x(\tau_i^+-0)-\tau_i(y(\tau_i^-0)|\le\mu|x(\tau_i^+-0)-y(\tau_i^--0)|\le\mu\left(\delta_i^-+C(\tau_i^+-\tau_i^-)\right)$ and hence

$$\tau_i^+-\tau_i^- \le \frac{\mu\delta_i^-}{1-C\mu}. \tag{3.3}$$

Therefore, $|x(\tau_i^+ - 0) - y(\tau_i^- - 0)| \le \delta_i^- + C\frac{\mu\delta_i^-}{1-C\mu}$. For δ_i^+ we have

$$\begin{aligned}
\delta_i^+ &\le \delta_i^- + |x(\tau_i^+) - x(\tau_i^- - 0)| + |y(\tau_i^+) - y(\tau_i^- - 0)| \\
&\le \delta_i^- + |S(x(\tau_i^+ - 0) - S(y(\tau_i^+ - 0)| + \Big| \int_{\tau_i^-}^{\tau_i^+} \dot{x}(t) - \dot{y}(t)\,dt \Big| \\
&\le \delta_i^- + 2C(\tau_i^+ - \tau_i^-) + N|x(\tau_i^+ - 0) - y(\tau_i^- - 0)| \\
&\le \delta_i^- + \frac{2C\mu\delta_i^-}{1-C\mu} + N\big(\delta_i^- + C\frac{\mu\delta_i^-}{1-C\mu}\big) \\
&= a_1\delta_i^-.
\end{aligned}$$

Due the symmetry one can conclude that $\delta_i^+ \le a_1\delta_i^-$ also when $\tau_1^x \le \tau_1^y$. This is true for $i = 1, 2, \dots, p$. It follows from Lemma 3.2 that $\delta_i^- \le b\sum_{j=0}^{i-1}(a_1a_2)^j + \delta_0(a_1a_2)^i$.

We have only to see that $\tau_i^x < \tau_{i+1}^y$ and $\tau_i^y < \tau_{i+1}^x$ to complete the proof. For sufficiently small ε it follows that

$$|\tau_i^x - \tau_i^y| < \min_{0<i<p-1} \frac{\tau_{i+1}^x - \tau_i^x}{4}$$

thanks to (3.3). Note that $x(\cdot)$ is fixed and hence τ_i^x are known. It is evidently also that for every $\delta > 0$ there exists $\varepsilon(\delta)$ such that for every $0 < \varepsilon < \varepsilon(\delta)$ there exists a solution $y(\cdot)$ of (3.1) with $\sum_{i=1}^p |\tau_i^x - \tau_i^y| < \delta$. $\square$

Further, in this section we assume that $E \equiv \mathbb{R}^n$. We will study (1.1)–(1.2) with the help of the assumption

(H3) There exists a null set $\mathcal{N} \subset I$ with $h_F(t, x, \zeta) \le 0$, for all $\zeta \in N_D^P(x)$, for all $x \in S$, for all $t \in I \setminus \mathcal{N}$.

Remark 3.4. The condition (H3) is weaker than (H2) when E is a Hilbert space, however it is not applicable in more general spaces.

We assume further that $F(\cdot,\cdot)$ is almost USC with nonempty convex compact values. The following theorem is proved for autonomous $F(\cdot)$ in [12, 13] and extended to non-autonomous case in [17].

Theorem 3.5. *Under assumptions (A1)–(A3), the system* (1.1)–(1.2) *has a solution if and only if (H3) holds.*

Proof. Assume that $t(x_0)$ is not impulsive point. Therefore, there exists a neighborhood $x_0 + \varepsilon\mathbb{B}$ where $t(x)$ is not a jump point for every $x \in x_0 + \varepsilon\mathbb{B}$. Consequently from Theorem 1 of [17] we know that there exists $t' > t$ such that the system (1.1)–(1.2) has a solution on $[t, t']$. One can continue as in the proof of Theorem 2.3. If x_0 is an impulsive point then we consider the system (1.1)–(1.2) with x_0 replaced by $x(0) + S_1(x_0)$ (the solution after impulse).

The proof of only if part is omitted, because it is the same as the proof in case without impulses (cf. [17]). $\square$

When $F(\cdot,\cdot)$ is defined on the whole space the invariance problem becomes simpler. We will call the solutions $x(\cdot)$ which belong to D *viable*.

The system (1.1)–(1.2) is called weakly invariant when there exists a viable solution $x(\cdot)$. The system (1.1)–(1.2) is said to be (strongly) invariant when all the solutions are viable.

We will say that the multivalued map $G(t,x) \subset F(t,x)$ is a submultifunction (of F) when $G(\cdot,\cdot)$ is almost USC with nonempty convex compact values.

The following lemma is an extension of [17, Proposition 3].

Lemma 3.6 (Invariance principle)**.** *Suppose that (A1)–(A3) hold. If $F(t,\cdot)$ is OSL, then the system* (1.1)–(1.2) *is invariant if and only if the system*

$$\dot{x}(t) \in G(t, x(t)), \quad x(0) = x_0 \in D, \quad a.e.\ t \in I = [0,1],\ t \neq \tau_i(x), \tag{3.4}$$

$$\Delta x|_{t=\tau_i(x)} = S_i(x), \quad i = 1, \dots, p,\ x(t) \in D, \tag{3.5}$$

is weakly invariant for every submultifunction G.

Proof. Let (1.1)–(1.2) be strongly invariant. Since the system (3.4)–(3.5) has a solution (thanks to Theorem 3.5, one has that it is strongly (and hence weakly) invariant. Let (3.4)–(3.5) be weakly invariant for every submultifunction G. Let $x(\cdot)$ be a solution of (1.1)–(1.2). We define the multifunction

$$G(t,y) = \left\{ v \in F(t,y) :\ \langle x(t) - y, \dot{x}(t) - v \rangle \leq L|x(t) - y|^2 \right\}.$$

Since $F(t,\cdot)$ is OSL, one has that $G(\cdot,\cdot)$ is nonempty valued. Let $u, v \in G(t,y)$ then $\langle x(t) - y, \dot{x}(t) - \lambda u + (1-\lambda)v \rangle \leq (\lambda + (1-\lambda)v)\, L|x(t) - y|^2$, i.e. G is convex valued. Let F be USC on $\mathcal{A} \times \mathbb{R}^n$, let $(t,y) \in \mathcal{A} \times \mathbb{R}^n$ and let $\dot{x}(\cdot)$ is continuous on $\mathcal{A}$. If $\mathcal{A} \ni t_i \to t$, $y_i \to y$ and $G(t_i, y_i) \ni v_i \to v$, then

$$\lim_{i\to\infty} \langle x(t_i) - y_i, \dot{x}(t_i) - v_i \rangle = \langle x(t) - y, \dot{x}(t) - v \rangle.$$

Moreover, $\lim_{i\to\infty} L|x(t_i) - y_i|^2 = L|x(t) - y|^2$. Thus $G(\cdot,\cdot)$ is almost USC (because $G(t,x) \subset F(t,x)$).

Therefore, G is a submultifunction of F. Let $y(\cdot)$ be viable solution of (3.4)–(3.5). If $[\mu,\nu]$ is an interval without impulses of $x(\cdot)$ and $y(\cdot)$, then $\langle x(t) - y(t), \dot{x}(t) - \dot{y}(t) \rangle \leq L|x(t) - y(t)|^2$. Thus $\frac{d}{dt}|x(t) - y(t)| \leq 2L|x(t) - y(t)|^2$. If $x(\mu) = y(\mu)$, then $x(t) \equiv y(t)$ on $[\mu,\nu]$ thanks to Gronwall inequality.

However, $x(0) = y(0) = x_0$. Consequently $x(t) \equiv y(t)$ on the whole interval I and hence $x(t) \in D$ for every $t \in I$. The last implies that (1.1)–(1.2) is invariant. $\square$

The following theorem is an immediate corollary of Theorem 3.5 and Lemma 3.6. It extends [17, Proposition 3] to impulsive systems.

Theorem 3.7. *Let the conditions of Lemma 3.6 hold. Then system* (1.1)–(1.2) *is invariant if and only if for every submultifunction G there exists a null set $\mathcal{N}_G$ such that $h_G(t,x,\zeta) \leq 0$, for all $\zeta \in N_D^P(x)$, for all $x \in S$, for all $t \in I \setminus \mathcal{N}_G$.*

Recall that A is said to be absolute (metric) retract [14, p. 83] if, given a metric space Ω, closed $B \subset \Omega$ and continuous $f : B \to A$, there exists a continuous extension $\tilde{f} : \Omega \to A$ of f. A is said to be R_δ if $A = \cap_{k\geq 1} A_k$ for decreasing sequence of compact absolute retracts A_k.

The set B is said to be contractible if there is $x_0 \in B$ and continuous $h : [0,1] \times B \to B$ such that $h(0,x) = x$ and $h(1,x) = x_0$ on B. It is well known that A is R_δ if and only if $A = \cap_{n\geq 1} B_n$ with decreasing sequence of closed contractible sets (cf. [21]).

Theorem 3.8. *Let $D \equiv \mathbb{R}^n$ and let $F(\cdot,\cdot)$ be almost USC with nonempty convex compact values. Under assumption (A1)–(A3) and (A6) the solution set of differential inclusion* (1.1)–(1.2) *is nonempty R_δ in $\mathfrak{Im}_{p,L}$.*

Proof. Due to Lemma 2.2 the solution set of (1.1)–(1.2) is in $\mathfrak{Im}_{p,L}$ (it is nonempty thanks to Theorem 2.3). Now we will use the locally Lipschitz approximation of USC multifunctions.

Let $I \setminus \mathcal{I} = \cup_{j=1}^{\infty} I_j$ be a sequence of pairwise disjoint compacts such that F is USC on $I_j \times E$ and $\text{meas}(\mathcal{I}) = 0$. Define $F_n(t,x) = \sum_{\lambda\in\Lambda} \psi_\lambda(t,x) C_\lambda$ on $I_j \times E$ with $C_\lambda := \overline{\text{co}}\, F(t, x + 2r_n\mathbb{B})$, where $r_n = 3^{-n}$. It is easy to see that $F(t,x) \subset F_{n+1}(t,x) \subset F_n(t,x) \subset \overline{\text{co}}\, F(t, x + 2r_n\mathbb{B})$. We can take a strongly measurable selection g_λ of $F(\cdot, x_\lambda)$ and define $f(t,x) = \sum_\lambda \varphi_\lambda(x) g_\lambda(t)$. Therefore, $f(\cdot,x)$ is strongly measurable and $f(t,\cdot)$ is locally Lipschitz (cf. [14, Lemma 2.2]). Consequently, the system

$$\dot{x}(t) = f(t, x(t)), \text{ a.e. on } I, \; x(t) = y \; t \neq \tau_i(x), \tag{3.6}$$

$$\Delta x|_{t=\tau_i(x)} = S_i(x(\tau_i(x) - 0)), \; i = 1, \dots, p, \tag{3.7}$$

admits a unique solution, which depends continuously on (t, y) (cf. [5, 28]). Thus the solution set of (1.1)–(1.2) with F replaced by F_n has a nonempty contractible solution set Sol_n (cf. [14, p. 82]). Furthermore it is easy to see that the solution set Sol of (1.1)–(1.2) satisfies $Sol = \cap_{n=1}^{\infty} Sol_n$. Consequently Sol is R_δ set. □

References

[1] Aubin J.-P., *Impulsive Differential Inclusions and Hybrid Systems: A Viability Approach* Lecture Notes, Univ. Paris 2002.

[2] Aubin J.-P., *Viability Theory*, Birkhäuser, Boston, 1991.

[3] Aubin J.-P., Cellina A., *Differential Inclusions*, Springer, Berlin, 1984.

[4] Aubin J.-P., Lygeros J., Quincampoix M., Sastry S., Seube N., Impulse differential inclusions: a viability approach to hybrid systems, *IEEE Tras. Aotomat. Control* **47(1)** (2002) 2-20.

[5] Bainov D., Simeonov P., *Systems with Impulsive Effect. Stability, Theory and Applications*, Ellis Series in Maths and Appl., Ellis Horwood, Chicester 1989.

[6] Benchohra M., Boucherif A., On first order initial value problems for impulsive differential inclusions in Banach spaces, Dyn. Syst. Appl. **8** (1999) 119-126.

[7] Benchohra M., Henderson J., Ntouyas S., *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Company, New York (in press).

[8] Benchohra M., Henderson J., Ntouyas S., Ouahabi A., existence results for impulsive lower semicontinuous differential inclusions, *Int. J. Pure Appl. Math.* **1** (2002) 431-443.

[9] Bothe D., *Multivalued Differential Equations on Graphs and Applications*, Ph.D. Thesis, Paderborn, 1992.

[10] Bressan A., Colombo G., Selections and representations of multifunctions in paracompact spaces, *Studia Math.* **102** (1992) 209-216.

[11] Cardinali T., Servadei R., Periodic solutions of nonlinear impulsive differential inclusions with constrains, *Proc. Amer. Math. Soc.* **132** (2004) 2339-2349.

[12] Clarke F., Ledyaev Yu., and Radulescu M., Approximate invariance and differential inclusions in Hilbert spaces, *J. Dynam. Control Syst.* **3** (1997) 493-518.

[13] Clarke F., Ledyaev Yu., Stern R. and Wolenski P., *Nonsmooth Analysis and Control Theory,* Springer, New York, 1998.

[14] Deimling K., *Multivalued Differential Equations*, De Gruyter Berlin, 1992.

[15] Donchev T., One Sided Lipschitz Multifunctions and Applications, Proc. Louisiana Math. Control Conf., in *Optimal Control, Stabilization and Nonsmooth Analysis*, de Querioz, Malisoff, Wolenski (Eds.), *Lect. Notes Contr. Inf. Sci.* **301**, Springer, Berlin, 2004, pp. 333-342.

[16] Donchev T., Mixed type semicontinuous differential inclusions in Banach spaces, *Ann. Polon. Math.* **LXXVII.3** (2001) 245-259.

[17] Donchev T., Rios V., Wolenski P., Strong Invariance and one-sided Lipschitz multifunctions, *Nonlinear Analysis TMA* **60** (2005) 849-862

[18] Frigon M., O'Regan D., First order impulsive initial and periodic problems with variable moments, *J. Math. Anal. Appl.* **233** (1999) 730-739.

[19] Fryszkowski A., Gornievicz L., Mixed Semicontinuous Mappings and Their Applications to Differential Inclusions, *Set-Valued Analysis* **8** (2000) 203-217.

[20] Hu S., Papageorgiou N., *Handbook of Multivalued Analysis*, vol. I *Theory* 1997; vol. II *Applications* 2000, Kluwer Dodrecht.

[21] Hyman D., On decreasing sequences of compact absolute retracts, *Fundamenta Math.* **64** (1969) 91-97.

[22] Kaul S., Liu X., Vector Lyapunov functions for impulsive differential systems with variable times, *Dynam. Contin. Discrete Impuls. Systems* **6** (1999) 25-38.

[23] Lakshmikantham V., Bainov D., Simeonov P., *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[24] Pianigiani G., On the fundamental theory of multivalued differential equations, *J. Differential Equations* **25** (1977) 30-38.

[25] Plotnikov V., Kitanov N., On Continuous dependence of solutions of impulsive differential inclusions and impulse control problems, *Kybern. Syst. Analysis* (2002) **No 5** 143 -154 (in Russian); English translation: Cybernetics System Analysis **38** (2002) 749-758.

[26] Plotnikov V., Ivanov R., Kitanov N., Method of averaging for impulsive differential inclusions, *Pliska. Stud. Math. Bulgar.* **12** (1998) 43-55.

[27] Plotnikov V., Plotnikov A., Vityuk A., *Differential Equations with Multivalued Right-Hand Side. Asymptotic Methods*, Astro Print Odessa, 1999. (Russian)

[28] Samoilenko A., Peresyuk N., *Differential Equations with Impulsive Effects*, World Scientific, Singapore, 1995.

[29] Tolstonogov A., *Differential Inclusions in a Banach Space*, Kluwer, Dordrecht, 2000.

[30] Watson P., Impulsive differential inclusions, *Nonlin. World* **4** (1997) 395-402.

Tzanko Donchev

Department of Mathematics, University of Architecture and Civil Engineering, 1 "Hr. Smirnenski" str., 1046 Sofia, Bulgaria

E-mail address: tdd51us@yahoo.com