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STRONG GLOBAL ATTRACTOR FOR A QUASILINEAR NONLOCAL WAVE EQUATION ON $\mathbb{R}^N$

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Abstract. We study the long time behavior of solutions to the nonlocal quasilinear dissipative wave equation

$$u_{tt} - \phi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t + |u|^a u = 0,$$

in $\mathbb{R}^N$, $t \geq 0$, with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$. We consider the case $N \geq 3$, $\delta > 0$, and $(\phi(x))^{-1}$ a positive function in $L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. The existence of a global attractor is proved in the strong topology of the space $\mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$.

1. Introduction

Our aim in this work is to study the quasilinear hyperbolic initial-value problem

$$u_{tt} - \phi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t + |u|^a u = 0, \quad x \in \mathbb{R}^N,\ t \geq 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with initial conditions u_0, u_1 in appropriate function spaces, $N \geq 3$, and $\delta > 0$. Throughout the paper we assume that the functions $\phi, g : \mathbb{R}^N \to \mathbb{R}$ satisfy the condition

(G1) $\phi(x) > 0$, for all $x \in \mathbb{R}^N$ and $(\phi(x))^{-1} := g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

For the modelling process we refer the reader to some of our earlier papers [11, 13] or to the original paper by Kirchhoff in 1883 [8]. There he proposed the so called Kirchhoff string model in the study of oscillations of stretched strings and plates.

In bounded domains there is a vast literature concerning the attractors of semilinear waves equations. We refer the reader to the monographs [3, 14]. Also in the paper [4], the existence of global attractor in a weak topology is discussed for a general dissipative wave equation. Ono [9], for $\delta \geq 0$, has proved global existence, decay estimates, asymptotic stability and blow up results for a degenerate non-linear wave equation of Kirchhoff type with a strong dissipation. On the other hand, it seems that very few results are achieved for the unbounded domain case. In our previous work [11], we proved global existence and blow-up results for an equation of Kirchhoff type in all of $\mathbb{R}^N$. Also, in [13] we proved the existence of

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compact invariant sets for the same equation. Recently, in [12] we studied the stability of the origin for the generalized equation of Kirchhoff strings on $\mathbb{R}^N$, using central manifold theory. Also, Karahalios and Stavrakakis [5], [7] proved existence of global attractors and estimated their dimension for a semilinear dissipative wave equation on $\mathbb{R}^N$.

The presentation of this paper is follows: In Section 2, we discuss the space setting of the problem and the necessary embeddings for constructing the evolution triple. In Section 3, we prove existence of an absorbing set for our problem in the energy space $\mathcal{X}_0$. Finally in Section 4, we prove that there exists a global attractor $\mathcal{A}$ in the strong topology of the energy space $\mathcal{X}_1 := \mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$, so extending some earlier results of us on the asymptotic behavior of the problem (see [13]).

Notation. We denote by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R. Sometimes for simplicity we use the symbols C_0^∞, $\mathcal{D}^{1,2}$, L^p, $1 \leq p \leq \infty$, for the spaces $C_0^\infty(\mathbb{R}^N)$, $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$, respectively; $\|\cdot\|_p$ for the norm $\|\cdot\|_{L^p(\mathbb{R}^N)}$, where in case of $p = 2$ we may omit the index. The symbol $:=$ is used for definitions.

2. Space Setting. Formulation of the Problem

As it is already shown in the paper [11], the space setting for the initial conditions and the solutions of problem (1.1)-(1.2) is the product space

$$\mathcal{X}_0 := D(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N), \quad N \geq 3.$$

Also the space $\mathcal{X}_1 := \mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$, with the associated norm $e_1(u(t)) := \|u\|^2_{\mathcal{D}^{1,2}} + \|u_t\|^2_{L^2_g}$ is introduced, where the space $L^2_g(\mathbb{R}^N)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the inner product

$$(u,v)_{L^2_g(\mathbb{R}^N)} := \int_{\mathbb{R}^N} guv dx. \tag{2.1}$$

It is clear that $L^2_g(\mathbb{R}^N)$ is a separable Hilbert space and the embedding $\mathcal{X}_0 \subset \mathcal{X}_1$ is compact. The homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is defined, as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the following energy norm $\|u\|^2_{\mathcal{D}^{1,2}} := \int_{\mathbb{R}^N} |\nabla u|^2 dx$. It is known that

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \nabla u \in (L^2(\mathbb{R}^N))^N \right\}$$

and $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is embedded continuously in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, that is, there exists $k > 0$ such that

$$\|u\|_{\frac{2N}{N-2}} \leq k \|u\|_{\mathcal{D}^{1,2}}. \tag{2.2}$$

The space $D(A)$ is going to be introduced and studied later in this section. The following generalized version of Poincaré's inequality is going to be frequently used

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \alpha \int_{\mathbb{R}^N} g u^2 dx, \tag{2.3}$$

for all $u \in C_0^\infty$ and $g \in L^{N/2}$, where $\alpha := k^{-2} \|g\|^{-1}_{N/2}$ (see [1, Lemma 2.1]). It is shown that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is a separable Hilbert space. Moreover, the following compact embedding is useful.

Lemma 2.1. *Let $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then the embedding $\mathcal{D}^{1,2} \subset L^2_g$ is compact. Also, let $g \in L^{\frac{2N}{2N-pN+2p}}(\mathbb{R}^N)$. Then the following continuous embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^p_g(\mathbb{R}^N)$ is valid, for all $1 \le p \le 2N/(N-2)$.*

For the proof of the above lemma, we refer to [6, Lemma 2.1]. To study the properties of the operator $-\phi\Delta$, we consider the equation

$$-\phi(x)\Delta u(x) = \eta(x), \quad x \in \mathbb{R}^N, \tag{2.4}$$

without boundary conditions. Since for every $u, v \in C_0^\infty(\mathbb{R}^N)$ we have

$$(-\phi\Delta u, v)_{L^2_g} = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx, \tag{2.5}$$

we may consider (2.4) as an operator equation of the form

$$A_0 u = \eta, \quad A_0 : D(A_0) \subseteq L^2_g(\mathbb{R}^N) \to L^2_g(\mathbb{R}^N), \quad \eta \in L^2_g(\mathbb{R}^N). \tag{2.6}$$

The operator $A_0 = -\phi\Delta$ is a symmetric, strongly monotone operator on $L^2_g(\mathbb{R}^N)$. Hence, the theorem of Friedrichs is applicable. The energy scalar product given by (2.5) is

$$(u, v)_E = \int_{\mathbb{R}^N} \nabla u \nabla v dx$$

and the energy space X_E is the completion of $D(A_0)$ with respect to $(u, v)_E$. It is obvious that the energy space is the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The energy extension $A_E = -\phi\Delta$ of A_0,

$$-\phi\Delta : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.7}$$

is defined to be the duality mapping of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. We define $D(A)$ to be the set of all solutions of equation (2.4), for arbitrary $\eta \in L^2_g(\mathbb{R}^N)$. Using the theorem of Friedrichs we have that the extension A of A_0 is the restriction of the energy extension A_E to the set $D(A)$. The operator $A = -\phi\Delta$ is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the graph scalar product

$$(u, v)_{D(A)} = (u, v)_{L^2_g} + (Au, Av)_{L^2_g}, \quad \text{for all } u, v \in D(A).$$

The norm induced by the scalar product is

$$\|u\|_{D(A)} = \Big\{ \int_{\mathbb{R}^N} g|u|^2 \, dx + \int_{\mathbb{R}^N} \phi|\Delta u|^2 \, dx \Big\}^{1/2},$$

which is equivalent to the norm

$$\|Au\|_{L^2_g} = \big\{ \int_{\mathbb{R}^N} \phi|\Delta u|^2 \, dx \big\}^{1/2}.$$

So we have established the evolution quartet

$$D(A) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^2_g(\mathbb{R}^N) \subset \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.8}$$

where all the embeddings are dense and compact. Finally, the definition of weak solutions for the problem (1.1)–(1.2) is given.

Definition 2.2. A *weak solution* of (1.1)-(1.2) is a function u such that the following three conditions are satisfied:

(i) $u \in L^2[0, T; D(A)]$, $u_t \in L^2[0, T; \mathcal{D}^{1,2}(\mathbb{R}^N)]$, $u_{tt} \in L^2[0, T; L^2_g(\mathbb{R}^N)]$,

(ii) for all $v \in C_0^\infty([0,T] \times (\mathbb{R}^N))$, satisfies the generalized formula

$$\begin{aligned}&\int_0^T (u_{tt}(\tau), v(\tau))_{L_g^2} d\tau + \int_0^T \Big(\|\nabla u(t)\|^2 \int_{\mathbb{R}^N} \nabla u(\tau) \nabla v(\tau) dx\, d\tau \Big) \\ &+\delta \int_0^T (u_t(\tau), v(\tau))_{L_g^2} d\tau + \int_0^T (|u(\tau)|^a u(\tau), v(\tau))_{L_g^2} d\tau = 0,\end{aligned} \tag{2.9}$$

(iii) u satisfies the initial conditions $u(x,0) = u_0(x)$, $u_0 \in D(A)$, $u_t(x,0) = u_1(x)$, $u_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

3. Existence of an Absorbing Set.

In this section we prove existence of an absorbing set for our problem (1.1)-(1.2) in the energy space $\mathcal{X}_0$. First, we give existence and uniqueness results for the problem (1.1)-(1.2) using the space setting established previously.

Theorem 3.1 (Local Existence)**.** *Consider that $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}$ and satisfy the nondegeneracy condition*

$$\|\nabla u_0\| > 0. \tag{3.1}$$

Then there exists $T = T(\|u_0\|_{D(A)}, \|\nabla u_1\|) > 0$ such that the problem (1.1)-(1.2) admits a unique local weak solution u satisfying

$$u \in C(0,T; D(A)) \text{ and } u_t \in C(0,T; \mathcal{D}^{1,2}).$$

Moreover, at least one of the following two statements holds:

(i) $T = +\infty$,

(ii) $e(u(t)) := \|u(t)\|_{D(A)}^2 + \|u_t(t)\|_{\mathcal{D}^{1,2}}^2 \to \infty$, *as* $t \to T_-$.

For the proof of the above theorem, we refer to [11, Theorem 3.2].

Remark 3.2. The nondegeneracy condition (3.1) is imposed by the method which is used even for the proof of existence of local solutions of the problem (1.1)-(1.2). For more details we refer to the proof of Theorem 3.2 in [11]. Also we must notice that this condition is necessary even in the case of bounded domains (e.g., see [9] and [10]).

Lemma 3.3. *Assume that $a \geq 0$, $N \geq 3$. If the initial data $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}$ and satisfy the condition*

$$\|\nabla u_0\| > 0, \tag{3.2}$$

then

$$\|\nabla u(t)\| > 0, \quad \text{for all } t \geq 0. \tag{3.3}$$

Proof. Let $u(t)$ be a unique solution of the problem (1.1)-(1.2) in the sense of Theorem 3.1 on $[0,T)$. Multiplying (1.1) by $-2\Delta u_t$ (in the sense of the inner product in the space L^2) and integrating it over $\mathbb{R}^N$, we have

$$\begin{aligned}&\frac{d}{dt} \|\nabla u_t(t)\|^2 + \|\nabla u(t)\|^2 \frac{d}{dt} \|u(t)\|_{D(A)}^2 \\ &+2\|\nabla u_t(t)\|^2 + 2(|u|^a u, \Delta u_t(t)) = 0\end{aligned} \tag{3.4}$$

Since $\|\nabla u_0\| > 0$, we see that $\|\nabla u(t)\| > 0$ near $t = 0$. Let

$$T := \sup\{t \in [0, +\infty) : \|\nabla u(s)\| > 0 \quad \text{for } 0 \leq s < t\},$$

then $T > 0$ and $\|\nabla u(t)\| > 0$ for $0 \leq t < T$. By contradiction we may prove that $T = +\infty$. $\square$

Theorem 3.4 (Absorbing Set)**.** *Assume that* $0 \le a < 2/(N-2)$, $N \ge 3$, $M_0 := \frac{1}{2}\|\nabla u_0\|^2 > 0$, $(u_0,\ u_1) \in D(A) \times \mathcal{D}^{1,2}$ *and*

$$\frac{\delta}{4} > 4\alpha^{-1/2} R^2 c_3^2, \tag{3.5}$$

where $c_3 := (\max\{1, M_0^{-1}\})^{1/2}$ *and* R *a given positive constant. Then the ball* $\mathcal{B}_0 := B_{\mathcal{X}_0}(0, \bar{R}_*)$, *for any* $\bar{R}_* > R_*$, *is an absorbing set in the energy space* $\mathcal{X}_0$, *where*

$$R_*^2 := \frac{2k_2 R^{2(a+1)}}{\delta}\big(\frac{\delta}{4} - \frac{4R^2c_3^2}{\sqrt{\alpha}}\big)^{-1}.$$

Proof. Given the constants $T > 0,\ R > 0$, we introduce the two parameter space of solutions

$$X_{T,R} := \{u \in C(0,T; D(A)) : u_t \in C(0,T;\mathcal{D}^{1,2}),\ e(u) \le R^2,\ t \in [0,T]\},$$

where $e(u) := \|u_t\|^2_{\mathcal{D}^{1,2}} + \|u\|^2_{D(A)}$. The set $X_{T,R}$ is a complete metric space under the distance $d(u,v) := \sup_{0\le t\le T} e(u(t)-v(t))$. Following [9] we introduce the notation

$$T_0 := \sup\{t \in [0,\infty) :\ \|\nabla u(s)\|^2 > M_0,\ 0 \le s \le t\}.$$

Condition $\frac{1}{2}\|\nabla u_0\|^2 = M_0 > 0$ implies $T_0 > 0$ and $\|\nabla u(t)\|^2 > M_0 > 0$, for all $t \in [0, T_0]$. Next, we set $v = u_t + \varepsilon u$ for sufficiently small ε. Then, for calculation needs, equation (1.1) is rewritten as

$$v_t + (\delta - \varepsilon)v + (-\phi(x)\|\nabla u\|^2\Delta - \varepsilon(\delta-\varepsilon))u + f(u) = 0. \tag{3.6}$$

Multiplying equation (3.6) by

$$gAv = g(-\varphi\Delta)v = -\Delta v = -\Delta(u_t + \varepsilon u),$$

and integrating over $\mathbb{R}^N$, we obtain (using Hölder inequality with $p^{-1} = \frac{1}{N}$, $q^{-1} = \frac{N-2}{2N}$, $r^{-1} = \frac{1}{2}$)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\{\|u\|^2_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)} + \|v\|^2_{\mathcal{D}^{1,2}} + \frac{\varepsilon(\delta-\varepsilon)}{2}\|u\|^2_{\mathcal{D}^{1,2}}\big\}\\
&+ (\delta-\varepsilon)\|v\|^2_{\mathcal{D}^{1,2}} + \varepsilon\|u\|^2_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)} + \varepsilon^2(\delta-\varepsilon)\|u\|^2_{\mathcal{D}^{1,2}}\\
&\le \big|\big(\frac{d}{dt}\|u\|^2_{\mathcal{D}^{1,2}}\big)\|u\|^2_{D(A)}\big| + \|u\|^a_{L^{Na}}\|\nabla u\|_{L^{\frac{2N}{N-2}}}\|\nabla v\|.
\end{aligned} \tag{3.7}$$

We observe that

$$\begin{aligned}
\theta(t) &:= \|u\|^2_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)} + \|v\|^2_{\mathcal{D}^{1,2}} + \frac{\varepsilon(\delta-\varepsilon)}{2}\|u\|^2_{\mathcal{D}^{1,2}}\\
&\ge M_0\|u\|^2_{D(A)} + \|u_t\|^2_{\mathcal{D}^{1,2}} \ge c_3^{-2}e(u).
\end{aligned} \tag{3.8}$$

Also

$$\begin{aligned}
\big|\big(\frac{d}{dt}\|u\|^2_{\mathcal{D}^{1,2}}\big)\|u\|^2_{D(A)}\big| &= \big|\big(2\int_{\mathbb{R}^N}\Delta u u_t\varphi g\,dx\big)\|u\|^2_{D(A)}\big|\\
&\le 2\big(\|u\|^2_{D(A)}\big)^{1/2}\big(\|u_t\|^2_{L^2_g}\big)^{1/2}\|u\|^2_{D(A)}\\
&\le 2\alpha^{-1/2}R^2 e(u) \le 2\alpha^{-1/2}R^2c_3^2\theta(t).
\end{aligned} \tag{3.9}$$

Applying Young's inequality in the last term of (3.7) and using relations (3.8), (3.9) and the estimates

$$\|u\|^a_{L^{Na}} \le R^a \quad\text{and}\quad \|\nabla u\|_{L^{\frac{2N}{N-2}}} \le \|u\|_{D(A)} \le R, \tag{3.10}$$

inequality (3.7) becomes (for suitably small ε)

$$\frac{d}{dt}\theta(t) + C_*\theta(t) \le \frac{C(R)}{\delta}, \tag{3.11}$$

where $C_* = \frac{1}{2}\left(\delta/4 - 4\alpha^{-1/2}R^2c_3^2\right) > 0$ and $C(R) = R^{2(a+1)}$. An application of Gronwall's inequality in the relation (3.11) gives

$$\theta(t) \le \theta(0)e^{-C_*t} + \frac{1-e^{-C_*t}}{C_*}\frac{C(R)}{\delta}. \tag{3.12}$$

Following the reasoning developed by K. Ono (see [9]), the nondegeneracy condition $\|\nabla u_0\| > 0$ and the relation (3.3), imply that $\|\nabla u(s)\| > M_0 > 0$, $0 \le s \le t$, $t \in [0, +\infty)$. Now, letting $t \to \infty$, in the relation (3.12) conclude that

$$\lim_{t\to\infty} \sup \theta(t) \le \frac{R^{2(a+1)}}{\delta C_*} := R_*^2. \tag{3.13}$$

So, the ball $B_0 := B_{\mathcal{X}_0}(0, \bar{R}_*)$, for any $\bar{R}_* > R_*$, is an absorbing set for the associated semigroup $S(t)$ in the energy space of solutions $\mathcal{X}_0$. □

Corollary 3.5 (Global Existence)**.** *The unique local solution the problem* (1.1)-(1.2) *defined by Theorem 3.1 exists globally in time.*

Proof. Combining inequality (3.13) and the arguments developed in the proof of [11, Theorem 3.2], we conclude that the solution of the problem (1.1)-(1.2) exists globally in time. □

4. Strong Global Attractor in the space $\mathcal{X}_1$

In this section we study the problem (1.1)-(1.2) from a dynamical system point of view. We need the following results.

Theorem 4.1. *Assume that* $0 \le a \le 4/(N-2)$, *where* $N \ge 3$. *If* $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}$ *and satisfy the nondegeneracy condition*

$$\|\nabla u_0\| > 0, \tag{4.1}$$

then there exists $T > 0$ *such that the problem* (1.1)-(1.2) *admits local weak solutions* u *satisfying*

$$u \in C(0,T;\mathcal{D}^{1,2}) \quad \text{and} \quad u_t \in C(0,T;L_g^2). \tag{4.2}$$

Proof. The proof follows the lines of [11, Theorem 3.2], so we just sketch the proof. The compactness of the embedding $\mathcal{X}_0 \subset \mathcal{X}_1$ implies $e_1(u(t)) \le e(u(t))$, where the associated norms are

$$e_1(u(t)) := \|u\|_{\mathcal{D}^{1,2}}^2 + \|u_t\|_{L_g^2}^2 \quad \text{and} \quad e(u(t)) := \|u\|_{D(A)}^2 + \|u_t\|_{\mathcal{D}^{1,2}}^2.$$

Then, for some positive constant R an a priori bound can be found of the form

$$e_1(u(t)) \le e(u(t)) \le R^2.$$

Hence the solutions u of the problem (1.1)-(1.2) satisfy

$$u \in L^\infty(0,T;\ \mathcal{D}^{1,2}), \quad u_t \in L^\infty(0,T;L_g^2).$$

Finally, the continuity properties (4.2), are proved following ideas from [14, Sections II.3 and II.4]. □

Next, the strong continuity of the semigroup $S(t)$ is proved in the space $\mathcal{X}_1$.

Lemma 4.2. *The mapping $S(t) : \mathcal{X}_1 \to \mathcal{X}_1$ is continuous, for all $t \in \mathbb{R}$.*

Proof. Let u, v two solutions of the problem (1.1)-(1.2) such that

$$\begin{gathered} u_{tt} - \phi(x)\|\nabla u\|^2 \Delta u + \delta u_t = -|u|^a u, \\ v_{tt} - \phi(x)\|\nabla v\|^2 \Delta v + \delta v_t = -|v|^a v. \end{gathered}$$

Let $w = u - v$. So, we have

$$\begin{gathered} w_{tt} - \phi\|\nabla u\|^2 \Delta w + \delta w_t = \phi\{\|\nabla u\|^2 - \|\nabla v\|^2\}\Delta v - (|u|^a u - |v|^a v) \\ w(0) = 0, \quad w_t(0) = 0. \end{gathered}$$

Multiplying the previous equation by $2gw_t$ and integrating over $\mathbb{R}^N$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} g w_t w_{tt} dx - 2\int_{\mathbb{R}^N} \|\nabla u\|^2 \Delta w w_t dx + 2\delta \int_{\mathbb{R}^N} g w_t^2 dx \\ &= \{\|\nabla u\|^2 - \|\nabla v\|^2\} \int_{\mathbb{R}^N} \Delta v w_t dx - 2\int_{\mathbb{R}^N} g(|u|^a u - |v|^a v)) w_t dx. \end{aligned} \tag{4.3}$$

Hence

$$\begin{aligned} &\frac{d}{dt} e^*(w) + 2\delta \|w_t\|^2_{L^2_g} \\ &= (\frac{d}{dt}\|\nabla u\|^2)\|\nabla w\|^2 + 2\{\|\nabla u\|^2 - \|\nabla v\|^2\}(\Delta v\ , w_t) - 2(|u|^a u - |v|^a v, w_t)_{L^2_g} \\ &\equiv I_1(t) + I_2(t) + I_3(t). \end{aligned} \tag{4.4}$$

So

$$\frac{d}{dt} e^*(w) \le I_1(t) + I_2(t) + I_3(t), \tag{4.5}$$

where $e^*(w) = \|w_t\|^2_{L^2_g} + C_u\|w\|^2_{\mathcal{D}^{1,2}}$ and $C_u = \|u\|^2_{\mathcal{D}^{1,2}}$. To estimate the above integrals, more smoothness of the solutions u, v is needed. Theorem 3.1 guarantees the uniqueness of local solutions in the space $\mathcal{X}_0$, if the initial conditions $(u_0, u_1) \in \mathcal{X}_0$. To improve these results, it is assumed that $(u_0, u_1) \in \mathcal{X}_1$. Then, applying again Theorem 3.1, it could be proved the existence of a local solution (u, u_t) in $\mathcal{X}_1$. Furthermore, we may obtain

$$\begin{aligned} I_1(t) &= (2\int_{\mathbb{R}^N} \Delta u u_t \phi(x) g(x) dx)\|\nabla w\|^2 \\ &\le 2(\|u\|^2_{D(A)})^{1/2}(\|u_t\|^2_{L^2_g})^{1/2}\|\nabla w\|^2 \\ &\le 2R_* k(\|u_t\|^2_{\mathcal{D}^{1,2}})^{1/2}\|\nabla w\|^2 \\ &\le 2R_*^2 k\|\nabla w\|^2 \le C_2 e^*(w), \end{aligned} \tag{4.6}$$

where $C_2 = 2R_*^2 k$. Also, the following estimation is valid

$$\begin{aligned} I_3(t) \le |I_3(t)| &\le \alpha^{-1}(\|\nabla u\|^2 - \|\nabla v\|^2)\|\nabla(u - v)\|\ \|w_t\|_{L^2_g} \\ &\le \alpha^{-1} 2R_*^2 \|w\|_{\mathcal{D}^{1,2}}\|w_t\|_{L^2_g} \\ &\le C_A(\frac{C_u}{2C_u}\|w\|^2_{\mathcal{D}^{1,2}} + \frac{1}{2}\|w_t\|^2_{L^2_g}) \\ &\le C_A C_B e^*(w), \end{aligned} \tag{4.7}$$

where we have used Young's inequality and $C_A = 2\alpha^{-1}R_*^2$, $C_B = \max(\frac{1}{2}, \frac{1}{2C_u})$. Hence,

$$
\begin{aligned}
I_2(t) &\le (\|\nabla u\| + \|\nabla v\|)(\|\nabla(u-v)\|)\Big(\int_{\mathbb{R}^N} \Delta v w_t dx\Big) \\
&\le 2R_*\|w\|_{\mathcal{D}^{1,2}}(\|v\|^2_{D(A)})^{1/2}(\|w_t\|^2_{L^2_g})^{1/2} \\
&\le 2R_*^2\|w\|_{\mathcal{D}^{1,2}}(\|w_t\|^2_{L^2_g})^{1/2} \\
&\le 2R_*^2(\frac{C_u}{2C_u}\|w\|^2_{\mathcal{D}^{1,2}} + \frac{1}{2}\|w_t\|^2_{L^2_g}) \le C_\Gamma C_B e^*(w),
\end{aligned}
\tag{4.8}
$$

where $C_\Gamma = 2R_*^2$. Finally, using relations (4.6)-(4.8), estimation (4.5) becomes

$$
\frac{d}{dt}e^*(w) \le (C_2 + C_AC_B + C_\Gamma C_B)e^*(w) \le C_{**}e^*(w), \tag{4.9}
$$

where $C_{**} = C_2 + C_AC_B + C_\Gamma C_B$ and the proof is completed. □

Remark 4.3 (Continuity in $\mathcal{X}_1$). It is important to state that the operator $S(t) : \mathcal{X}_0 \to \mathcal{X}_0$ associated to the problem (1.1)-(1.2) is weakly continuous in the space $\mathcal{X}_0$, but it is strongly continuous in the space $\mathcal{X}_1$. Therefore, we will study problem (1.1)-(1.2) as a dynamical system in the space $\mathcal{X}_1 := \mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$.

Remark 4.4 (Uniqueness in $\mathcal{X}_1$). Assuming that the initial data are from the space $\mathcal{X}_1$, relation (4.9) guarantees the uniqueness of the solutions for the problem (1.1)-(1.2). Indeed, if $\widehat{u}_a = (u_0, u_1)$, $\widehat{u}_b = (u_0', u_1')$, from inequality (4.9) take

$$
\|S(t)\widehat{u}_a - S(t)\widehat{u}_b\|_{\mathcal{X}_1} \le C(\|\widehat{u}_a\|_{\mathcal{X}_1}, \|\widehat{u}_b\|_{\mathcal{X}_1})\|\widehat{u}_a - \widehat{u}_b\|_{\mathcal{X}_1}. \tag{4.10}
$$

Remark 4.5. According to Theorem 3.4 we have that the ball $\mathcal{B}_0 := B_{\mathcal{X}_0}(0, \overline{R}_*)$ is an absorbing set in the space $\mathcal{X}_0$, so and in $\mathcal{X}_1$ by the compact embedding.

So, we obtain the following theorem.

Theorem 4.6. *The dynamical system given by the semigroup $(S_t)_{t\ge0}$, possesses an invariant set, which attracts all bounded sets of $\mathcal{X}_1$, denoted by*

$$
\mathcal{A} = \cap_{t\ge0} \cup_{s\ge t} \overline{S_s\mathcal{B}_0} \subset \mathcal{X}_1.
$$

The above set is also compact, so it is global attractor *for the strong topology of $\mathcal{X}_1$.*

Proof. First, we have that operators $(S_t)_{t\ge0}$ form a semigroup on $\mathcal{X}_1$ and that $S_t : \mathcal{X}_1 \to \mathcal{X}_1$ is continuous, for all $t \in \mathbb{R}$ (Lemma 4.2). Also, we have that the ball $\mathcal{B}_0$, is an absorbing set in $\mathcal{X}_1$ (Remark 4.5). Our goal is to prove that the functional invariant set $\mathcal{A}$ is compact for the strong topology of $\mathcal{X}_1$. So, we must show that for a point $w_1 \in \mathcal{A}$, the sequence $S(t_j)u_0^j$ converges strongly to w_1 in $\mathcal{X}_1$. Here, we have that $(u_0^j)_{j\in N}$ and $(t_j)_{j\in N}$, are two sequences such that (u_0^j) is bounded in $\mathcal{X}_1$, t_j goes to $+\infty$, as j goes to $+\infty$ and $S(t_j)u_0^j$ converges weakly to w_1 in the space $\mathcal{X}_1$, as j goes to $+\infty$ (for more details we refer to [2] and [3]). We fix $T > 0$ and note that the sequence $S(t_j - T)u_0^j$ is bounded in $\mathcal{X}_1$ thanks to the existence of an absorbing set in $\mathcal{X}_1$. Hence from this sequence we may extract a subsequence j_1 such that, for some $v_1 \in \mathcal{X}_1$,

$$
S(t_{j_1} - T)u_0^{j_1} \rightharpoonup v_1, \quad \text{as } j_1 \to \infty. \tag{4.11}
$$

Introducing the notation

$$
u_{j_1}(t) := S(t_{j_1} + t - T)u_0^{j_1}, \tag{4.12}
$$

we deduce from (4.11) that

$$u_{j_1}(t) \rightharpoonup S(t)v_1, \quad \text{as } j_1 \to \infty, \tag{4.13}$$

since $S(t)$ is weakly continuous on $\mathcal{X}_1$. Using the energy type estimate (3.12) and the fact that the sequence $\theta(u_{j_1}(0)) = \theta(S(t_{j_1} - T)u_0^{j_1})$ is bounded by a constant, let say C, we obtain

$$\lim_{j_1\to\infty} \sup \theta(S(t_{j_1})u_0^{j_1}) \le Ce^{-C_*T} + \frac{1-e^{-C_*T}}{C_*}\frac{C(R)}{\delta}. \tag{4.14}$$

Applying the invariance of the set $\mathcal{A}$, for $v_1(t) = S(t)v_1$, we get

$$\theta(w_1) = \theta(S(T)v_1) \le e^{-C_*T}\theta(v_1) + \frac{1-e^{-C_*T}}{C_*}\frac{C(R)}{\delta}. \tag{4.15}$$

Subtracting by parts relations (4.14) and (4.15) we get

$$\lim_{j_1\to\infty} \sup \theta(S(t_{j_1})u_0^{j_1}) \le \theta(w_1) + e^{-C_*T}(C - \theta(v_1)). \tag{4.16}$$

Since T is chosen arbitrarily, for $T = 0$ we have

$$\lim_{j_1\to\infty} \sup \theta(S(t_{j_1})u_0^{j_1}) \le \theta(w_1). \tag{4.17}$$

On the other hand, since $S(t_{j_1})u_0^{j_1}$ converges weakly to w_1 in $\mathcal{X}_1$, we have that $\liminf_{j_1\to\infty} \theta(S(t_0^{j_1}) \ge \theta(w_1)$. So we get

$$\lim_{j\to\infty} \theta(S(t_j)u_0^j) = \theta(w_1). \tag{4.18}$$

Using again the fact that $S(t)\mathcal{A} = \mathcal{A}$ and that $\theta(t)$ is weakly continuous, we obtain

$$\lim_{j\to\infty} \|S(t_j)u_0^j\|^2_{\mathcal{X}_1} = \|w_1\|^2_{\mathcal{X}_1}. \tag{4.19}$$

Therefore, $S(t_j)u_0^j$ converges strongly to w_1 in the space $\mathcal{X}_1$ as $j \to \infty$. Thus, we obtain that $\mathcal{A}$ is a global attractor in the strong topology of $\mathcal{X}_1$ (see also [14]). □

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References

[1] K. J. Brown and N. M. Stavrakakis; *Global Bifurcation Results for a Semilinear Elliptic Equation o n all of* $\mathbb{R}^N$, Duke Math. J., 85, (1996), 77-94.

[2] J. M. Ghidaglia, *A Note on the Strong towards Attractors of Damped Forced KdV Equations*, J. Differential Equations, 110, (1994), 356-359.

[3] J. K. Hale, *Asymptotic Behaviour of Dissipative Systems*, Mathematical Surveys and Monographs 25, AMS, Providence, R.I., 1988.

[4] J. K. Hale and N M Stavrakakis, *Compact Attractors for Weak Dynamical Systems*, Applicable Analysis, Vol 26, (1998), 271-287.

[5] N. I. Karachalios and N. M. Stavrakakis, *Existence of Global Attractors for semilinear Dissipative Wave Equations on* $\mathbb{R}^N$, J. Differential Equations, 157, (1999), 183-205.

[6] N. I. Karachalios and N M Stavrakakis; *Global Existence and Blow-Up Results for Some Nonlinear Wave Equations on* $\mathbb{R}^N$*, Adv. Diff. Equations*, Vol 6, (2001), 155-174.

[7] N. I. Karachalios and N. M. Stavrakakis; *Estimates on the Dimension of a Global Attractor for a Semilinear Dissipative Wave Equation on* $\mathbb{R}^N$, Discrete and Continuous Dynamical Systems, Vol 8, (2002), 939-951.

[8] G. Kirchhoff , *Vorlesungen Über Mechanik*, Teubner, Leipzig, 1883.

[9] K. Ono, *On Global Existence, Asymptotic Stability and Blowing - Up of Solutions for some Degenerate Non-Linear Wave Equations of Kirchhoff Type with a Strong Dissipation*, Math. Methods Appl. Sci., 20 (1997), 151-177.

[10] K. Ono, *Global Existence and Decay Properties of Solutions for Some Mildly Degenerate Nonlinear Dissipative Kirchhoff Strings*, Funkcial. Ekvac., 40, (1997), 255-270.

[11] P. G. Papadopoulos and N. M. Stavrakakis, *Global Existence and Blow-Up Results for an Equation of Kirchhoff Type on* $\mathbb{R}^N$, Topological Methods in Nonlinear Analysis, 17, (2001), 91-109.

[12] P. G. Papadopoulos and N. M. Stavrakakis, *Central Manifold Theory for the Generalized Equation of Kirchhoff Strings on* $\mathbb{R}^N$, Nonlinear Analysis TMA, 61, (2005), 1343-1362.

[13] P. G. Papadopoulos and N. M. Stavrakakis, *Compact Invariant Sets for Some Quasilinear Nonlocal Kirchhoff Strings on* $\mathbb{R}^N$, submitted.

[14] R. Temam, *Infinite- Dimensional Dynamical Systems in Mechanics and Physics*, Appl. Math. Sc., 68, (2nd Edition), Springer-Verlag, 1997.

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