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POSITIVE SOLUTIONS OF A BOUNDARY-VALUE PROBLEM FOR A DIFFERENTIAL EQUATION WITH DAMPING AND ACTIVELY BOUNDED DELAYED FORCING TERM

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Abstract. Sufficient conditions are given for the existence of positive solutions of a boundary-value problem concerning a second-order delay differential equation with damping and forcing term whose the delayed part is an actively bounded function, a meaning which is introduced in this paper. The Krasnoselskii fixed point theorem on cones in Banach spaces is used.

1. Introduction

We deal with the existence of positive solutions of a boundary-value problem concerning a second order delay differential equation of the form

$$x''(t) + p(t)x'(t) + q(t)x(t) + f(t, x_t) = 0, \quad t \in I := [0, 1], \tag{1.1}$$

where the delayed part $f(t, x_t)$ of the forcing term is an actively bounded function, a meaning which is introduced in this paper. Also, it is assumed that the coefficient $p(t)$ of the damping term can be written as the sum of two (suitable) functions, $p_1(t) + p_2(t)$. Such a split depends on the coefficient $q(t)$ of the instantaneous part of the forcing term and it affects our conditions.

As it is noted elsewhere (see, e.g. [9, 15]), boundary-value problems associated with delay differential equations are generated from topics of physics and variational problems of control theory, as well as from applied mathematics appeared early in the literature. The literature contains a relatively great number of works dealing with the existence of solutions of boundary-value problems which are associated not necessary with ordinary differential equations. For example, in the book [1] one can find such problems for difference and integral equations, in [6] for equations whose the solutions depend on the past and on the future, in [12] for equations with deviating arguments, etc. Moreover a great deal can be met in the literature for the case of delay differential equations. We refer, for instance to [2, 4, 8, 10, 11, 13, 16, 17, 18, 19, 20, 21, 24, 25, 26] and to the references therein. Significant results are also given in [3, 7, 23, 27].

Most of the works mentioned above apply the fixed point of Krasnoselskii [22], which states as follows:

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Theorem 1.1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$$

be a completely continuous operator such that either

$$\|Au\| \leq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \geq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_2,$$

or

$$\|Au\| \geq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \leq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_2.$$

Then A has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Recall that an operator $A : X \to Y$ is *completely continuous* if it is continuous and maps bounded sets into precompact sets.

We notice that when the (Fixed Point) Theorem 1.1 is applied to boundary-value problems for functional diferential equations the most crucial point is to provide such conditions on the forcing term (which depends on the history of the solution) which (conditions) guarantee the fact that the corresponding integral operator satisfies the two alternatives of Krasnoselskii's fixed point theorem. In this article, in order to cover the autonomous and nonautonomous cases, the continuous and discrete delay, as well as the atomic and the nonatomic response, we give an improvement of these conditions by introducing the meaning of what we call *actively bounded* function. Before we explain it, some notation is needed.

Let $\mathbf{R}$ be the real line $(-\infty, +\infty)$ and let $\mathbf{R}^+$ be the set of nonnegative reals. We shall denote by $C_0(I)$ the space of all continuous functions $x : I \to \mathbf{R}$, with $x(0) = 0$. This is a Banach space, when it is furnished with the usual sup-norm $\|\cdot\|_I$. We set

$$C_0^+(I) := \{x \in C_0(I) : x(t) \geq 0, t \in I\}.$$

Fix any $r \geq 0$, set $J := [-r, 0]$ and consider the Banach space $C(J)$ of all continuous functions $\psi : J \to \mathbf{R}$ furnished with the sup-norm $\|\cdot\|_J$. Let $C^+(J)$ be the set of all $\psi \in C(J)$ such that $\psi(s) \geq 0$, for all $s \in J$. Also let $C_0^+(J)$ be the subset of $C^+(J)$ whose elements vanish at 0.

2. The class of Actively Bounded Functions

Here we introduce the meaning of the actively bounded functions and give some examples.

Definition 2.1. We call a function $f(\cdot,\cdot) : I \times C^+(J) \to \mathbf{R}^+$ *actively bounded*, if there are two measurable real nonnegative functions $L_0(t, m, M)$ and $\omega(t, m, M)$, $t \in I$ and $0 < m < M < +\infty$, as well as for each $t \in I$ a nonempty closed set $\Theta_t \subseteq J$ such that

$$\omega(t, m, M) \leq f(t, \psi) \leq L_0(t, m, M),$$

for all $t \in I$ and $\psi \in P(t, m, M)$, where

$$P(t, m, M) := \{\psi \in C^+(J) : m \leq \inf_{s \in \Theta_t} \psi(s), \|\psi\|_J \leq M\}.$$

Let $\Theta_t(f)$ be the smallest set of the form Θ_t. We denote by $\mathcal{F}_{AB}$ the class of all actively bounded functions.

Remark 2.2. It is easy to see that for two elements f, g of $\mathcal{F}_{AB}$ it holds

$$\Theta_t(f+g) = \Theta_t(fg) = \Theta_t(f) \cup \Theta_t(g).$$

Thus the sum and the product of two elements f, g of $\mathcal{F}_{AB}$ is again an element of $\mathcal{F}_{AB}$.

The class of actively bounded functions is wide. In the next paragraph we give some examples of classes of general forms of actively bounded functions, which, by using Remark 2.2, may produce new forms of such functions. A specific example of an actively bounded function is presented in the last section.

Example 2.3. Consider points $r = s_1 > s_2 > \dots > s_k \geq 0$ and a function of the form

$$f(t,\psi) := \hat{f}(t, \psi(-s_1), \psi(-s_2), \dots, \psi(-s_k)), \quad \psi \in C^+(J),$$

where $J := [-r, 0]$, $\hat{f} : I \times (\mathbf{R}^+)^{\mathbf{k}} \to \mathbf{R}^+$ is continuous. This is an element of the class $\mathcal{F}_{AB}$ with

$$\begin{gathered}\Theta_t(f) := \{-s_1, -s_2, \dots, -s_k\}, \\ \omega(t,m,M) := \min\{\hat{f}(t,\xi_1,\xi_2,\dots,\xi_k) : m \leq \xi_j \leq M, \ \ j = 1,2,\dots,k\}, \\ L_0(t,m,M) := \max\{\hat{f}(t,\xi_1,\xi_2,\dots,\xi_k) : \ \ m \leq \xi_j \leq M, \ \ j = 1,2,\dots,k\},\end{gathered}$$

Example 2.4. Consider the continuous functions $\tau_j : I \to (-\infty, 1]$, $j = 1,2,\dots,k$ such that for some $r > 0$ it holds $-r \leq \tau_j(t) - t \leq 0$, for all $t \in I$ and $j \in \{1,2,\dots,k\}$. Let the function

$$f(t,\psi) := \hat{f}(t, \psi(\tau_1(t)-t), \psi(\tau_2(t)-t), \dots, \psi(\tau_k(t)-t)), \quad \psi \in C^+(J),$$

where $J := [-r, 0]$ and the function $\hat{f} : I \times (\mathbf{R}^+)^{\mathbf{k}} \to \mathbf{R}^+$ is continuous. Then f is an actively bounded function, with

$$\Theta_t(f) := \{\tau_1(t)-t, \tau_2(t)-t, \dots, \tau_k(t)-t\}$$

and ω, L_0 as in Example 2.3.

Example 2.5. Let $r > r' > 0$ and consider a function

$$f(t,\psi) := \int_{-r}^{-r'} K(t,s,\psi(s))ds, \quad \psi \in C^+(J),$$

where $K : I \times [-r, -r'] \times \mathbf{R}^+ \to \mathbf{R}^+$ is continuous and such that for some closed nonempty $\Theta \subseteq J := [-r, 0]$ it satisfies $\int_\Theta K(t,s,\xi)ds > 0$, for all $t \in I$ and $\xi \geq 0$. This is an actively bounded function, with $\Theta_t(f)$ being the intersection of all such Θ, for all $t \in I$.

Example 2.6. Assume that $f_1 : I \times C^+(J) \to \mathbf{R}^+$ and $f_2 : C^+(J) \to (0, +\infty)$ are two actively bounded functions and consider the function

$$f(t,\psi) := \frac{f_1(t,\psi)}{f_2(\psi)} : I \times C^+(J) \to \mathbf{R}^+.$$

It is not hard to see that this is an actively bounded function with $\Theta_t(f) := \Theta_t(f_1) \cup \Theta_t(f_2)$.

3. On the Formulation of the BVP

The basic theory of delay differential equations is exhibited in several places of the literature. Especially we refer to the (already) classical books [9, 14, 15], which are basic sources on the subject.

For any continuous function y defined on the interval $[-r, 1]$ and any $t \in [0, 1] =: I$, the symbol y_t is used to denote the element of $C(J)$ defined by

$$y_t(s) = y(t + s), \quad s \in J.$$

The initial condition which associates (1.1) is of the form

$$x_0 = \phi, \tag{3.1}$$

and the boundary condition is

$$ax(1) + bx'(1) = 0. \tag{3.2}$$

Here ϕ is an element of $C_0^+(J)$ and a, b are nonnegative real numbers, with

$$a + b > 0.$$

The latter is motivated mainly by the works [16, 20, 21].

Our purpose is to establish sufficient conditions for the existence of positive solutions of the boundary-value problem (BVP) (1.1), (3.1), (3.2).

Here we want to make clear what makes the difference between the ordinary and the delay case and, in particular, what is going to be proved for the delay boundary-value problem. It is well known that in the ordinary case, namely, when $r = 0$, (thus $J = \{0\}$ and (1.1) is an ordinary differential equation), we are called to give conditions which guarantee the validity of the following fact: *There is a solution x of the (ordinary differential equation)* (1.1) *with* $x(0) = 0$ *and satisfying condition* (3.2). It follows that uniqueness of such a solution means that there is exactly one function with these properties. On the other hand in the (nontrivial) delay case the problem is quite different. Indeed, here we are called to give our response to the following challenge: *Determine a class S of initial functions with the property that for each $\phi \in S$ there is a solution x of* (1.1) *satisfying* (3.2) *and having initial value equal to ϕ, i.e. satisfying condition* (3.1). (Notice that some authors use to extend the situation from the ordinary case by simply assuming that $\phi(s) = 0$, for all $s \in J$, see, e.g. [5].) Therefore uniqueness of solutions of the BVP (1.1), (3.1), (3.2) presupposes that there is only one solution with initial value the fixed initial function ϕ. Any new initial function from the class S implies new solution of the boundary-value problem (1.1), (3.1), (3.2). As we shall see later, in this paper the set S will be a closed ball in the family $C_0^+(J)$.

4. Reformulation of the problem and the main conditions

We shall reformulate the problem (1.1), (3.1), (3.2) by transforming it into a fixed point problem. The solution of the latter is guaranteed by Theorem 1.1.

Fix a $\phi \in C_0^+(J)$. For each function $x \in C_0(I)$ we shall denote by $T(\cdot, x; \phi)$ the function defined on $[-r, 1]$ by $T(s, x; \phi) := x(s)$, if $s > 0$ and $= \phi(s)$, if $s \le 0$. It is easy to see that it holds

$$\|T_t(\cdot, x_1; \phi) - T_t(\cdot, x_2; \phi)\|_J \le \|x_1 - x_2\|_I, \tag{4.1}$$

for all $t \in I$ and $x_1, x_2 \in C_0(I)$. (Recall that for each $t \in I$ the symbol $T_t(\cdot, x; \phi)$ denotes the element of $C(J)$ defined by $T_t(s, x; \phi) := T(t+s, x; \phi), \quad s \in J$.) Thus the function

$$x \to T_t(\cdot, x; \phi) : C_0(I) \to C(J)$$

is continuous (uniformly with respect to t).

By a solution of the boundary-value problem (1.1), (3.1), (3.2) we mean a function $x \in C_0(I)$ satisfying (3.1), x satisfies condition (3.2) and moreover its second derivative $x''(t)$ exists for all $t \in I$ and the relation

$$x''(t) + p(t)x'(t) + q(t)x(t) + f(t, T_t(\cdot, x; \phi)) = 0 \tag{4.2}$$

is true for all $t \in I$.

Our basic conditions of the problem are stated as follows:

(H1) The functions $p, q : I \to \mathbf{R}$ are continuous and such that p can be written in the form

$$p = p_1 + p_2,$$

where p_1 continuous, p_2 positive and differentiable and moreover they satisfy the inequality

$$q(t) \geq p_2'(t) + p_1(t)p_2(t),$$

for all $t \in I$.

The presence of the instantaneous factor $q(t)$ of the forcing term in (1.1) is quite technical. One can see that condition (H1) is satisfied even in case $p = q = 0$. Indeed, take any $c > 0$ and put

$$p_2 = c = -p_1.$$

This fact will be used in the example in the last section.

It is, also, assumed that the functions p_1 and p_2 presented in (H1) are related to the coefficients a and b appeared in condition (3.2) as follows:

(H2) It holds $bp_2(1) > a$ and

$$b > (bp_2(1) - a)\int_0^1 e^{\int_s^1 (p_1(v) - p_2(v))dv} ds.$$

To proceed, we set $y(t) := x'(t)$ and write equation (4.2) in the form

$$y'(t) + p_1(t)y(t) + p_2(t)x'(t) + q(t)x(t) + f(t, T_t(\cdot, x; \phi)) = 0.$$

Integrating from t ($t \geq 0$) to 1 we obtain

$$\begin{aligned} y(t) &= y(1)e^{\int_t^1 p_1(s)ds} \\ &\quad + \int_t^1 [p_2(u)x'(u) + q(u)x(u) + f(u, T_u(\cdot, x; \phi))]e^{\int_t^u p_1(s)ds} du. \end{aligned}$$

This leads to

$$\begin{aligned} &x'(t) + p_2(t)x(t) \\ &= [x'(1) + p_2(1)x(1)]^{\int_t^1 p_1(s)ds} + \int_t^1 F(u, T_u(\cdot, x; \phi))e^{\int_t^u p_1(s)ds} du, \end{aligned}$$

where we have put, for $u \in I$,

$$F(u, T_u(\cdot, x; \phi)) := f(u, T_u(\cdot, x; \phi)) + \big[q(t) - p_1(u)p_2(u) - p_2'(u)\big]x(u).$$

Also, for simplicity, we set

$$V(u,s,t) := e^{\int_u^s p_1(u)du - \int_u^t p_2(u)du},$$

$$E(t) := \int_0^t V(u,1,t)du.$$

Thus the solution x satisfies

$$x(t) = [x'(1) + p_2(1)x(1)]E(t) + \int_0^t \int_u^1 V(u,s,t)F(s, T_s(\cdot, x; \phi))ds\, du, \quad t \in I. \tag{4.3}$$

Next keeping in mind condition (3.2) we set $t = 1$ in Eq. (4.3) and get a system of two linear equations with unknowns $x(1)$ and $x'(1)$. Solving the system, we obtain

$$x(1) = \frac{b}{b + aE(1) - bp_2(1)E(1)} \int_0^1 \int_u^1 V(u,s,1)F(s, T_s(\cdot, x; \phi))ds\, du$$

$$x'(1) = \frac{-a}{b + aE(1) - bp_2(1)E(1)} \int_0^1 \int_u^1 V(u,s,1)F(s, T_s(\cdot, x; \phi))ds\, du.$$

Hence (4.3) becomes

$$\begin{aligned} x(t) = \gamma E(t) &\int_0^1 \int_u^1 V(u,s,1)F(s, T_s(\cdot, x; \phi))ds\, du \\ &+ \int_0^t \int_u^1 V(u,s,t)F(s, T_s(\cdot, x; \phi))ds\, du, \quad t \in I, \end{aligned} \tag{4.4}$$

where

$$\gamma := \frac{-a + p_2(1)b}{b + aE(1) - bp_2(1)E(1)}.$$

Because of condition (H2), the constant γ is positive.

Lemma 4.1. *A function x is a solution of the boundary-value problem* (1.1), (3.1), (3.2) *if and only if it satisfies the operator equation*

$$x = A_\phi x, \tag{4.5}$$

where A_ϕ is the operator defined by

$$(A_\phi x)(t) := \int_0^1 G(t,s)F(s, T_s(\cdot, x; \phi))ds, \quad x \in C_0^+(I). \tag{4.6}$$

Here the kernel is

$$G(t,s) := \zeta(t \vee s)e^{\int_t^1 p_2(v)dv} \int_0^{t \wedge s} V(u,s,1)du, \tag{4.7}$$

where $t \vee s := \max\{t,s\}$, $t \wedge s := \min\{t,s\}$ and ζ is the increasing function defined by

$$\zeta(t) := \gamma \int_0^t V(u,1,1)du + 1. \tag{4.8}$$

Proof. Assume that x is a solution. Then it satisfies (4.4) and so we have

$$x(t) = \int_0^1 \int_u^1 U(u,s,t)F(s, x_s(\cdot; \phi))ds\, du, \tag{4.9}$$

where

$$U(u,s,t) := \gamma E(t)V(u,s,1) + V(u,s,t)\chi_{[0,t]}(u).$$

Here for any set B of real numbers the symbol $\chi_B(\cdot)$ stands for the characteristic function of B, i.e. $\chi_B(u) = 1$, if $u \in B$ and $= 0$, otherwise. We apply Fubini's Theorem in the second part of (4.9) to obtain

$$x(t) = \int_0^1 G(t,s)F(s, T_s(\cdot, x; \phi))ds,$$

where $G(t,s) := \int_0^s U(u,s,t)du$. If we assume that $s \le t$, then we obtain

$$\begin{aligned} G(t,s) &= \int_0^s \Big[\gamma E(t)V(u,s,1) + V(u,s,t)\Big]du \\ &= \zeta(t)e^{\int_t^1 p_2(v)dv}\int_0^s V(u,s,1)du, \end{aligned} \tag{4.10}$$

where the function ζ is defined in (4.8). If we assume that $s \ge t$, then we get

$$\begin{aligned} G(t,s) &= \int_0^t \Big[\gamma V(u,1,t)\int_0^s V(v,s,1)dv + V(u,s,t)\Big]du \\ &= \zeta(s)e^{\int_t^1 p_2(v)dv}\int_0^t V(u,s,1)du. \end{aligned}$$

Therefore, x satisfies the operator equation (4.5), with A_ϕ being given by (4.6).

To show the inverse, assume that x satisfies (4.5), with A_ϕ as in (4.6). Apply, again, Fubibi's Theorem in the second part of (4.4) and differentiate both sides to see that x is a solution of the boundary-value problem (1.1), (3.1), (3.2). □

5. Main Results

Now we are ready to present our main results of this article.

Theorem 5.1. *Suppose that $f(t,\phi)$ is an actively bounded continuous function and let $\Theta_t(f), t \in I$ be the set-valued function in Definition 2.1. Let also $L_0(t,m,M)$ and $\omega(t,m,M)$ be the functions defined in Section 2 and let*

$$L(t,m,M) := [q(t) - (p_2'(t) + p_1(t)p_2(t))]M + L_0(t,m,M).$$

Also assume that there are $0 < \alpha < \beta \le 1$ and two (distinct) real numbers ρ_1, ρ_2 such that

$$\begin{aligned} &\frac{1}{\rho_1}\int_0^1 G(s,s)L(s, \frac{\mu}{\eta}\rho_1, \rho_1)ds \le \frac{1}{\eta}, \\ &\frac{1}{\rho_2}\sup_{t\in I}\int_\Sigma G(t,s)\omega(s, \frac{\mu}{\eta}\rho_2, \rho_1 \vee \rho_2)ds \ge 1, \end{aligned} \tag{5.1}$$

where

$$\Sigma := \{s \in [0,1] : \quad s + \theta \in [\alpha, \beta], \quad \theta \in \Theta_s(f)\}, \tag{5.2}$$

$$\mu := \min\Big\{e^{-\int_0^\beta p_2(v)dv}, \quad \frac{\int_0^\alpha V(u,1,1)du}{\int_0^1 V(u,1,1)du}\Big\}, \tag{5.3}$$

$$\eta := \zeta(1)e^{\int_0^1 p_2(v)dv}. \tag{5.4}$$

Then, for any $\phi \in C_0^+(J)$ with $\|\phi\| \le \rho_1$, there is a positive solution of the boundary-value problem (1.1), (3.1), (3.2) *having norm in the interval with ends the numbers ρ_1, ρ_2.*

Proof. First notice that $\mu < \eta$. We shall elaborate a little on the Green's function G defined by (4.7). Let $t \le s$. Then we have

$$G(t,s) = \zeta(s)e^{-\int_s^1 p_1(v)dv}e^{\int_t^1 p_2(v)dv}\int_0^t V(u,1,1)du,$$

where ζ is defined by (4.8), and therefore

$$\frac{G(t,s)}{G(s,s)} = e^{\int_t^s p_2(v)dv}\frac{\int_0^t V(u,1,1)du}{\int_0^s V(u,1,1)du} \le e^{\int_0^s p_2(v)dv}. \tag{5.5}$$

Assume that $s \le t$. Then

$$G(t,s) = \zeta(t)e^{\int_t^1 p_2(v)dv}\int_0^s V(u,s,1)du.$$

Thus we have

$$\begin{aligned}\frac{G(t,s)}{G(s,s)} &= \frac{\zeta(t)\exp\Big(\int_t^1 p_2(v)dv\Big)\int_0^s V(u,s,1)du}{\zeta(s)\exp\Big(\int_s^1 p_2(v)dv\Big)\int_0^s V(u,s,1)du}\\ &\le \frac{\zeta(1)\exp\Big(\int_0^1 p_2(v)dv\Big)}{\zeta(s)\exp\Big(\int_s^1 p_2(v)dv\Big)} \le \zeta(1)e^{\int_0^s p_2(v)dv}.\end{aligned} \tag{5.6}$$

Therefore, from (5.5) and (5.6) we conclude that for all s, t in $[0,1]$ it holds

$$G(t,s) \le G(s,s)\zeta(1)e^{\int_0^1 p_2(v)dv} = \eta G(s,s). \tag{5.7}$$

Next fix $t \in [\alpha,\beta]$. Then for each $s \in [0,t]$ we obtain

$$\frac{G(t,s)}{G(s,s)} = \frac{\zeta(t)\exp\Big(\int_t^1 p_2(v)dv\Big)}{\zeta(s)\exp\Big(\int_s^1 p_2(v)dv\Big)} \ge e^{\int_t^1 p_2(v)dv-\int_s^1 p_2(v)dv}$$

and, since p_2 is nonnegative, we have

$$G(t,s) \ge G(s,s)e^{-\int_s^t p_2(v)dv} \ge G(s,s)e^{-\int_0^\beta p_2(v)dv}. \tag{5.8}$$

Take $s \in [t,1]$. Then we obtain

$$\frac{G(t,s)}{G(s,s)} = e^{\int_t^s p_2(v)dv}\frac{\int_0^t V(u,1,1)du}{\int_0^s V(u,1,1)du} \ge \frac{\int_0^\alpha V(u,1,1)du}{\int_0^1 V(u,1,1)du}. \tag{5.9}$$

From (5.8) and (5.9) we see that for all $t \in [\alpha,\beta]$ and $s \in [0,1]$ it holds

$$G(t,s) \ge \mu G(s,s), \tag{5.10}$$

where μ is defined in (5.3). Define the set

$$\mathcal{K} := \{x \in C_0^+(I) : x(t) \ge \frac{\mu}{\eta}\|x\|,\ t \in [\alpha,\beta]\}$$

and observe that it is a cone in the space $C_0(I)$.

Now consider a initial function $\phi \in C_0^+(J)$ with $\|\phi\|_J \le \rho_1$, where ρ_1 satisfies the system of inequalities (5.1). Let A_ϕ be the corresponding operator defined by (4.6). Because of Lemma 4.1 it is enough to show that the operator A_ϕ has a fixed point. To this end we let any $x \in \mathcal{K}$. Then we have $(A_\phi x)(0) = 0$, because of (4.7). Also from (5.6) we get

$$\|A_\phi x\|_I = \sup_{t\in I} \int_0^1 G(t,s)F(s,T_s(\cdot,x;\phi))ds \leq \eta \int_0^1 G(s,s)F(s,T_s(\cdot,x;\phi))ds. \tag{5.11}$$

From (H1) and the definition of f, we have $F(s,T_s(\cdot,x;\phi)) \geq 0$, for all $s \in I$. Also, it is clear that $(A_\phi x)(t) \geq 0$ for all $t \in I$.

Let $t \in [\alpha,\beta]$. Then from (5.10) and (5.11) we get

$$\begin{aligned}
(A_\phi x)(t) &= \int_0^1 G(t,s)F(s,T_s(\cdot,x;\phi))ds \\
&\geq \mu \int_0^1 G(s,s)F(s,T_s(\cdot,x;\phi))ds \\
&\geq \frac{\mu}{\eta}\|A_\phi x\|_I.
\end{aligned} \tag{5.12}$$

Relation (5.12) and the comments given before it guarantee that the operator A_ϕ maps the cone $\mathcal{K}$ into itself. Furthermore from (4.1) and the first argument in Definition 2.1 we conclude that the function $y \to F(\cdot,T_\cdot(\cdot,y;\phi))$ is continuous and it maps bounded sets into bounded sets; thus the operator A_ϕ is completely continuous.

Next take $x \in \mathcal{K}$. By definition, for any $s \in \Sigma$ we have $s + \theta \in [\alpha,\beta] \subseteq I$, for all $\theta \in \Theta_s$. Thus we have

$$T_s(\theta,x;\phi) = x(s+\theta) \geq \frac{\mu}{\eta}\|x\|_I. \tag{5.13}$$

Let $x \in \mathcal{K}$ with $\|x\|_I = \rho_1$. Taking it into account together with the choice of $\|\phi\|_J$, we have $\|T_s(\cdot,x;\phi)\|_J \leq \rho_1$. Thus, because of (5.7) and (5.13) for all $t \in I$ we have

$$\begin{aligned}
\|A_\phi x\|_I &= \sup_{t\in I} \int_0^1 G(t,s)F(s,T_s(\cdot,x;\phi))ds \\
&\leq \eta \int_0^1 G(s,s)F(s,T_s(\cdot,x;\phi))ds \\
&\leq \eta \int_0^1 G(s,s)L(s,\frac{\mu}{\eta}\rho_1,\rho_1)ds \\
&\leq \rho_1 = \|x\|_I.
\end{aligned} \tag{5.14}$$

Also, let $x \in \mathcal{K}$, with $\|x\|_I = \rho_2$. Then we have

$$\|T_s(\cdot;x,\phi)\|_J \leq \rho,$$

where $\rho := \rho_1 \vee \rho_2$. Consequently, from (H1) and (5.13), we obtain

$$\begin{aligned}
\|A_\phi x\|_I &= \sup_{t\in I} \int_0^1 G(t,s)F(s,T_s(\cdot,x;\phi))ds \\
&\geq \sup_{t\in I} \int_\Sigma G(t,s)F(s,T_s(\cdot,x;\phi))ds \\
&\geq \sup_{t\in I} \int_\Sigma G(t,s)\omega(s;\frac{\mu}{\eta}\|x\|_I,\rho)ds \geq \|x\|_I.
\end{aligned} \tag{5.15}$$

Finally, consider as Ω_1 and Ω_2 the open balls with radius $\rho_1 \wedge \rho_2$ and $\rho_1 \vee \rho_2$ respectively. The previous arguments together with (5.14) and (5.15) permit us to apply Theorem 1.1 to get the result. □

6. An Application

Consider the delay differential equation

$$x''(t) + te^{-x(t-\frac{1}{2})}\sqrt{x(\frac{t}{2})} = 0, \quad t \in [0,1], \tag{6.1}$$

associated with the initial condition (3.1) and the boundary condition

$$x(1) + 2x'(1) = 0. \tag{6.2}$$

Here we set $r = 1/2$, $q(t) := 0$ and $p(t) := 0 = (-1) + (1)$. Note that one can write p as $p(t) := (-c) + c$ and then to consider c as parameter, but this job is far from our purpose for the moment. Observe that the function

$$f(t, \psi) := te^{-\psi(-\frac{1}{2})}\sqrt{\psi(-\frac{t}{2})}$$

is actively bounded with

$$\begin{gathered}\Theta_t(f) := \{-\frac{1}{2}, -\frac{t}{2}\},\\ \omega(t; m, M) := te^{-M}\sqrt{m},\\ L(t, m, M) := M + te^{-m}\sqrt{M}.\end{gathered}$$

Fixing $\alpha \in (0, \frac{1}{2})$ and taking $\beta := 1$ we get $\Sigma = [2\alpha, 1]$. Thus the constants μ and η are

$$\mu := \frac{e^{2\alpha} - 1}{e^2 - 1}, \quad \eta := \frac{4e^{-1}}{3 + e^{-2}}.$$

Therefore, the first condition in (5.1) is satisfied for any $\rho_1 > 0$ with

$$\rho_1 e^{2\frac{\mu}{\eta}\rho_1} \geq \frac{(7e - 24e^{-1} - 3e^{-3})^2}{16(18 - 8e^{-1} + 12e^{-2} + 3e^{-3} + 2e^{-4})^2}.$$

Also, the second condition in (5.1) is satisfied for any $\rho_2 \in (0, \rho_1)$ with

$$\rho_2 \leq \frac{\mu}{\eta} e^{-2\rho_1} \frac{16\big(1 - e\alpha\cosh(2\alpha) + e\sin(2\alpha)\big)^2}{(3 + e^{-2})^2}.$$

For instance, if we take

$$\rho_1 = \frac{(7e - 24e^{-1} - 3e^{-3})^2}{16(18 - 8e^{-1} + 12e^{-2} + 3e^{-3} + 2e^{-4})^2} \approx 0.0221868\,,$$

and choose $\alpha = 0.015$, then we obtain that the best value of ρ_2 is 0.0171286. The conclusion is that given any continuous function $\phi : [-\frac{1}{2}, 0] \to [0, 0.0221868]$ with $\phi(0) = 0$ there is a solution x of the problem (6.1), (6.2) having ϕ as initial function and being such that $x(0) = 0 \leq x(t)$ for all $t \in [0,1]$ and $0.0171286 \leq \|x\|_I \leq 0.0221868$.

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