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EXISTENCE RESULTS FOR NONLOCAL MULTIVALUED BOUNDARY-VALUE PROBLEMS

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Abstract. In this paper we establish some existence results for nonlocal multivalued boundary-value problems. Our approach is based on existence results for operator inclusions involving a suitable closed-valued multifunction; see [2, 3]. Some applications are given.

1. Introduction

Let $(X, \|\cdot\|_X)$ be a separable real Banach space and let $(\mathbb{R}^n, \|\cdot\|)$ be the real Euclidean n-space with the norm $\|z\| = \max_{1\leq i\leq n} |z_i|$ and induced metric d. Denote by $M([0,1], X)$ the family of all (equivalence classes of) strongly Lebesgue measurable functions from $[0,1]$ to X. The papers [2] and [3] provide some existence results for operator inclusions of the type

$$\begin{gathered} u \in U, \\ \Psi(u)(t) \in F(t, \Phi(u)(t)) \quad \text{a.e. in } [0,1], \end{gathered} \tag{1.1}$$

where U is a nonempty set, F is a multifunction from $[0,1]\times X$ into $\mathbb{R}^n$, and $\Phi : U \to M([0,1], X)$, $\Psi : U \to L^s([0,1], \mathbb{R}^n)$ are two abstract operators (see Theorems 2.1, 2.2, and 2.3 below). Their approach is chiefly based on the following conditions:

(U1) Ψ is bijective and for any $v \in L^s([0,1], \mathbb{R}^n)$ and any sequence $\{v_h\} \subset L^s([0,1], \mathbb{R}^n)$ weakly converging to v in $L^q([0,1], \mathbb{R}^n)$ there exists a subsequence of $\{\Phi(\Psi^{-1}(v_h))\}$ which converges a.e. in $[0,1]$ to $\Phi(\Psi^{-1}(v))$. Furthermore, a nondecreasing function $\varphi : [0, +\infty[\to [0, +\infty]$ can be defined in such a way that

$$\operatorname*{ess\,sup}_{t\in[0,1]} \|\Phi(u)(t)\| \leq \varphi(\|\Psi(u)\|_p) \quad \forall u \in U. \tag{1.2}$$

Here $p, q, s \in [0, +\infty]$ with $q < +\infty$ and $q \leq p \leq s$.

(U2) To each $\rho \in L^s([0,1], \mathbb{R}^+_0)$ there corresponds a nonnegative measurable function ρ^* so that if $u \in U$ and $\|\Psi(u)(t)\| \leq \rho(t)$ a.e. in $[0,1]$, then $\Phi(u)$ is Lipschitz continuous with constant $\rho^*(t)$ at almost all $t \in [0,1]$.

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(F1) There exists $r > 0$ such that the function
$$M(t) := \sup_{\|x\|_X \le \varphi(r)} d(0, F(t,x)), \quad t \in [0,1],$$
belongs to $L^s([0,1], \mathbb{R}_0^+)$ and $\|M\|_p \le r$.

Let us denote by $W^{2,s}([0,1], \mathbb{R}^n)$ the space of all $u \in C^1([0,1], \mathbb{R}^n)$ such that u' is absolutely continuous in $[0,1]$ and $u'' \in L^s([0,1], \mathbb{R}^n)$.

The aim of this paper is to establish, under suitable assumptions, the existence of at least one generalized solution in $W^{2,s}([0,1], \mathbb{R}^n)$ to the problem
$$\begin{gathered} u'' \in F(t,u,u') \quad \text{a.e. } t \in [0,1] \\ u(0) - k_1 u'(0) = H_1(u), \\ u(1) + k_2 u'(1) = H_2(u), \end{gathered} \tag{1.3}$$
where $F : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a multifunction with nonempty closed values, for $i = 1, 2$, L_{H_i} and k_i are nonnegative constants, $H_i : W^{2,s}([0,1], \mathbb{R}^n) \to \mathbb{R}^n$ fulfill
$$\|H_i(u) - H_i(v)\| \le L_{H_i} \|u - v\|_\infty, \quad \forall u, v \in W^{2,s}([0,1], \mathbb{R}^n). \tag{1.4}$$
The key to solve problem (1.3) is to show that the operators Φ, Ψ and F satisfy the structural hypotheses (U1), (U2) and (F1). This study is motivated by many nonlocal boundary-value problems discussed in [1, 4, 5, 10]. For more details on these topics, see also [7, 8, 9]. Recent references are furnished in [6, 11, 13]. In our opinion this method exhibits at least two interesting features: owing to (F1), no compactness condition on the values of F is required; we can treat both the case when F takes convex values, $F(\cdot, x, y)$ is measurable, while $F(t, \cdot, \cdot)$ is upper semicontinuous, and the one where F is measurable and $F(t, \cdot, \cdot)$ is lower semicontinuos. However, in the latter case, in addition to (U1), we also need (U2). In all concrete situations, to verify (U1) and (U2) we exploit only the boundary conditions of (1.3) and we do not use the assumptions on F.

Basic definitions and preliminary results are given in Section 2. Main results are contained in Section 3, while Section 4 is devoted to some applications.

2. Basic definitions and preliminary results

Given a separable real Banach space $(X, \|\cdot\|_X)$, the symbol $\mathcal{B}(X)$ indicates the Borel σ-algebra of X. If W is a nonempty subset of X, $x_0 \in W$ and $\epsilon > 0$, we write $d(x_0, W) := \inf_{w \in W} \|x_0 - w\|_X$ as well as
$$B(x_0, \epsilon) := \{z \in X : \|x_0 - z\|_X \le \epsilon\}, \quad B^0(x_0, \epsilon) := \{z \in X : \|x_0 - z\|_X < \epsilon\}.$$
A function ψ from $[0,1]$ into X is said to be Lipschitz continuous at the point $t \in [0,1]$ when there exist a neighborhood V_t of t and a constant $k_t \ge 0$ such that $\|\psi(\tau) - \psi(t)\|_X \le k_t |\tau - t|$ for every $\tau \in [0,1] \cap V_t$. Given any $p \in [1, +\infty]$, we write p' for the conjugate exponent of p besides $L^p([0,1], X)$ for the space of $u \in M([0,1], X)$ satisfying $\|u\|_p < +\infty$, where
$$\|u\|_p := \begin{cases} \left(\int_0^1 \|u(t)\|_X^p d\mu \right)^{1/p} & \text{if } p < +\infty, \\ \operatorname{ess\,sup}_{t \in [0,1]} \|u(t)\|_X & \text{if } p = +\infty, \end{cases}$$
and μ is the Lebesgue measure on $[0,1]$. Let F be a multifunction from W into $\mathbb{R}^n$ (briefly, $F : W \to 2^{\mathbb{R}^n}$), namely a function which assigns to each point $x \in W$ a nonempty subset $F(x)$ of $\mathbb{R}^n$. If $V \subseteq W$ we write $F(V) := \cup_{x \in V} F(x)$ and $F|_V$ for

the restriction of F to V. The graph of F is the set $\{(x,z) \in W \times \mathbb{R}^n : z \in F(x)\}$. If $Y \subseteq \mathbb{R}^n$ we define $F^-(Y) := \{x \in W : F(x) \cap Y \neq \emptyset\}$. If $(W, \mathfrak{F})$ is a measurable space and $F^-(Y) \in \mathfrak{F}$ for any open subset Y of $\mathbb{R}^n$, we say that F is $\mathfrak{F}$-measurable, or simply measurable as soon as no confusion can arise. We denote with $\mathcal{L}$ the Lebesgue $\sigma-$algebra in $\mathbb{R}^n$. We say that F is upper semicontinuous at the point $x_0 \in W$ if to every open set $Y \subseteq \mathbb{R}^n$ satisfying $F(x_0) \subseteq Y$ there corresponds a neighborhood W_0 of x_0 such that $F(W_0) \subseteq Y$. The multifunction F is called upper semicontinuous when it is upper semicontinuous at each point of W. In such a case its graph is clearly closed in $W \times \mathbb{R}^n$ provided that $F(x)$ is closed for all $x \in W$. We say that F has a closed graph at x_0 if the condition $\{x_k\} \subseteq W$, $\{z_k\} \subseteq \mathbb{R}^n$, $lim_{k\to+\infty} x_k = x_0$, $lim_{k\to+\infty} z_k = z_0$, $z_k \in F(x_k)$, $k \in \mathbb{N}$, imply $z_0 \in F(x_0)$. We say that F is lower semicontinuous at the point x_0 if to every open set $Y \subseteq \mathbb{R}^n$ satisfying $F(x_0) \cap Y \neq \emptyset$ there corresponds a neighborhood V_0 of x_0 such that $F(x) \cap Y \neq \emptyset$, $x \in V_0$. The multifunction F is called lower semicontinuous when it is lower semicontinuous at each point of W. Finally, for $B \subseteq [0,1] \times X$, $\mathrm{proj}_X(B)$ indicates the projection of B onto X. We say that a multifunction $F : B \to 2^{\mathbb{R}^n}$ has the lower Scorza Dragoni property if to every $\epsilon > 0$ there corresponds a closed subset I_ϵ of $[0,1]$ such that $\mu([0,1] \setminus I_\epsilon) < \epsilon$ and $F|_{(I_\epsilon \times X)\cap B}$ is lower semicontinuous. Let D be a nonempty closed subset of X, let $A \subseteq [0,1] \times D$, and let $C := ([0,1] \times D) \setminus A$. We always suppose that the set A complies with

- (A1) $A \in \mathcal{L} \otimes \mathcal{B}(X)$ and $A_t = \{x \in D : (t,x) \in A\}$ is open in D for every $t \in [0,1]$.

Moreover, let F be a closed-valued multifunction from $[0,1] \times D$ into $\mathbb{R}^n$, let $m \in L^s([0,1], \mathbb{R}_0^+)$, and let $N \in \mathcal{L}$ with $\mu(N) = 0$. The conditions below will be assumed in what follows.

- (A2) $F|_A$ has the lower Scorza Dragoni property.
- (A3) $F(t,x) \cap B^0(0, m(t)) \neq \emptyset$ whenever $(t,x) \in A \cap [([0,1] \setminus N) \times D]$.
- (A4) The set $\{x \in \mathrm{proj}_X(C) : F|_C(\cdot, x) \text{ is measurable}\}$ is dense in $\mathrm{proj}_X(C)$.
- (A5) For every $(t,x) \in C \cap [([0,1] \setminus N) \times D]$ the set $F(t,x)$ is convex, $F(t,\cdot)$ has a closed graph at x, and $F(t,x) \cap B(0, m(t)) \neq \emptyset$.

Combining opportunely the above conditions we point out the abstract results that we will apply to study problem (1.1). More precisely we have the following results.

Theorem 2.1 ([2, Theorem 3.1]). *Let Φ, Ψ, φ be as in (U1) and let $F : [0,1] \times X \to 2^{\mathbb{R}^n}$ be a multifunction with convex closed values satisfying (F1). Suppose that*

- (A4') *The set $\{x \in X : F(\cdot, x)$ is $\mathfrak{F}$-measurable$\}$ is dense in X*
- (A5') *For almost every $t \in [0,1]$ and every $x \in X$, $F(t,\cdot)$ has closed graph at x.*

Then (1.1) *has at least one solution $u \in U$ with $\|\Psi(u)(t)\| \leq M(t)$ a.e. in $[0,1]$.*

Theorem 2.2 ([3, Theorem 2.3]). *Let $r > 0$ be such that $\|m\|_p \leq r$, and let $D = B(0_X, \varphi(r))$. Suppose that F, Φ, Ψ satisfy (A2)-(A5), (U1) and (U2). Then problem* (1.1) *has at least one solution $u \in U$ with $\|\Psi(u)(t)\| \leq M(t)$ a.e. in $[0,1]$.*

Theorem 2.3 ([3, Theorem 2.5]). *Suppose that $H : [0,1] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a closed-valued multifunction with the following properties:*

- (C1) *For almost all $t \in [0,1]$ and for all $x \in \mathbb{R}^n$, either $H(t,x)$ is convex or $H(t,\cdot)$ is lower semicontinuous at x.*
- (C2) *The set $\{x \in \mathbb{R}^n : H(\cdot, x)$ is measurable$\}$ is dense in $\mathbb{R}^n$.*

(C3) *For almost every $t \in [0,1]$ and every $x \in X$, $H(t,\cdot)$ has closed graph at x.*
(C4) *There is $m_1 \in L^s([0,1],\mathbb{R}_0^+)$ such that $H(t,x) \cap B(0,m_1(t)) \neq \emptyset$ a.e. in $[0,1]$, for all $x \in \mathbb{R}^n$.*

Then there exists a set $A \subseteq [0,1] \times \mathbb{R}^n$ and a closed-valued multifunction F from $[0,1] \times \mathbb{R}^n$ into $\mathbb{R}^n$ satisfying (A1)-(A5). Moreover, for almost every $t \in [0,1]$ and every $x \in \mathbb{R}^n$, one has $F(t,x) \subseteq H(t,x)$.

Moreover, to problem (1.3) we associate the following functions: $G : [0,1] \times [0,1] \to \mathbb{R}$ (the Green function associated with (1.3)),

$$G(t,s) := \frac{g(t,s)}{1+k_1+k_2}, \quad \text{with} \quad g(t,s) := \begin{cases} (k_1+t)(s-1-k_2), & 0 \le t \le s \le 1 \\ (k_1+s)(t-1-k_2), & 0 \le s \le t \le 1; \end{cases}$$

for every $r \in]0,+\infty[$, put

$$\varphi'(r) = m_{\varphi'} r + q_{\varphi'}, \quad \varphi''(r) = m_{\varphi''} r + q_{\varphi''}, \quad \varphi(r) = \max\{\varphi(r), \varphi'(r)\},$$

where

$$m_{\varphi'} := \frac{\sup_{[0,1]^2} |g(t,s)|}{1+k_1(1-L_{H_2})+k_2(1-L_{H_1})-(L_{H_1}+L_{H_2})},$$
$$q_{\varphi'} := \frac{((1+k_2)\|H_1(0)\|+(1+k_1)\|H_2(0)\|)}{1+k_1(1-L_{H_2})+k_2(1-L_{H_1})-(L_{H_1}+L_{H_2})},$$
$$m_{\varphi''} := (L_{H_1}+L_{H_2}) m_{\varphi'} + \sup_{[0,1]^2} |g(t,s)|,$$
$$q_{\varphi''} := (L_{H_1}+L_{H_2}) q_{\varphi'} + \|H_1(0)\| + \|H_2(0)\|.$$

3. Main results

Let F be a closed-valued multifunction from $[0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ fulfilling (F1).

Theorem 3.1. *Suppose that:*

(I1) *There exist $c, d \in \mathbb{R}^n$ such that*
$$H_1(ct+d) = d - ck_1, \qquad H_2(ct+d) = d + c(1+k_2).$$
(I2)
$$\frac{(1+k_2)L_{H_1}+(1+k_1)L_{H_2}}{1+k_1+k_2} < 1.$$
(I3) *For almost all $t \in [0,1]$ and all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $F(t,x,y)$ is convex.*
(I4) *The set $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : F(\cdot,x,y)$ is measurable$\}$ is dense in $\mathbb{R}^n \times \mathbb{R}^n$.*
(I5) *For almost every $t \in [0,1]$ the graph of $F(t,\cdot,\cdot)$ is closed.*

Then problem (1.3) admits at least one solution $u \in W^{2,s}([0,1])$ such that

$$\|u\|_\infty \le \varphi'(r), \quad \|u'\|_\infty \le \varphi''(r), \quad \|u''(t)\| \le M(t) \quad \text{for a.e. } t \in [0,1].$$

Proof. We apply Theorem 2.1 by choosing $X = \mathbb{R}^n \times \mathbb{R}^n$, $\Phi(u)(t) = (u(t), u'(t))$ and $\Psi(u)(t) = u''(t)$ for each $u \in U$, where

$$U = \Big\{ u \in W^{2,s}([0,1],\mathbb{R}^n) : \exists\, \sigma \in L^s([0,1],\mathbb{R}^n) \text{ such that}$$
$$u(t) = \frac{1}{1+k_1+k_2}\Big[(1+k_2-t)H_1(u) + (k_1+t)H_2(u)\Big] + \int_0^1 G(t,s)\sigma(s)ds \Big\}.$$

Clearly, since Y has finite dimension and (F1) holds, we only need to show that U_1 is verified. With this aim, we first observe that U is not empty. Indeed, by (I1), the function $w(t) := ct + d \in U$. Moreover, for each $u \in U$, arguing in standard way it results that $u'' \equiv \sigma$ and, by (1.4), we get the following inequalities

$$\|H_i(u)\| \le \|H_i(0)\| + L_{H_i}\|u\|_\infty. \quad \text{for } i = 1, 2,$$

$$\begin{aligned} \|u\|_\infty \le {} & \frac{1}{1+k_1+k_2}(1+k_2)[\|H_1(0)\| + L_{H_1}\|u\|_\infty] \\ & + \frac{1+k_1}{1+k_1+k_2}[L_{H_2}\|u\|_\infty + \|H_2(0)\|] + \int_0^1 \sup_{[0,1]^2} |G(t,s)|\|u''(s)\|\,ds. \end{aligned}$$

Hence

$$\begin{aligned} & \|u\|_\infty\Big(1 - \frac{(1+k_2)L_{H_1} + (1+k_1)L_{H_2}}{(1+k_1+k_2)}\Big) \\ & \le \frac{(1+k_1)\|H_2(0)\| + (1+k_2)\|H_2(0)\|}{1+k_1+k_2} + \frac{\sup_{[0,1]^2}|g(t,s)|}{1+k_1+k_2}\|u''\|_p, \end{aligned}$$

which implies

$$\|u\|_\infty \le \varphi'(\|u''\|_p). \tag{3.1}$$

Similarly, we have

$$\begin{aligned} \|u'\|_\infty \le {} & \frac{1}{1+k_1+k_2}\Big[(L_{H_1} + L_{H_2})\varphi'(\|u''\|_p) \\ & + \|H_1(0)\| + \|H_2(0)\| + \sup_{[0,1]^2}\Big|\frac{\partial g}{\partial t}(t,s)\Big|\|u''\|_p\Big], \end{aligned}$$

which clearly ensures,

$$\|u'\|_\infty \le \varphi''(\|u''\|_p). \tag{3.2}$$

So, (1.2) holds. Let us next prove that $\Psi : U \to L^s([0,1],\mathbb{R}^n)$ is injective. Arguing by contradiction, suppose that there exist two functions $u, v \in U$ and $B \subseteq [0,1]$ with $\mu(B) > 0$, such that $u(t) \ne v(t)$ for each $t \in B$ and $u''(t) = v''(t)$ a.e. in $[0,1]$. Pick $t \in [0,1]$. By (1.4), one has

$$\|u(t) - v(t)\| \le \frac{(1+k_2)L_{H_1} + (1+k_1)L_{H_2}}{1+k_1+k_2}\|u - v\|_\infty.$$

From this, taking into account (I2), we have $\|u - v\|_\infty = 0$, that is $u(t) = v(t)$ a.e. in $[0,1]$, which is absurd. Hence we get a contradiction. Next, fix $v \in L^s([0,1],\mathbb{R}^n)$ and a sequence $\{v_h\}$ weakly converging to v in $L^s([0,1],\mathbb{R}^n)$. To simplify the notation we put $u_h = \Psi^{-1}(v_h)$ and $u = \Psi^{-1}(v)$, i.e., $u_h'' = v_h$, $u'' = v$, $u_h'' \rightharpoonup u''$ in $L^s([0,1],\mathbb{R}^n)$ and for a.e. t in $[0,1]$ one has

$$\Phi(\Psi^{-1}(v_h))(t) = (u_h(t), u_h'(t)).$$

We claim that

$$\Phi(\Psi^{-1}(v_h))(t) \to \Phi(\Psi^{-1}(v))(t), \quad \text{a.e. } t \in [0,1]. \tag{3.3}$$

To see this, we first prove that

$$\lim_{h\to+\infty} u_h(t) = u(t) \quad \text{for all } t \in [0,1]. \tag{3.4}$$

Fix $t \in [0,1]$. By (1.4) it is easy to show that

$$
\begin{aligned}
&\|u_h(t) - u(t)\| \\
&\le \frac{1}{1+k_1+k_2}\Big((1+k_2)\|H_1(u_h) - H_1(u)\| + (k_1+1)\|H_2(u_h) - H_2(u)\|\Big) \\
&\quad + \Big\| \int_0^1 G(t,s)(u_h''(s) - u''(s))ds \Big\| \\
&\le \frac{(1+k_2)L_{H_1} + (1+k_1)L_{H_2}}{1+k_1+k_2}\|u_h - u\|_\infty + \Big\| \int_0^1 G(t,s)(u_h''(s) - u''(s))ds \Big\|.
\end{aligned}
$$

Moreover, since u_h'' weakly converges to u'' in $L^s([0,1],\mathbb{R}^n)$, and $G(t,\cdot) \in L^s([0,1])$ if $s \ge 1$, it results

$$
\lim_{h\to+\infty} \int_0^1 G(t,s)(u_h''(s) - u''(s))ds = 0.
$$

Therefore, we have

$$
\limsup_{h\to\infty} \|u_h - u\|_\infty \le \frac{(1+k_2)L_{H_1} + (1+k_1)L_{H_2}}{1+k_1+k_2} \limsup_{h\to\infty} \|u_h - u\|_\infty.
$$

Furthermore, since the sequence $\{\|u_h\|_p\}$ is bounded, from (3.1), it is easy to show that $\limsup_{h\to\infty} \|u_h - u\|_\infty < +\infty$. Hence, on account of (I2), the preceding inequality provides

$$
\lim_{h\to+\infty} \|u_h - u\|_\infty = 0. \tag{3.5}
$$

Now we prove that

$$
\lim_{h\to+\infty} u_h'(t) = u'(t) \quad \text{a.e. in } [0,1].
$$

To do this, we observe that if $u \in U$, then an easy computation ensures that

$$
u'(t) = \frac{1}{1+k_1+k_2}(H_2(u) - H_1(u)) + \int_0^1 \frac{\partial G(t,s)}{\partial t} u''(s)ds.
$$

Hence, for every $t \in [0,1]$, one has

$$
\begin{aligned}
\|u_h'(t) - u'(t)\| &\le \frac{1}{1+k_1+k_2}\Big(\|H_2(u_h) - H_2(u)\| \\
&\quad + \|H_1(u_h) - H_1(u)\|\Big) + \Big\| \int_0^1 \frac{\partial G(t,s)}{\partial t}(u_h''(s) - u''(s))ds \Big\| \\
&\le \frac{L_{H_1} + L_{H_2}}{1+k_1+k_2}\|u_h - u\|_\infty + \Big\| \int_0^1 \frac{\partial G(t,s)}{\partial t}(u_h''(s) - u''(s))ds \Big\|.
\end{aligned}
$$

Thus by (3.5) and taking into account that $\frac{\partial G(t,\cdot)}{\partial t} \in L^s([0,1])$, exploiting again that $u_h'' \rightharpoonup u''$ we also get

$$
\lim_{h\to+\infty} \|u_h'(t) - u'(t)\| = 0. \tag{3.6}
$$

From (3.5) and (3.6), we conclude that (3.3) holds, and the proof is complete. $\square$

Remark 3.2. If we take $k_1 = k_2 = 0$ and $H_1(\cdot) = H_2(\cdot) = 0$, problem (1.3) reduces to homogeneous Dirichlet problem and Theorem 3.1 takes the form of [12, Theorem 2.1] provided $\mathbb{R}^n$ is equipped with the norm used here.

Theorem 3.3. *Suppose that (I1) and (I2) hold. In addition, assume that:*

(A1') *F has the lower Scorza Dragoni property.*

(A2') *There exist two positive constants r and δ with $\delta < r$ such that $\|M\|_p \leq r - \delta$.*

Then the conclusion of Theorem 3.1 holds.

Proof. We apply Theorem 2.2 by putting $A = D = [0,1] \times B(0, \varphi(r))$ and $m = M + \delta$. Clearly, (A2') yields $\|m\|_p \leq r$ and since $C = \emptyset$, (A2)-(A5) hold. Moreover the same arguments used in the previous proof ensure (U1). Hence, to achieve the conclusion we only need to verify (U2). To do this, let $\rho \in L^s([0,1], \mathbb{R}_0^+)$ be such that $\|u''(t)\| \leq \rho(t)$ a.e. in $[0,1]$ and let $u \in U$, $t_1, t_2 \in [0,1]$ with $t_1 < t_2, (t_2 < t_1)$. We observe that

$$\|u_j'(t_1) - u_j'(t_2)\| = \|u''(\xi)\||t_1 - t_2| \leq \rho(\xi)|t_1 - t_2| \quad \xi \in (t_1, t_2), ((t_2, t_1)).$$

Further, by (3.2), for every $j = 1, \dots, n$, one has

$$|u_j(t_1) - u_j(t_2)| \leq \varphi''(r)|t_1 - t_2|, \quad \forall t_1, t_2 \in [0,1].$$

Hence, putting $\rho^*(t) = \max\{\rho(t), \varphi''(r)\}$, for all $t \in [0,1]$, one has

$$\|\Phi(u)(t_1) - \Phi(u)(t_2)\| = \max_{j=1,\dots,n} \{|u_j(t_1) - u_j(t_2)|, |u_j'(t_1) - u_j'(t_2)|\} \leq \rho^*(t)|t_1 - t_2|.$$

So the proof is complete. $\square$

Finally, we have the result.

Theorem 3.4. *Assume that H_1 and H_2 are two bounded functions satisfying* (1.4) *and replace (I3) in Theorem 3.1 and (F1), respectively, with*

(Q1) *For almost every $t \in [0,1]$ and for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $F(t,x,y)$ is convex or $F(t,\cdot,\cdot)$ is lower semicontinuous at (x,y).*

(Q2) *The function*

$$m'(t) := \sup\Big\{d(0, F(t,x,y)), (x,y) \in \mathbb{R}^n \times \mathbb{R}^n\Big\} \quad \text{for a.e. } t \in [0,1],$$

lies in $L^s([0,1], \mathbb{R}_0^+)$.

Then problem (1.3) *admits at least one generalized solution in $W^{2,s}([0,1], \mathbb{R}^n)$.*

Proof. Arguing as in the proof of Theorem 2.3 we obtain a closed-valued multifunction $F' : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ satisfying properties (A1)-(A5). Moreover, in this framework, by [3, Theorem 2.1], we can associate to F' a convex closed-valued multifunction $G : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ fulfilling the assumptions of Theorem 2.2 and such that any solution of problem (1.1), with $F = G$, is also solution of (1.1) with F' instead of F. Bearing in mind that H_1 and H_2 are bounded and that a weakly convergent sequence in L^p is bounded, arguing as above it is easy to show that (U1) and (U2) hold. Then due to Theorem 3.1, problem (1.3) with G instead of F has a solution, thereby implying that problem (1.3) with F' instead of F has a solution. Therefore the conclusion follows taking into account that $F'(t,x,y) \subseteq H(t,x,y)$. $\square$

4. Applications

This section is devoted to study some boundary-value problems by means of the results obtained before. In this order of ideas, let $T : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ be a closed-valued multifunction such that $T(\cdot, x)$ is measurable for every $x \in \mathbb{R}$ and let

$h_1, h_2 : \mathbb{R} \to \mathbb{R}$ be two Lipschitz continuous functions with constants L_1 and L_2 respectively. For every $u \in W^{2,s}([0,1]$, put

$$H_i(u) = \int_0^1 h_i(u(s))ds, \quad i = 1, 2.$$

We first study a second order differential inclusion with boundary integral conditions.

Theorem 4.1. *Assume that:*

(C1) *There exist two constants $c, d \in \mathbb{R}$ such that*
$$\int_d^{d+c} h_1(s)ds = cd - c^2 k_1, \quad \int_d^{d+c} h_2(s)ds = c^2(1+k_2)^2 + cd$$
or $h_1(d) = h_2(d) = d$ according to whether $c \neq 0$ or $c = 0$.

(C2)
$$\frac{(1+k_2)L_1 + (1+k_1)L_2}{1+k_1+k_2} < 1.$$

(C3) *For almost every $t \in [0,1]$, $T(t,\cdot)$ is upper semicontinuous and takes convex values.*

(C4) *There exist α, $\beta \in L^1([0,1])$ with $\|\alpha\|_1 m_{\varphi'} < 1$ such that for a.e. $t \in [0,1]$ one has*
$$m(t) := \sup\Big\{d(0, T(t,x)) : |x| \le \frac{\varphi'(\|\beta\|_1)}{1 - \|\alpha\|_1 m_{\varphi'}}\Big\} \le \alpha(t)|x| + \beta(t).$$

Then problem
$$\begin{gathered} u'' \in T(t,u) \quad a.e.\ t \in [0,1] \\ u(0) - k_1 u'(0) = \int_0^1 h_1(u(s))ds, \\ u(1) + k_2 u'(1) = \int_0^1 h_2(u(s))ds, \end{gathered} \tag{4.1}$$
admits at least one generalized solution $u \in W^{2,1}([0,1], \mathbb{R}^2)$ such that
$$\|u\|_\infty \le \frac{\varphi'(\|\beta\|_1)}{1 - \|\alpha\|_1 m_{\varphi'}} \quad and \quad \|u''(t)\| \le m(t) \quad a.e\ in\ [0,1].$$

Proof. Taking $r = \frac{\|\alpha\|_1 q_{\varphi'} + \|\beta\|_1}{1 - \|\alpha\|_1 m_{\varphi'}}$, an easy computation shows that
$$\varphi'(r) = \frac{\varphi'(\|\beta\|_1)}{1 - \|\alpha\|_1}.$$
Moreover, by (C4), one has $\|M\|_1 \le r$. Then the conclusion follows at once from Theorem 3.1. $\square$

Remark 4.2. We point out that Theorem 4.1 and [4, Theorem 3.3] are mutually independent. Indeed, here, T does not take compact values, whereas there h_1 and h_2 are two continuous and bounded functions.

Arguing as above and using Theorem 3.4 it is easy to verify the following result.

Theorem 4.3. *Assume that h_1 and h_2 are bounded. Let $T : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ be a closed-valued multifunction satisfying (C1), (C2) such that $T(\cdot, x)$ is measurable for every $x \in \mathbb{R}$. Further, we require*

(C5) *For almost every $t \in [0,1]$, and for all $u \in \mathbb{R}$, $T(t,x)$ is convex or $T(t,\cdot)$ is lower semicontinuous;*

(C6) *There exist α, $\beta \in L^1([0,1])$ with $\|\alpha\|_1 m_{\varphi'} < 1$, such that for a.e. $t \in [0,1]$ and for every $x \in \mathbb{R}$ one has*
$$d(0,T(t,x)) \leq \alpha(t)|x| + \beta(t).$$

Then problem (T) admits at least one solution $u \in W^{2,1}([0,1])$.

Remark 4.4. The following is a sufficient condition for (C5) and (C6) hold true:

(C6') There exist α and $\beta \in L^1([0,1])$ such that for almost every $t \in [0,1]$ one has
$$d_H(T(t,x),T(t,y)) \leq \alpha(t)|x-y| \quad \text{and} \quad d_H(0,T(t,0)) \leq \beta(t),$$

where $d_H(T(t,x),T(t,y)) = \max\{\sup_{z\in T(t,x)} d(z,T(t,y)), \sup_{w\in T(t,y)} d(w,T(t,x))\}$ indicates the Hausdorff distance in $\mathbb{R}^n$. Here, with respect to [4, Theorem 3.5], h_1 and h_2 are bounded. However, in this framework on the data we only require that (C2) is satisfied. To be precise, there they need the following condition
$$\frac{(1+k_2)L_{H_1} + (1+k_1)L_{H_2}}{1+k_1+k_2} + \sup_{[0,1]^2} |G(t,s)| \|\beta\|_1 < 1.$$

Now as consequence of Theorem 3.3 we have the following theorem.

Theorem 4.5. *Let $Q : [0,1] \times \mathbb{R}^2 \to 2^{\mathbb{R}}$ be a closed-valued multifunction fulfilling the Scorza Dragoni property. Assume that*

(C7) *There exist α, β, $\gamma \in L^1([0,1])$ with $\|\alpha\|_1 + \|\beta\|_1 < 1/4$ end a positive constant ρ, such that for almost every $t \in [0,1]$ and for all $(x,y) \in \mathbb{R}^2$ with*
$$\max\{|x|,|y|\} \leq 4\Big(\frac{\rho + \|\gamma\|_1}{1-4(\|\alpha\|_1 + \|\beta\|_1)}\Big),$$
one has $M(t) \leq \alpha(t)|x| + \beta(t)|y| + \gamma(t)$.

Then the Nicoletti problem
$$\begin{gathered} u'' \in Q(t,u,u') \quad \text{a.e. } t \in [0,1] \\ u(0) = 0, \quad u'(1) = 0, \end{gathered} \tag{4.2}$$
admits at least one generalized solution $u \in W^{2,1}([0,1],\mathbb{R}^2)$ such that $\|u''(t)\| \leq M(t)$ a.e in $[0,1]$,
$$\|u\|_\infty \leq 2\Big(\frac{\rho + \|\gamma\|_1}{1-4(\|\alpha\|_1 + \|\beta\|_1)}\Big) \quad \text{and} \quad \|u'\|_\infty \leq 4\Big(\frac{\rho + \|\gamma\|_1}{1-4(\|\alpha\|_1 + \|\beta\|_1)}\Big).$$

Proof. Choose $H_1 = 0$, $H_2(u) = u(1)$, $k_1 = 0$, $k_2 = 1$ and
$$r = \frac{\rho + \|\gamma\|_1}{1-(\|\alpha\|_1 + \|\beta\|_1)}.$$
Taking into account that $\varphi(r) \leq 4r$, the conclusion follows immediately from Theorem 3.3. □

The last application is devoted to study the following three-point boundary-value problem with nonlinear boundary conditions
$$\begin{gathered} u'' = f(t,u) \quad \text{a.e. in } [0,1] \\ u(0) = a, \quad u(1) = g(u(\eta)), \quad \eta \in]0,1[, \end{gathered} \tag{4.3}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $a \in \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant L_g. Obviously, problem (4.3) can be rewritten in the framework of problem (1.1). Due to Theorem 3.1 and by means of arguments similar to those used above we obtain the following result.

Theorem 4.6. *Assume that* $L_g < 1$ *and*

- (C8) *There exists* $\xi \in \mathbb{R}$ *such that* $g(\xi) = a + \frac{\xi - a}{\eta}$.
- (C9) *There is* $r \in]0, +\infty[$ *such that for a.e.* $t \in [0,1]$ *and all* $x \in \mathbb{R}$ *with* $|x| \leq \frac{1}{1-L_g}(|a| + |g(0)| + \frac{r}{4})$ *one has* $|f(t,x)| \leq r$.

Then problem (4.3) *admits at least one generalized solution* $u \in W^{2,1}([0,1])$ *such that*

$$\|u\|_\infty \leq \frac{1}{1-L_g}\Big(|a| + |g(0)| + \frac{r}{4}\Big) \quad \textit{and} \quad \|u''\|_\infty \leq r.$$

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