Electronic Journal of Differential Equations, Vol. 2006(2006), No. 76, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MULTIPLICITY OF SOLUTIONS FOR A CLASS OF ELLIPTIC SYSTEMS IN $\mathbb{R}^N$

GIOVANY M. FIGUEIREDO

Abstract. This article concerns the multiplicity of solutions for the system of equations

$$-\Delta u + V(\epsilon x)u = \alpha|u|^{\alpha-2}u|v|^{\beta},$$
$$-\Delta v + V(\epsilon x)v = \beta|u|^{\alpha}|v|^{\beta-2}v$$

in $\mathbb{R}^N$, where V is a positive potential. We relate the number of solutions with the topology of the set where V attains its minimum. The results are proved by using minimax theorems and Ljusternik-Schnirelmann theory.

1. Introduction

The purpose of this article is to investigate the multiplicity of solutions for the system

$$\begin{gathered}
-\Delta u + V(\epsilon x)u = \alpha|u|^{\alpha-2}u|v|^{\beta} \quad \text{in } \mathbb{R}^N, \\
-\Delta v + V(\epsilon x)v = \beta|u|^{\alpha}|v|^{\beta-2}v \quad \text{in } \mathbb{R}^N, \\
u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \quad \text{for all } x \in \mathbb{R}^N,
\end{gathered} \tag{1.1}$$

where $\epsilon > 0$, $\alpha, \beta > 1$ such that $\alpha + \beta = p$, $2 < p < 2N/(N-2)$, $N \geq 3$ and the potential $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfies

$$0 < V_0 := \inf_{x\in\mathbb{R}^N} V(x) < V_\infty := \liminf_{|x|\to\infty} V(x). \tag{1.2}$$

In this work, we will consider the cases $V_\infty < \infty$ or $V_\infty = \infty$. This kind of hypothesis was introduced by Rabinowitz [16] in the study of a nonlinear Schrödinger equation.

We say that $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is a weak solution of the system in (1.1) if

$$\int_{\mathbb{R}^N} \Big[\nabla u\nabla\phi + \nabla v\nabla\psi + V(\epsilon x)(u\phi + v\psi)\Big] = \int_{\mathbb{R}^N} \Big[\alpha|u|^{\alpha-2}u|v|^{\beta}\phi + \beta|u|^{\alpha}|v|^{\beta-2}v\psi\Big]$$

for all $(\phi, \psi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

2000 *Mathematics Subject Classification.* 35J20, 35J50, 35J60.
Key words and phrases. Variational methods; Palais-Smale condition; Ljusternik-Schnirelmann theory.

Submitted May 24, 2005. Published July 12, 2006.

In this paper we also relate the number of solutions of (1.1) with the topology of the set of minima of the potential V. In order to present our result we introduce the set of global minima of V, given by

$$M = \{x \in \mathbb{R}^N : V(x) = V_0\}.$$

Note that, by (1.2), M is compact. For any $\delta > 0$, let $M_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\}$ be the closed δ-neighborhood of M. Our main result is as follows.

Theorem 1.1. *Suppose that V satisfies* (1.2)*. Then, for any $\delta > 0$ given, there exists $\epsilon_\delta > 0$ such that, for any $\epsilon \in (0, \epsilon_\delta)$, the system* (1.1) *has at least* $\operatorname{cat}_{M_\delta}(M)$ *solutions.*

We recall that, if Y is a closed set of a topological space X, $\operatorname{cat}_X(Y)$ is the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible set in X which cover Y.

Existence and concentration of positive solutions for the problem

$$-\epsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N \tag{1.3}$$

have been extensively studied in recent years, see for example, Ambrosetti, Badiale and Cingolani [4], Del Pino & Felmer [8], Floer [9], Lazzo [11], Oh [13, 14, 15], Rabinowitz [16] , Wang [17] and their references.

Cingolani and Lazzo in [6] studied positive solutions for the Schrödinger equation (1.3) with $f(u) = |u|^{q-2}u$, $\epsilon > 0$, $2 < q < 2^*$, V satisfying (1.2) and proved a multiplicity result similar to Theorem 1.1. Alves and Monari in [3], proved only the existence and concentration of a nontrivial solutions (u, v) to problem (1.1).

In this work, motivated by [6], [3], and using some recent ideas from [2] and [10], we prove the multiplicity of solutions to (1.1). Our main result completes the study made in [6] in the following sense: We are working with a system of equations and here, in the proof of some lemmas and propositions, we use different arguments than those in [6], for example the proposition 3.6, Lemma 4.4 for appearing along the text and we prove a compactness result on Nehari manifolds. Moreover, we do not know if the problem below has a unique positive solution,

$$\begin{gathered}
-\Delta u + \mu u = \alpha |u|^{\alpha-2} u |v|^\beta \quad \text{in } \mathbb{R}^N, \\
-\Delta v + \mu v = \beta |u|^\alpha |v|^{\beta-2} v \quad \text{in } \mathbb{R}^N, \\
u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \mu > 0\,.
\end{gathered} \tag{1.4}$$

This fact is used in a lot of papers in the scalar case.

The paper is organized as follows: In Section 2 we present the abstract framework of the problem as well as some remarks on the autonomous problem. In Section 3 we obtain some compactness properties of the functional associated to the system (1.1). Theorem 1.1 is proved in Section 4.

2. The variational framework

Throughout this paper we suppose that the function V satisfies the conditions (1.2). We write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$. For any $\epsilon > 0$, we denote by X_ϵ the Sobolev space

$$X_\epsilon = \{(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int V(\epsilon x)(|u|^2 + |v|^2) < \infty\}$$

endowed with the norm

$$\|(u,v)\|_\epsilon^2 = \int (|\nabla u|^2 + |\nabla v|^2) + \int V(\epsilon x)(|u|^2 + |v|^2),$$

We will look for solutions of (1.1) by finding critical points of the C^2-functional $I_\epsilon : X_\epsilon \to \mathbb{R}$ given by

$$I_\epsilon((u,v)) = \frac{1}{2}\int \big[|\nabla u|^2 + |\nabla v|^2 + V(\epsilon x)(|u|^2 + |v|^2)\big] - \int Q(u,v),$$

where $Q(u,v) = (u^+)^\alpha (v^+)^\beta$ and $w^\pm = \max\{\pm w, 0\}$ the positive (negative) part of w. By definition of Q, we see that, if (u,v) is a nontrivial critical point of I_ϵ, then u,v are positive in $\mathbb{R}^N$. Indeed, since that

$$\begin{aligned}\langle I_\epsilon'((u,v)),(\phi,\psi)\rangle &= \int \Big[\nabla u \nabla \phi + \nabla v \nabla \psi + V(\epsilon x)(u\phi + v\psi)\Big]\\ &\quad - \alpha \int |u|^{\alpha-2}u|v|^\beta \phi - \beta \int |u|^\alpha |v|^{\beta-2} v\psi,\end{aligned}$$

we have

$$0 = \langle I_\epsilon'((u,v)),(u^-,v^-)\rangle = \|(u^-,v^-)\|_\epsilon^2$$

and therefore $u,v \ge 0$ in $\mathbb{R}^N$. By the Maximum Principle in $\mathbb{R}^N$, $u,v > 0$ in $\mathbb{R}^N$.

We introduce the Nehari manifold of I_ϵ by setting

$$\mathcal{N}_\epsilon = \{(u,v) \in X_\epsilon \setminus \{(0,0)\} : \langle I_\epsilon'((u,v)),(u,v)\rangle = 0\}.$$

Note that, if $(u,v) \in \mathcal{N}_\epsilon$, we have

$$I_\epsilon((u,v)) = \frac{1}{2}\|(u,v)\|_\epsilon^2 - \int Q(u,v) = \big(\frac{1}{2} - \frac{1}{p}\big)\|(u,v)\|_\epsilon^2 \ge 0,$$

and therefore the following minimization problem is well defined

$$c_\epsilon = \inf_{(u,v)\in\mathcal{N}_\epsilon} I_\epsilon((u,v)).$$

Moreover, we can easily conclude that there exists $r > 0$, independent of ϵ, such that

$$\|(u,v)\|_\epsilon \ge r > 0 \quad \text{for any } \epsilon > 0,\ (u,v) \in \mathcal{N}_\epsilon. \tag{2.1}$$

We now present some important properties of c_ϵ and $\mathcal{N}_\epsilon$. The proofs can be adapted from [18, Chapter 4] (see also [12, Lemmas 3.1 and 3.2]). First we observe that, for any $(u,v) \in X_\epsilon \setminus \{(0,0)\}$ there exists a unique $t_{u,v} > 0$ such that $t_{u,v}(u,v) \in \mathcal{N}_\epsilon$. The maximum of the function $t \mapsto I_\epsilon(t(u,v))$ for $t \ge 0$ is achieved at $t = t_{u,v}$ and the function $(u,v) \mapsto t_{u.v}$ is continuous from $X_\epsilon \setminus \{(0,0)\}$ to $(0,\infty)$. Note that by conditions on α and β, we have

$$Q(u,v) \le \frac{\alpha}{p}|u|^p + \frac{\beta}{p}|v|^p. \tag{2.2}$$

Standard calculations imply that I_ϵ satisfies the geometry of the Mountain Pass theorem. Arguing as in [18, Theorem 4.2] we can prove that c_ϵ is positive, it coincides with the mountain pass level of I_ϵ and satisfies

$$c_\epsilon = \inf_{\gamma\in\Gamma_\epsilon} \max_{t\in[0,1]} I_\epsilon(\gamma(t)) = \inf_{(u,v)\in X_\epsilon\setminus\{(0,0)\}} \max_{t\ge 0} I_\epsilon(t(u,v)) > 0, \tag{2.3}$$

where $\Gamma_\epsilon = \{\gamma \in C([0,1],X_\epsilon) : \gamma(0) = (0,0),\ I_\epsilon(\gamma(1)) < 0\}$.

We will denote by $\|I'_\epsilon((u,v))\|_*$ the norm of the derivative of I_ϵ restricted to $\mathcal{N}_\epsilon$ at the point (u,v). This norm is given by (see [18, Proposition 5.12])

$$\|I'_\epsilon((u,v))\|_* = \min_{\lambda\in\mathbb{R}} \|I'_\epsilon((u,v)) - \lambda J'_\epsilon((u,v))\|_{X^*_\epsilon},$$

where X^*_ϵ denotes the dual space of X_ϵ and $J_\epsilon : X_\epsilon \to \mathbb{R}$ is defined as

$$J_\epsilon((u,v)) = \|(u,v)\|^2_\epsilon - p\int Q(u,v). \tag{2.4}$$

As we will see, it is important to compare c_ϵ with the minimax level of the autonomous problem (1.4). The solutions of (1.4) are precisely the positive critical points of the functional $E_\mu : H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$E_\mu((u,v)) = \frac{1}{2}\int \left(|\nabla u|^2 + |\nabla v|^2\right) + \frac{1}{2}\int \mu\left(|u|^2+|v|^2\right) - \int Q(u,v).$$

We also define the autonomous minimization problem

$$m(\mu) = \inf_{(u,v)\in\mathcal{M}_\mu} E_\mu((u,v)),$$

where $\mathcal{M}_\mu$ is the Nehari manifold of E_μ, that is

$$\mathcal{M}_\mu = \left\{(u,v)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\setminus\{(0,0)\} : \langle E'_\mu((u,v)),(u,v)\rangle = 0\right\}.$$

The number $m(\mu)$ and the manifold $\mathcal{M}_\mu$ have properties similar to those of c_ϵ and $\mathcal{N}_\epsilon$. Moreover, Alves and Monari in [3, Theorem 4.11] showed that $m(\mu)$ is attained by a solution $(u,v)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ of the problem (1.4).

3. A compactness condition

In this section we obtain some compactness properties of the functional I_ϵ. We start by recalling the definition of the Palais-Smale condition. So, let E be a Banach space, $\mathcal{V}$ be a C^1-manifold of E and $I : E\to\mathbb{R}$ a C^1-functional. We say that $I|_\mathcal{V}$ satisfies the Palais-Smale condition at level c ($(\mathrm{PS})_c$) if any sequence $(u_n)\subset\mathcal{V}$ such that $I(u_n)\to c$ and $\|I'(u_n)\|_*\to 0$ contains a convergent subsequence.

The next lemma shows a property involving $(PS)_c$ sequences for I_ϵ. Its proof uses well-know arguments and will be omitted.

Lemma 3.1. *Let $((u_n,v_n))\subset X_\epsilon$ be a $(\mathrm{PS})_c$ sequence for I_ϵ. Then*

(i) $((u_n,v_n))$ is bounded in X_ϵ,
(ii) there exists $(u,v)\in X_\epsilon$ such that, up to a subsequence, $(u_n,v_n)\rightharpoonup(u,v)$ weakly in X_ϵ and $I'_\epsilon((u,v))=0$,
(iii) $((u_n^+,v_n^+))$ is also a $(\mathrm{PS})_c$ sequence for I_ϵ.

Moreover, the same holds if we replace I_ϵ and X_ϵ which E_μ and $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, respectively.

Remark 3.2. Let $((u_n,v_n))$ be a Palais-Smale sequence for I_ϵ (or E_μ). Since we are always interested in the existence of convergent subsequences, we may use the above lemma to suppose that $u_n\ge 0$ and $v_n\ge 0$ for all $n\in\mathbb{N}$. This will be made from now on.

Lemma 3.3. *Let $((u_n,v_n))\subset X_\epsilon$ be a $(\mathrm{PS})_d$ sequence for I_ϵ. Then we have either*

(i) $\|(u_n,v_n)\|_\epsilon\to 0$, or

(ii) there exist a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R,\ \gamma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \geq \gamma > 0.$$

The above lemma follows by adapting the arguments of [3, page 171] (see also [12, Theorem 2.1]).

Remark 3.4. For future reference we note that, if $\epsilon_n \to 0$ and $((u_n, v_n)) \subset \mathcal{N}_{\epsilon_n}$ is a bounded sequence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that $I_{\epsilon_n}((u_n, v_n)) \to d$, then we can argue along the same lines of the above proof to conclude that either $\|(u_n, v_n)\|_{\epsilon_n} \to 0$ or (ii) holds. We also have a similar result if $((u_n, v_n)) \subset H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is a *(PS)$_d$* sequence for the autonomous functional E_μ.

Lemma 3.5. *Consider $V_\infty < \infty$ and let $((u_n, v_n)) \subset X_\epsilon$ be a* (PS)$_d$ *sequence for I_ϵ such that $(u_n, v_n) \rightharpoonup (0,0)$ weakly in X_ϵ. If $(u_n, v_n) \not\to (0,0)$ in X_ϵ, then $d \geq m(V_\infty)$.*

Proof. Let $(t_n) \subset (0, +\infty)$ be such that $(t_n(u_n, v_n)) \subset \mathcal{M}_{V_\infty}$. We start by proving that $\limsup_{n\to\infty} t_n \leq 1$. Arguing by contradiction, we suppose that there exist $\lambda > 0$ and a subsequence, which we also denote by (t_n), such that

$$t_n \geq 1 + \lambda \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Since $((u_n, v_n))$ in bounded in X_ϵ, $\langle I'_\epsilon((u_n, v_n)), (u_n, v_n)\rangle \to 0$, that is,

$$\int \Big[|\nabla u_n|^2 + |\nabla v_n|^2 + V(\epsilon x)(|u_n|^2 + |v_n|^2)\Big] = p \int Q(u_n, v_n) + o_n(1).$$

Moreover, recalling that $(t_n(u_n, v_n)) \subset \mathcal{M}_{V_\infty}$, we get

$$\int \Big[|\nabla u_n|^2 + |\nabla v_n|^2 + V_\infty(|u_n|^2 + |v_n|^2)\Big] = p(t_n^{p-2}) \int Q(u_n, v_n).$$

These two equalities imply

$$p(t_n^{p-2} - 1) \int Q(u_n, v_n) = \int [V_\infty - V(\epsilon x)](|u_n|^2 + |v_n|^2) + o_n(1). \tag{3.2}$$

Using the condition (1.2), we have that given $\delta > 0$, there exists $R > 0$ such that

$$V(\epsilon x) \geq V_\infty - \delta \quad \text{for any } |x| \geq R. \tag{3.3}$$

Let $C > 0$ be such that $\|(u_n, v_n)\|_\epsilon \leq C$. Since $\|(u_n, v_n)\|_\epsilon \to 0$ in $H^1(B_R(0)) \times H^1(B_R(0))$ we can use (3.2) and (3.3) to obtain

$$p(t_n^{p-2} - 1) \int Q(u_n, v_n) \leq \delta C + o_n. \tag{3.4}$$

for any $\delta > 0$. Since $(u_n, v_n) \not\to (0,0)$, we may invoke Lemma 3.3 to obtain $(y_n) \subset \mathbb{R}^N$ and $R, \gamma > 0$ such that

$$\int_{B_R(y_n)} (u_n^2 + v_n^2) \geq \gamma > 0. \tag{3.5}$$

If we define $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + y_n), v_n(x + y_n))$ we may suppose that, up to a subsequence,

$$\begin{gathered} (\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v) \quad \text{weakly in } X_\epsilon, \\ (\tilde{u}_n, \tilde{v}_n) \to (u, v) \quad \text{in } L^p(B_R(0)) \times L^p(B_R(0)), \\ (\tilde{u}_n(x), \tilde{v}_n(x)) \to (u(x), v(x)) \quad \text{for a.e. } x \in \mathbb{R}^N, \end{gathered}$$

for some nonnegative functions u, v. Moreover, in view of (3.5), there exists a subset $\Omega \subset \mathbb{R}^N$ with positive measure such that u, v are strictly positive in Ω.

We can use (3.1) to rewrite (3.4) as

$$0 < p((1+\lambda)^{p-2} - 1) \int_\Omega |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta \leq \delta C, \quad \forall \delta > 0.$$

for any $\delta > 0$. Letting $n \to \infty$, using Fatou's lemma, we obtain

$$0 < p((1+\lambda)^{p-2} - 1) \int_\Omega |u|^\alpha |v|^\beta \leq \delta C.$$

for any $\delta > 0$. We obtain a contradiction by taking $\delta \to 0$. Thus, $\limsup_{n\to\infty} t_n \leq 1$, as claimed.

Setting $t_0 = \limsup_{n\to\infty} t_n$, we consider two complementary cases:

Case 1: $t_0 < 1$. In this case we may suppose, without loss of generality, that $t_n < 1$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned}
m(V_\infty) &\leq E_{V_\infty}(t_n(u_n, v_n)) - \frac{1}{2} E'_{V_\infty} \langle (t_n(u_n, v_n))(t_n(u_n, v_n)) \rangle \\
&= (\frac{p}{2} - 1) t_n^p \int Q(u_n, v_n) \leq (\frac{p}{2} - 1) \int Q(u_n, v_n) \\
&= I_\epsilon((u_n, v_n)) - \frac{1}{2} \langle I'_\epsilon((u_n, v_n)), (u_n, v_n) \rangle \\
&= d + o_n(1).
\end{aligned}$$

Taking the limit we conclude that $d \geq m(V_\infty)$.

Case 2: $t_0 = 1$. Up to a subsequence, we may suppose that $t_n \to 1$. We first note that

$$d + o_n(1) \geq m(V_\infty) + I_\epsilon((u_n, v_n)) - E_{V_\infty}(t_n(u_n, v_n)).$$

Note that

$$\begin{aligned}
&I_\epsilon((u_n, v_n)) - E_{V_\infty}(t_n(u_n, v_n)) \\
&= \int \frac{(1 - t_n^2)}{2} (|\nabla u_n|^2 + |\nabla v_n|^2) + \frac{1}{2} \int V(\epsilon x)(|u_n|^2 + |v_n|^2) \\
&\quad - \frac{t_n^2}{2} \int V_\infty (|u_n|^p + |v_n|^p) - (1 - t_n^p) \int Q(u_n, v_n).
\end{aligned}$$

Since $(\|(u_n, v_n)\|_\epsilon)$ is bounded, we have

$$\int \frac{(1 - t_n^2)}{2} (|\nabla u_n|^2 + |\nabla v_n|^2) = o_n(1),$$
$$(1 - t_n^p) \int Q(u_n, v_n) = o_n(1).$$

Using the condition (1.2), we obtain

$$d + o_n(1) \geq m(V_\infty) - \delta C + o_n(1),$$

for any $\delta > 0$. By taking $n \to \infty$ and $\delta \to 0$, we conclude that $d \geq m(V_\infty)$. □

We present below two compactness results which we will need for the proof of the main theorem.

Proposition 3.6. *The functional I_ϵ satisfies the $(PS)_c$ condition at any level $c < m(V_\infty)$ if $V_\infty < \infty$ and at any level $c \in \mathbb{R}$ if $V_\infty = \infty$.*

Proof. Let $((u_n, v_n)) \subset X_\epsilon$ be such that $I_\epsilon((u_n, v_n)) \to c$ and $I'_\epsilon((u_n, v_n)) \to 0$ in X^*_ϵ. By Lemma 3.1 the weak limit (u, v) of $((u_n, v_n))$ is such that $I'_\epsilon((u, v)) = 0$. Thus,

$$I_\epsilon(u, v) = I_\epsilon(u, v) - \frac{1}{2} I'_\epsilon((u, v))(u, v) = (\frac{p}{2} - 1) \int Q(u, v) \geq 0.$$

Let $\tilde{u}_n = u_n - u$ and $\tilde{v}_n = v_n - v$. Arguing as in [1, Lemma 3.3] we can show that $I'_\epsilon(\tilde{u}_n, \tilde{v}_n) \to 0$ and

$$I_\epsilon((\tilde{u}_n, \tilde{v}_n)) \to c - I_\epsilon((u, v)) = d < m(V_\infty),$$

where we used that $c < m(V_\infty)$ and $I_\epsilon((u, v)) \geq 0$. Since $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (0, 0)$ weakly in X_ϵ and $d < m(V_\infty)$, it follows from Lemma 3.5 that $(\tilde{u}_n, \tilde{v}_n) \to (0, 0)$ in X_ϵ, i.e., $(u_n, v_n) \to (u, v)$ in X_ϵ.

The case $V_\infty = \infty$ follows from [7, Proposition 2.4]. This concludes the proof of the proposition. □

Proposition 3.7. *The functional I_ϵ restricted to $\mathcal{N}_\epsilon$ satisfies the $(PS)_c$ condition at any level $c < m(V_\infty)$ if $V_\infty < \infty$ and at any level $c \in \mathbb{R}$ if $V_\infty = \infty$.*

Proof. Let $((u_n, v_n)) \subset \mathcal{N}_\epsilon$ be such that $I_\epsilon((u_n, v_n)) \to c$ and $\|I'_\epsilon((u_n, v_n))\|_* \to 0$. Then there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$I'_\epsilon((u_n, v_n)) = \lambda_n J'_\epsilon((u_n, v_n))) + o_n(1), \tag{3.6}$$

where J_ϵ was defined in (2.4). Thus

$$0 = I'_\epsilon((u_n, v_n))(u_n, v_n) = \lambda_n J'_\epsilon((u_n, v_n))(u_n, v_n) + o_n(1).$$

Since

$$J'_\epsilon((u_n, v_n))(u_n, v_n) = (2 - p)\|(u_n, v_n)\|^2_\epsilon < 0,$$

and $\|(u_n, v_n)\|^2_\epsilon \not\to 0$ by (2.1), we have $\lambda_n = o_n(1)$. By using (3.6), we conclude that $I'_\epsilon((u_n, v_n)) \to 0$ in X^*_ϵ, that is, $((u_n, v_n))$ is a $(\mathrm{PS})_c$ sequence for I_ϵ. The result follows from Proposition 3.6. □

Corollary 3.8. *The critical points of functional I_ϵ on $\mathcal{N}_\epsilon$ are critical points of I_ϵ in X_ϵ*

The proof of the above corollary follows by using similar arguments explored in the previous proposition.

4. Multiplicity of solutions

For any $\mu > 0$, we denote by $\|\cdot\|_{H_\mu}$ the following norm in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$

$$\|(u, v)\|_{H_\mu} = \Big\{ \int \Big[|\nabla u|^2 + |\nabla v|^2 + \mu(|u|^2 + |v|^2)\Big] \Big\}^{1/2}$$

which is well defined and equivalent to the standard norm of $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Let (w_1, w_2) be a ground state solution of the problem (AP_{V_0}) and consider $\eta : [0, \infty) \to \mathbb{R}$ a cut-off function such that $0 \leq \eta \leq 1$, $\eta(s) = 1$ if $0 \leq s \leq \delta/2$ and $\eta(s) = 0$ if $s \geq \delta$. We recall that M denotes the set of global minima points of V and define, for each $y \in M$, $\Psi_{i,\epsilon,y} : \mathbb{R}^N \to \mathbb{R}$ by setting

$$\Psi_{i,\epsilon,y}(x) = \eta(|\epsilon x - y|) w_i\big(\frac{\epsilon x - y}{\epsilon}\big), \quad i = 1, 2.$$

Let t_ϵ be the unique positive number satisfying

$$\max_{t\geq 0} I_\epsilon(t(\Psi_{1,\epsilon,y},\Psi_{2,\epsilon,y})) = I_\epsilon(t_\epsilon(\Psi_{1,\epsilon,y},\Psi_{2,\epsilon,y})),$$

and define the map $\Phi_\epsilon : M \to \mathcal{N}_\epsilon$ in the following way

$$\Phi_\epsilon(y) = \Phi_{\epsilon,y} = (t_\epsilon(\Psi_{1,\epsilon,y},\Psi_{2,\epsilon,y})). \tag{4.1}$$

In view of the definition of t_ϵ we have that the above map is well defined. Moreover, the following holds.

Lemma 4.1. $\lim_{\epsilon\to 0} I_\epsilon(\Phi_{\epsilon,y}) = m(V_0)$, *uniformly in* $y \in M$.

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $\lambda > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0$ such that

$$|I_{\epsilon_n}(\Phi_{\epsilon_n,y_n}) - m(V_0)| \geq \lambda > 0. \tag{4.2}$$

Since $\langle I'_{\epsilon_n}(\Phi_{\epsilon_n,y_n}), \Phi_{\epsilon_n,y_n}\rangle = 0$, we have that

$$\|(\Psi_{1,\epsilon_n,y_n},\Psi_{2,\epsilon_n,y_n})\|^2_{\epsilon_n} = t^{p-2}_{\epsilon_n} p \int Q(\Psi_{1,\epsilon_n,y_n},\Psi_{2,\epsilon_n,y_n}).$$

Moreover, making the change of variables $z = (\epsilon_n x - y_n)/\epsilon_n$ and using the Lebesgue theorem, we can check that

$$\lim_{n\to\infty} \|(\Psi_{1,\epsilon_n,y_n},\Psi_{2,\epsilon_n,y_n})\|^2_{\epsilon_n} = \|(w_1,w_2)\|^2_{H_{V_0}},$$

$$\lim_{n\to\infty} \int Q(\Psi_{1,\epsilon_n,y_n},\Psi_{2,\epsilon_n,y_n}) = \int Q(w_1,w_2).$$

Thus, up to a subsequence, we have $t_n \to t_0 > 0$ and

$$\|(w_1,w_2)\|^2_{H_{V_0}} = t_0^{p-2} \int Q(w_1,w_2).$$

Since $(w_1,w_2) \in \mathcal{M}_{V_0}$, we obtain $t_0 = 1$. Letting $n \to \infty$, we get

$$\lim_{n\to\infty} I_{\epsilon_n}(\Phi_{\epsilon_n,y_n}) = E_{V_0}(w_1,w_2) = m(V_0),$$

which contradicts (4.2) and proves the lemma. ☐

For any $\delta > 0$, let $\rho = \rho_\delta > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) = x$ for $|x| < \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \geq \rho$. Finally, let us consider the barycenter map $\beta_\epsilon : \mathcal{N}_\epsilon \to \mathbb{R}^N$ given by

$$\beta_\epsilon(u,v) = \frac{\int \chi(\epsilon x)|u(x)|^2}{\int |u(x)|^2} + \frac{\int \chi(\epsilon x)|v(x)|^2}{\int |v(x)|^2}.$$

Lemma 4.2. $\lim_{\epsilon\to 0} \beta_\epsilon(\Phi_{\epsilon,y}) = y$ *uniformly for* $y \in M$.

Proof. Arguing by contradiction, we suppose that there exist $\lambda > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0$ such that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n,y_n}) - y_n| \geq \lambda > 0. \tag{4.3}$$

By using the change of variables $z = (\epsilon_n x - y_n)/\epsilon_n$, we get

$$\begin{aligned}\beta_\epsilon(\Phi_{\epsilon_n,y_n}) &= y_n + \frac{\int [\chi(\epsilon_n z + y_n) - y_n]|\eta(|\epsilon_n z|)w_1(z)|^2}{\int |\eta(|\epsilon_n z|)w_1(z)|^2} \\ &\quad + \frac{\int [\chi(\epsilon_n z + y_n) - y_n]|\eta(|\epsilon_n z|)w_2(z)|^2}{\int |\eta(|\epsilon_n z|)w_2(z)|^2}.\end{aligned}$$

Since $(y_n) \subset M \subset B_\rho(0)$ we have that $\chi(\epsilon_n z + y_n) - y_n = o_n(1)$. Hence, by the Lebesgue theorem, we conclude that

$$\beta_{\epsilon_n}(\Phi_{\epsilon_n, y_n}) - y_n = o_n(1),$$

which contradicts (4.3) and proves the lemma. $\square$

Lemma 4.3 (A Compactness Lemma)**.** *Let $((u_n, v_n)) \subset \mathcal{M}_\mu$ be a sequence satisfying $E_\mu(u_n, v_n) \to m(\mu)$. Then,*

(a) *$((u_n, v_n))$ has a subsequence strongly convergent in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, or*
(b) *there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence,*

$$(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + \tilde{y}_n), v_n(x + \tilde{y}_n))$$

converges strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

In particular, there exists a minimizer for $m(\mu)$.

Proof. Applying Ekeland's variational principle [18, Theorem 8.5], we may suppose that $((u_n, v_n))$ is a $(\mathrm{PS})_{m(\mu)}$ sequence for E_μ. Thus, going to a subsequence if necessary, we have that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with (u, v) being a critical point of E_μ.

If $(u, v) \neq (0, 0)$, it is easy to check that (u, v) is a ground state solution of the autonomous problem (1.4), that is, $E_\mu(u, v) = m(\mu)$.

We now consider the complementary case $(u, v) = (0, 0)$. In this case, by Remark 3.4, there exist a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} (|u_n|^2 + |v_n|^2) \geq \gamma > 0.$$

Defining $\tilde{u}_n(x) = u_n(x + \tilde{y}_n)$ and $\tilde{v}_n(x) = v_n(x + \tilde{y}_n)$ we have that $((\tilde{u}_n, \tilde{v}_n))$ is also a $(\mathrm{PS})_{m(\mu)}$ sequence of E_μ such that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v}) \neq (0, 0)$. It follows from the first part of the proof that, up to a subsequence, $((\tilde{u}_n, \tilde{v}_n))$ converges in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. The lemma is proved. $\square$

Lemma 4.4. *Let $\epsilon_n \to 0$ and $((u_n, v_n)) \subset \mathcal{N}_{\epsilon_n}$ be such that $I_{\epsilon_n}((u_n, v_n)) \to m(V_0)$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $(\tilde{u}_n, \tilde{v}_n)(x) = (u_n(x+\tilde{y}_n), v_n(x+\tilde{y}_n))$ has a convergent subsequence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Moreover, up to a subsequence, $(y_n) = (\epsilon_n \tilde{y}_n)$ is such that $y_n \to y \in M$.*

Proof. Arguing as in Remark 3.4, we obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in } H^1(\mathbb{R}^N) \text{ and } \tilde{v}_n \rightharpoonup \tilde{v} \quad \text{in } H^1(\mathbb{R}^N),$$

where $\tilde{u}_n = u_n(x + \tilde{y}_n)$ and $\tilde{v}_n = v_n(x + \tilde{y}_n)$ with $\tilde{u} \not\equiv 0$ and $\tilde{v} \not\equiv 0$.

Let $(t_n) \subset (0, +\infty)$ be such that $(\hat{u}_n, \hat{v}_n) = t_n(\tilde{u}_n, \tilde{v}_n) \in \mathcal{M}_{V_0}$. Defining $y_n = \epsilon_n \tilde{y}_n$, changing variables and recalling that $(u_n, v_n) \in \mathcal{N}_{\epsilon_n}$, we get

$$\begin{aligned}
&E_{V_0}((\hat{u}_n, \hat{v}_n)) \\
&\leq \frac{1}{2} \int \left[|\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 + V(\epsilon_n x + y_n)\left(|\hat{u}_n|^2 + |\hat{v}_n|^2\right)\right] - \int Q(\hat{u}_n, \hat{v}_n) \\
&= I_{\epsilon_n}((\hat{u}_n, \hat{v}_n)) \\
&= I_{\epsilon_n}(t_n(\tilde{u}_n, \tilde{v}_n)) \\
&\leq I_{\epsilon_n}((u_n, v_n)) = m(V_0) + o_n(1).
\end{aligned}$$

Hence

$$E_{V_0}((\hat{u}_n,\hat{v}_n)) \to m(V_0).$$

Since (t_n) is bounded, the sequence $((\hat{u}_n,\hat{v}_n))$ is also bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, thus for some subsequence, $(\hat{u}_n,\hat{v}_n) \rightharpoonup (\hat{u},\hat{v})$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Moreover, reasoning as in [10], up to some subsequence, still denote by (t_n),we can assume that $t_n \to t_0 > 0$, and this limit implies that $(\hat{u},\hat{v}) \not\equiv (0,0)$. From Lemma 4.3 $(\hat{u}_n,\hat{v}_n) \to (\hat{u},\hat{v})$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ and so, $(\tilde{u}_n,\tilde{v}_n) \to (\tilde{u},\tilde{v})$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

To complete the proof of the lemma, it suffices to check that $(y_n) = (\epsilon_n \tilde{y}_n)$ has a subsequence such that $y_n \to y \in M$. Indeed, suppose by contradiction that (y_n) is not bounded, then there exists a subsequence, still denoted by (y_n), such that $|y_n| \to \infty$. Considering firstly the case $V_\infty = \infty$, the inequality

$$\begin{aligned}
&\int V(\epsilon_n x + y_n)(|u_n|^2 + v_n|^2)\\
&\le \int (|\nabla u_n|^2 + \nabla v_n|^2) + \int V(\epsilon_n x + y_n)(|u_n|^2 + |v_n|^2)\\
&= p\int Q(u_n,v_n),
\end{aligned}$$

together with Fatou's Lemma imply

$$\infty = p\liminf_{n\to\infty} \int Q(u_n,v_n),$$

which is an absurd, because the sequence $Q(u_n,v_n)$ is bounded in $L^1(\mathbb{R}^N)$.

Now, let us consider the case $V_\infty < \infty$. Since $(\hat{u}_n,\hat{v}_n) \to (\hat{u},\hat{v})$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ and $V_0 < V_\infty$, we have

$$\begin{aligned}
m(V_0) &= \frac{1}{2}\int (|\nabla \hat{u}|^2 + \nabla \hat{v}|^2) + \frac{1}{2}\int V_0(|\hat{u}|^2 + |\hat{v}|^2) - \int Q(\hat{u},\hat{v})\\
&< \frac{1}{2}\int (|\nabla \hat{u}|^2 + \nabla \hat{v}|^2) + \frac{1}{2}\int V_\infty(|\hat{u}|^2 + |\hat{v}|^2) - \int Q(\hat{u},\hat{v})\\
&\le \liminf_{n\to\infty}\Big[\frac{1}{2}\Big(\int ((|\nabla \hat{u}_n|^2 + \nabla \hat{v}_n|^2) + V(\epsilon_n x + y_n)(|\hat{u}_n|^2 + |\hat{v}_n|^2))\Big)\\
&\quad - \int Q(\hat{u}_n,\hat{v}_n)\Big],
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
m(V_0) < \liminf_{n\to\infty}\Big[&\frac{t_n^2}{2}\Big(\int ((|\nabla \tilde{u}_n|^2 + \nabla \tilde{v}_n|^2) + V(\epsilon_n x + y_n)(|\tilde{u}_n|^2 + |\tilde{v}_n|^2))\Big)\\
&- \int Q(t_n\tilde{u}_n, t_n\tilde{v}_n)\Big].
\end{aligned}$$

The last inequality implies,

$$m(V_0) < \liminf_{n\to\infty} I_{\epsilon_n}((t_n u_n, t_n v_n)) \le \liminf_{n\to\infty} I_{\epsilon_n}((u_n,v_n)) = m(V_0),$$

which is impossible. Hence, (y_n) is bounded and, up to a subsequence, $y_n \to y \in \mathbb{R}^N$. If $y \notin M$, then $V(y) > V_0$ and we obtain a contradiction arguing as above. Thus, $y \in M$ and the lemma is proved. $\square$

Following [6], we introduce a subset of $\mathcal{N}_\epsilon$ which will be useful in the future. We take a function $h:[0,\infty)\to[0,\infty)$ such that $h(\epsilon)\to 0$ as $\epsilon\to 0$ and set

$$\Sigma_\epsilon = \{(u,v)\in\mathcal{N}_\epsilon : I_\epsilon((u,v)) \le m(V_0)+h(\epsilon)\}.$$

Given $y\in M$, we can use Lemma 4.1 to conclude that $h(\epsilon)=|I_\epsilon(\Phi_{\epsilon,y})-m(V_0)|$ is such that $h(\epsilon)\to 0$ as $\epsilon\to 0$. Thus, $\Phi_{\epsilon,y}\in\Sigma_\epsilon$ and we have that $\Sigma_\epsilon\neq\emptyset$ for any $\epsilon>0$.

Lemma 4.5. *For any $\delta>0$ we have that*

$$\lim_{\epsilon\to 0}\sup_{(u,v)\in\Sigma_\epsilon}\operatorname{dist}(\beta_\epsilon(u,v),M_\delta)=0.$$

Proof. Let $(\epsilon_n)\subset\mathbb{R}$ be such that $\epsilon_n\to 0$. By definition, there exists $((u_n,v_n))\subset\Sigma_{\epsilon_n}$ such that

$$\operatorname{dist}(\beta_{\epsilon_n}(u_n,v_n),M_\delta)=\sup_{(u,v)\in\Sigma_{\epsilon_n}}\operatorname{dist}(\beta_{\epsilon_n}(u,v),M_\delta)+o_n(1).$$

Thus, it suffices to find a sequence $(y_n)\subset M_\delta$ such that

$$|\beta_{\epsilon_n}(u_n,v_n)-y_n|=o_n(1). \tag{4.4}$$

To obtain such sequence, we note that $((u_n,v_n))\subset\Sigma_{\epsilon_n}\subset\mathcal{N}_{\epsilon_n}$. Thus, recalling that $m(V_0)\le c_{\epsilon_n}$, we get

$$m(V_0)\le c_{\epsilon_n}\le I_{\epsilon_n}((u_n,v_n))\le m(V_0)+h(\epsilon_n),$$

from which follows that $I_{\epsilon_n}((u_n,v_n))\to m(V_0)$. We may now invoke the Lemma 4.4 to obtain a sequence $(\tilde{y}_n)\subset\mathbb{R}^N$ such that $(y_n)=(\epsilon_n\tilde{y}_n)\subset M_\delta$ for n sufficiently large. Hence,

$$\beta_\epsilon(u_n,v_n)=y_n+\frac{\int[\chi(\epsilon_n z+y_n)-y_n]|\tilde{u}_n(z)|^2}{\int|\tilde{u}_n(z)|^2}+\frac{\int[\chi(\epsilon_n z+y_n)-y_n]|\tilde{v}_n(z)|^2}{\int|\tilde{v}_n(z)|^2},$$

Since $\epsilon_n z+y_n\to y\in M$, we have that $\beta_{\epsilon_n}(u_n,v_n)=y_n+o_n(1)$ and therefore the sequence (y_n) verifies (4.4). The lemma is proved. $\square$

We are now ready to present the proof of the multiplicity result and the technique used here is due to Benci and Cerami [5].

Proof of Theorem 1.1. Given $\delta>0$ we can use Lemmas 4.1, 4.2, 4.5 and argue as in [6, Section 6] to obtain $\epsilon_\delta>0$ such that, for any $\epsilon\in(0,\epsilon_\delta)$, the diagram

$$M\xrightarrow{\Phi_\epsilon}\Sigma_\epsilon\xrightarrow{\beta}M_\delta$$

is well defined and $\beta_\epsilon\circ\Phi_\epsilon$ is homotopically equivalent to the embedding $\iota:M\to M_\delta$. Moreover, using the definition of Σ_ϵ and taking ϵ_δ small if necessary, we may suppose that I_ϵ satisfies the Palais-Smale condition in Σ_ϵ. Standard Ljusternik-Schnirelmann theory and Corollary 3.8 provide at least $\operatorname{cat}_{\Sigma_\epsilon}(\Sigma_\epsilon)$ solutions of the problem (1.1). The inequality

$$\operatorname{cat}_{\Sigma_\epsilon}(\Sigma_\epsilon)\ge\operatorname{cat}_{M_\delta}(M)$$

follows from arguments used in [5, Lemma 4.3]. $\square$

Acknowledgments. The author would like to thank Professor F. J. S. A. Corrêa for his help and encouragement.

References

[1] C. O. Alves, P. C. Carrião and E. S. Medeiros, *Multiplicity of solutions for a class of quasilinear problems in exterior domains with Newmann conditions*, Abs. Appl. Analysis **3** (2004), 251–268.

[2] C. O. Alves and G. M. Figueiredo, *Existence and multiplicity of positive solutions to a p-Laplacian equation in* $\mathbb{R}^N$, Differential Integral Equations, 19-2(2006)143-162.

[3] C. O. Alves and S. H. M. Soares *Alguns fenômenos de concentração em equações elipticas*, Anais do 54 Seminário Brasileiro de Análise-SBA, Universidade Estadual de São Paulo-Ibilce-Unesp, 2001, 168-193.

[4] A. Ambrosetti,M. Badiale and S. Cingolani *Semiclassical states of nonlinear Schödinger equations*, Arch. Rat. Mech. Anal. 140(1997)285-300.

[5] V. Benci and G. Cerami, *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology*, Cal. Var. PDE, **2** (1994), 29-48.

[6] S. Cingolani and M. Lazzo, *Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations*, Topol. Methods Nonlinear Anal. **10** (1997), 1–13.

[7] David G. Costa *On a class of Elleptic systems in* $\mathbb{R}^N$. Electron. J. Diff. Eqns. Vol. 1994 (1994), No. 7, 1-14.

[8] M. Del Pino and P. L. Felmer, *Local Mountain Pass for semilinear elliptic problems in unbounded domains,* Calc. Var. 4 (1996), 121-137.

[9] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential,* Journal of Functional Analysis, 69 (1986), 397-408.

[10] G. M. Figueiredo, *Multiplicidade de soluções positivas para uma classe de problemas quasilineares*, Doct. dissertation, Unicamp, 2004.

[11] M. Lazzo, *Solutions positives multiples pour une equation elliptique non lineaire avec l'exposant critique de Sobolev*, C. R. Acad. Sci. Paris 314 (1992), 161-164.

[12] L. A. Maia, E. Montefusco and B. Pellacci. *Positive Solutions for a weakly coupled nonlinear Schrödinger system*, preprint, 2005.

[13] J. Y. Oh, *Existence of semi-classical bound state of nonlinear Schrödinger equations with potencial on the class* $(V)_a$, Comm. Partial Diff. Eq. v 13 (1988), 1499-1519.

[14] J. Y. Oh, *Corrections to existence of semi-classical bound state of nonlinear Schrödinger equations with potencial on the class* $(V)_a$, Comm. Partial Diff. Eq. v 14 (1989), 833-834.

[15] J. Y. Oh, *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential*, Comm. Partial Diff. Eq. v 131 (1990), 223-253.

[16] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. angew Math. Phys. **43** (1992), 270–291.

[17] X. Wang,*On concentration of positive bound states of nonlinear Schrödinger equations*, Comm. Math. Phys. 53 (1993), 229-244.

[18] M. Willem,*Minimax Theorems*, Birkhäuser, 1996.

Giovany M. Figueiredo
Universidade Federal do Pará, Departamento de Matemática, CEP: 66075-110 Belém - Pa, Brazil
E-mail address: giovany@ufpa.br