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PSEUDO ALMOST PERIODIC SOLUTIONS TO SOME DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. This paper studies some sufficient conditions for the existence and uniqueness of a pseudo almost periodic mild solution to abstract hyperbolic differential equations with infinite delay. Applications include the existence and uniqueness of a pseudo almost periodic mild solution to the heat equation.

1. Introduction

Let $(\mathcal{M}, \|\cdot\|)$ be a Banach space. In Burton and Zhang [5], the degree-theoretic by Granas [16] was used to obtain some sufficient conditions for the existence of T-periodic solutions to the differential equations with infinite delay given by

$$\frac{du}{dt} + Lu = \int_{-\infty}^{t} C(t,s)u(s)ds + f(u) + F(t), \tag{1.1}$$

where $-L$ is a sectorial linear operator on $\mathcal{M}$, $f : \mathcal{M} \mapsto \mathcal{M}$, $F : \mathbb{R} \mapsto \mathcal{M}$ are some $\mathcal{M}$-valued smooth functions with $F(t+T) = F(t)$, $(C(t,s))_{t\geq s}$ is a bounded linear operator on $\mathcal{M}$ with $C(t+T, s+T) = C(t,s)$, for some $T > 0$. Abstract results were then applied to some nonlinear heat equations with memory.

In this paper we study some sufficient condition for the existence and uniqueness of a pseudo almost periodic (mild) solution to (1.1). One should point out that the Banach fixed-point combined with techniques related to the theory of intermediate spaces will be preferred to that of Granas' theory used in [5]. In particular, we will be studying the case when the sectorial operator $-L$ is hyperbolic, equivalently,

$$\sigma(-L) \cap i\mathbb{R} = \{\emptyset\}.$$

Upon making some assumptions, it will be shown that (1.1) has a unique $\mathcal{M}_\alpha$-valued pseudo almost periodic mild solution ($\mathcal{M}_\alpha \subset \mathcal{M}$ being an abstract intermediate space). Applications include the study of pseudo almost periodic mild solutions to the Cauchy problem for the heat equation in $\mathcal{M} = C[0,1]$ given by the partial

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differential equations

$$u_t(t,x) = u_{xx}(t,x) + Hu(t,x) + f(u(t,x)) + F(t) + \int_{-\infty}^{t} C(t,s)u(s,x)ds \quad (1.2)$$

$$u(t,0) = u(t,1) = 0, \quad \forall \quad t \in \mathbb{R}, \quad (1.3)$$

where $H \in \mathbb{R}$ is a constant, and f, F are some appropriate functions.

In fact, to study (1.1) we consider the problem, which consists of studying pseudo almost periodic solutions to the differential equation

$$\frac{du}{dt} + Lu = Ku + f(u) + F(t), \quad \forall t \in \mathbb{R}, \quad (1.4)$$

where $-L$ is a sectorial linear operator on $\mathcal{M}$, and $K : \mathcal{M} \mapsto \mathcal{M}$ is the linear operator defined by

$$Ku(t) := \int_{-\infty}^{t} C(t,s)u(s)ds.$$

To deal with (1.4), as in [4, 13] we will make extensive use of some abstract intermediate spaces$\mathcal{M}_\alpha$ for $\alpha \in (0,1)$. In contrast with the fractional power spaces considered in some recent papers of the author et al. [10, 11], the interpolation and Hölder spaces, for instance, depend only on $D(L)$ and $\mathcal{M}$ and can be explicitly expressed in many concrete examples. The literature on those intermediate spaces is very extensive; thus we refer the reader only to the excellent book by Lunardi [18], which contains a comprehensive presentation on this topic and related issues.

The concept of the pseudo almost periodicity, which is the central question in this paper was introduced in the literature in the early nineties by Zhang [19, 20, 21] as a natural generalization of the well-known Bohr almost periodicity. Thus this new concept is welcome to implement another existing generalization of almost periodicity, the concept of asymptotically almost periodicity due to Fréchet [6, 15]. For more on these new and old concepts and related topics, see, e.g., [1, 2, 3, 7, 8, 9, 12, 19] and the references therein.

The existence of almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions is one of the most attracting topics in the qualitative theory of differential equations due to their significance and applications in physics, mathematical biology, chemistry, control theory, and many others.

Some contributions on almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions to abstract differential and partial differential equations have recently been made in [3, 7, 8, 9, 10, 17]. However, the existence of pseudo almost periodic to (1.1) in the case when $-L$ is hyperbolic is an untreated topic and this is the main motivation of the present paper. Among other things, some sufficient conditions for the existence and uniqueness of a pseudo almost periodic (mild) solution to (1.1) are obtained whenever $-L$ generates a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, which is not necessarily strongly continuous at 0.

2. Preliminaries

This section is devoted to some preliminary facts needed in the sequel. We basically use a similar setting as in [4, 13]. Throughout the rest of this paper, $(\mathcal{M}, \|\cdot\|)$ stands for a Banach space, $-L$ is a sectorial linear operator (Definition 2.1) which is not necessarily densely defined.

If L is a linear operator on $\mathcal{M}$, then $\rho(L)$, $\sigma(L)$, $D(L)$, $N(L)$, $R(L)$ stand for the resolvent, spectrum, domain, kernel, and range of L.

If $\mathcal{M}_0, \mathcal{M}_1$ are Banach spaces, then the space $(B(\mathcal{M}_0, \mathcal{M}_1), \|\cdot\|_{B(\mathcal{M}_0,\mathcal{M}_1)})$ denotes the Banach space of all bounded linear operators from $\mathcal{M}_0$ into $\mathcal{M}_1$ equipped with its operator topology. If $\mathcal{M}_0 = \mathcal{M}_1$, then this is simply denoted $B(\mathcal{M}_0)$.

2.1. **Sectorial Linear Operators and Their Semigroups.**

Definition 2.1. A linear operator $L : D(L) \subset \mathcal{M} \mapsto \mathcal{M}$ (not necessarily densely defined) is said to be sectorial if the following hold: There exist constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, and $M > 0$ such that

$$\begin{gathered} \rho(L) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ \|R(\lambda, L)\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\theta,\omega}, \end{gathered} \tag{2.1}$$

where $R(\lambda, L) = (\lambda I - L)^{-1}$ for each $\lambda \in \rho(L)$.

Remark 2.2. If $-L$ is sectorial, it then generates an analytic semigroup $(T(t))_{t\geq 0}$, which maps $(0,\infty)$ into $B(\mathcal{M})$ such that there exist $M_0, M_1 > 0$ with

$$\begin{gathered} \|T(t)\| \leq M_0 e^{\omega t}, \quad t > 0, \\ \|t(-L - \omega I)T(t)\| \leq M_1 e^{\omega t}, \quad t > 0. \end{gathered} \tag{2.2}$$

For the rest of the paper, we suppose that the analytic semigroup $(T(t))_{t\geq 0}$ associated with $-L$ is hyperbolic, that is, there exist a projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P, $N(P)$ is invariant with respect to $T(t)$, $T(t) : R(Q) \longrightarrow R(Q)$ is invertible and

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\| \quad \text{for } t \geq 0, \tag{2.3}$$

$$\|T(t)Qx\| \leq Me^{\delta t}\|x\| \quad \text{for } t \leq 0, \tag{2.4}$$

where $Q := I - P$ and, for $t \leq 0$, $T(t) := (T(-t))^{-1}$.

Recall that if a semigroup $(T(t))_{t\geq 0}$ is analytic, then $(T(t))_{t\geq 0}$ is hyperbolic if and only if

$$\sigma(-L) \cap i\mathbb{R} = \emptyset,$$

see for instance [14, Prop 1.15, p.305].

2.2. **Intermediate Spaces.**

Definition 2.3. Let $\alpha \in (0,1)$. A Banach space $(\mathcal{M}_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(-L)$ $(= D(L))$ and $\mathcal{M}$, or a space of class $\mathcal{J}_\alpha$, if $D(L) \subset \mathcal{M}_\alpha \subset \mathcal{M}$ and there is a constant $C > 0$ such that

$$\|x\|_\alpha \leq C.\|x\|^{1-\alpha}\|x\|_L^\alpha, \quad x \in D(L), \tag{2.5}$$

where $\|\cdot\|_L$ is the graph norm of L.

Concrete examples of $\mathcal{M}_\alpha$ include $D((-L)^\alpha)$ for $\alpha \in (0,1)$, the domains of the fractional powers of $-L$, the real interpolation spaces $D_L(\alpha,\infty)$, $\alpha \in (0,1)$, defined as follows

$$\begin{gathered} D_L(\alpha,\infty) := \{x \in \mathcal{M} : [x]_\alpha = \sup_{0<t\leq 1} \|t^{1-\alpha}(-L - \omega I)e^{-\omega t}T(t)x\| < \infty\} \\ \|x\|_\alpha = \|x\| + [x]_\alpha, \end{gathered}$$

and the abstract Hölder spaces $D_L(\alpha) := \overline{D(L)}^{\|\cdot\|_\alpha}$.

For the hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, we can easily check that estimations similar to (2.3) and (2.4) hold also with norms $\|\cdot\|_\alpha$. In fact, as the part of A in $R(Q)$ is bounded, it follows from the inequality (2.4) that

$$\|AT(t)Qx\| \leq C' e^{\delta t}\|x\| \quad \text{for } t \leq 0.$$

Hence, from (2.5) there exists a constant $C(\alpha) > 0$ such that

$$\|T(t)Qx\|_\alpha \leq C(\alpha) e^{\delta t}\|x\| \quad \text{for } t \leq 0. \tag{2.6}$$

We have also

$$\|T(t)Px\|_\alpha \leq \|T(1)\|_{B(\mathcal{M},\mathcal{M}_\alpha)}\|T(t-1)Px\| \quad \text{for } t \geq 1,$$

and then from (2.3), we obtain

$$\|T(t)Px\|_\alpha \leq M' e^{-\delta t}\|x\|, \quad t \geq 1,$$

where M' depends on α. For $t \in (0,1]$, by (2.2) and (2.5)

$$\|T(t)Px\|_\alpha \leq M'' t^{-\alpha}\|x\|.$$

Hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha) t^{-\alpha} e^{-\gamma t}\|x\| \quad \text{for } t > 0. \tag{2.7}$$

2.3. **Pseudo Almost Periodic Functions.** Let $C(\mathbb{R},\mathcal{M})$ denote the collection of continuous functions from $\mathbb{R}$ into $\mathcal{M}$. Let $(BC(\mathbb{R},\mathcal{M}), \|\cdot\|_\infty)$ denote the Banach space of all $\mathcal{M}$-valued bounded continuous functions equipped with the sup norm defined by $\|u\|_\infty := \sup_{t\in\mathbb{R}}\|u(t)\|$ for each $u \in B(\mathbb{R},\mathcal{M})$. (If $\mathcal{M}_\alpha$ for $\alpha \in (0,1)$ is an intermediate space, then $BC(\mathbb{R},\mathcal{M}_\alpha)$ will be equipped with the α- sup norm: $\|u\|_{\infty,\alpha} = \sup_{t\in\mathbb{R}}\|u(t)\|_\alpha$ for each $u \in BC(\mathbb{R},\mathcal{M}_\alpha)$.) Similarly, $BC(\mathbb{R}\times\Omega)$, where $\Omega \subset \mathcal{M}$ is an open subset, denotes the collection of all bounded continuous functions $G : \mathbb{R}\times\Omega \mapsto \mathcal{M}$.

Let $f \in BC(\mathbb{R},\mathcal{M})$. Define the linear shift operator σ_τ for some $\tau \in \mathbb{R}$ by $(\sigma_\tau f)(t) := f(t+\tau)$ for each $t \in \mathbb{R}$. Similarly, if $G \in B(\mathbb{R}\times\Omega)$, one defines the function $\sigma_\tau G(\cdot,x)$ for each $x \in \Omega$ by $\sigma_\tau G(t,x) := G(t+\tau,x)$ for each $t \in \mathbb{R}$.

Definition 2.4. A function $f \in BC(\mathbb{R},\mathcal{M})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the property: $\|\sigma_\tau f - f\|_\infty < \varepsilon$.

The number τ above is called an ε-*translation* number of f, and the collection of all such functions will be denoted $AP(\mathcal{M})$. The space $(AP(\mathcal{M}), \|\cdot\|_\infty)$ is a Banach space.

Definition 2.5. A function $F \in BC(\mathbb{R}\times\Omega)$ is called almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \Gamma \subset \Omega$ (Γ being a compact subset of Ω), if for each $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that every interval of length $l_\varepsilon > 0$ contains a number τ with the property: $\|\sigma_\tau F(\cdot,x) - F(\cdot,x)\|_\infty < \varepsilon$ for each $x \in \Gamma$.

Here again, the number τ above is called an ε-translation number of F and the class of such functions will be denoted $AP(\mathbb{R}\times\Omega)$.

Throughout the rest of the paper, we suppose $\Omega = \mathcal{M}$ and set

$$AP_0(\mathcal{M}) := \{f \in BC(\mathbb{R},\mathcal{M}) : \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|f(s)\|ds = 0\},$$

and define $AP_0(\mathbb{R} \times \mathcal{M})$ as the collection of functions $F \in BC(\mathbb{R} \times \mathcal{M})$ such that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|F(t,u)\| dt = 0$$

uniformly in $u \in \mathcal{M}$.

Definition 2.6. A function $f \in BC(\mathbb{R}, \mathcal{M})$ is called pseudo almost periodic if it can be expressed as $f = g + \phi$, where $g \in AP(\mathcal{M})$ and $\phi \in AP_0(\mathcal{M})$. The collection of such functions will be denoted by $PAP(\mathcal{M})$.

Remark 2.7.
(1) The functions g and ϕ in Definition 2.6 are respectively called the *almost periodic* and the *ergodic perturbation* components of f.
(2) The decomposition given in Definition 2.6 is unique.

From now on we equip $BC(\mathbb{R}, \mathcal{M})$, $AP(\mathcal{M})$, and $PAP(\mathcal{M})$ with the sup norm $\|\cdot\|_\infty$ previously defined. In view of the above, they all constitute Banach spaces.

Definition 2.8. A function $f \in BC(\mathbb{R} \times \mathcal{M})$ for is called pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathcal{M}$ if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{R} \times \mathcal{M})$ and $\phi \in AP_0(\mathbb{R} \times \mathcal{M})$. The collection of such functions will be denoted by $PAP(\mathbb{R} \times \mathcal{M})$.

3. Main results

To study (1.4) we first study pseudo almost periodic solutions to the inhomogeneous differential equation

$$u'(t) + Lu(t) = h(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $-L$ is sectorial on $\mathcal{M}$ and $h : \mathbb{R} \mapsto \mathcal{M}$ is pseudo almost periodic.

Let $(T(t))_{t\geq 0}$ denote the analytic semigroup defined on $\mathcal{M}$ whose infinitesimal generator is $-L$.

Definition 3.1. A mild solution to (3.1) is a continuous function $u : \mathbb{R} \to \mathcal{M}_\alpha$ satisfying

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\sigma)h(\sigma)\, d\sigma \tag{3.2}$$

for all $t \geq s$ and all $s \in \mathbb{R}$.

Proposition 3.2. *Let $-L$ be a sectorial linear operator on $\mathcal{M}$, which is hyperbolic, i.e., $\sigma(-L) \cap i\mathbb{R} = \{\emptyset\}$. If $h \in PAP(\mathcal{M})$, then (3.1) has a unique $\mathcal{M}_\alpha$-valued pseudo almost periodic mild solution $u(\cdot)$, which can be expressed as*

$$u(t) = \int_{-\infty}^{t} T(t-s)Ph(s)ds - \int_t^{+\infty} T(t-s)Qh(s)ds, \quad t \in \mathbb{R}. \tag{3.3}$$

Proof. It is not hard to see that

$$u(t) = \int_{-\infty}^{t} T(t-s)Ph(s)ds - \int_t^{+\infty} T(t-s)Qh(s)ds, \quad t \in \mathbb{R},$$

is well defined for each $t \in \mathbb{R}$, and satisfies

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\sigma)h(\sigma)\, d\sigma \tag{3.4}$$

for all $t \geq s$ and all $s \in \mathbb{R}$, and hence u is a mild solution to (3.1).

For the uniqueness, let v be another bounded continuous mild solution, i.e., v satisfies (3.4). Then, using both the projection operator P and the estimates (2.6)-(2.7) defined previously, it is not hard to see that

$$\begin{aligned} Pv(t) &= \int_{-\infty}^{t} T(t-s)Ph(s)ds, \quad t \in \mathbb{R}, \\ Qv(t) &= \int_{+\infty}^{t} T(t-s)Qh(s)ds, \quad t \in \mathbb{R}. \end{aligned}$$

Thus from the decomposition of the space $\mathcal{M}$ it follows that $v(t) = u(t)$.

It remains to prove that $u \in PAP(\mathcal{M}_\alpha)$. Since $h \in PAP(\mathcal{M})$, write $h = \phi + \zeta$ where $\phi \in AP(\mathcal{M})$ and $\zeta \in AP_0(\mathcal{M})$. Thus $Pu(t)$ and $Qu(t)$ can be rewritten as

$$\begin{aligned} Pu(t) &= \int_{-\infty}^{t} T(t-s)P\phi(s)ds + \int_{-\infty}^{t} T(t-s)P\zeta(s)ds \\ Qu(t) &= \int_{+\infty}^{t} T(t-s)Q\phi(s)ds + \int_{+\infty}^{t} T(t-s)Q\zeta(s)ds. \end{aligned}$$

Set $\Phi(t) = \int_{-\infty}^{t} T(t-s)P\phi(s)ds$, and $\Psi(t) = \int_{-\infty}^{t} T(t-s)P\zeta(s)ds$ for each $t \in \mathbb{R}$. The next step consists of showing that $\Phi \in AP(\mathcal{M}_\alpha)$ and $\Psi \in AP_0(\mathcal{M}_\alpha)$.

Clearly, $\Phi \in AP(\mathcal{M}_\alpha)$. Indeed, since $\phi \in AP(\mathcal{M})$, therefore for every $\varepsilon > 0$ there exists $\theta_\varepsilon > 0$ such that for all ξ there is $\tau \in [\xi, \xi + \theta_\varepsilon]$ with

$$\|\sigma_\tau \Phi - \Phi\|_\infty < \mu.\varepsilon$$

where $\mu = \frac{\gamma^{1-\alpha}}{M(\alpha)\Gamma(1-\alpha)}$ with Γ being the classical gamma function.

Now

$$\Phi(t+\tau) - \Phi(t) = \int_{-\infty}^{t} T(t-s)P\left(\phi(s+\tau) - \phi(s)\right)ds.$$

Thus, using (2.7) it easily follows that $\|\Phi(t+\tau) - \Phi(t)\|_\alpha < \varepsilon$ for each $t \in \mathbb{R}$, and hence $\|\sigma_\tau \Phi - \Phi\|_{\infty,\alpha} < \varepsilon$, that is, $\Phi \in AP(\mathcal{M}_\alpha)$.

Next, we show that $t \mapsto \Psi(t)$ is in $AP_0(\mathcal{M}_\alpha)$. First, note that $t \mapsto \Psi(t)$ is a bounded continuous function. It remains to show that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\Psi(t)\|_\alpha \, dt = 0.$$

Once again using (2.7) it follows that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\Psi(t)\|_\alpha \, dt \le I + J$$

where

$$I := \lim_{r\to\infty} \frac{M(\alpha)}{2r} \int_{-r}^{r} dt\mathcal{M}ig(\int_{-r}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} \|\zeta(s)\| ds\mathcal{M}ig),$$

and

$$J := \lim_{r\to\infty} \frac{M(\alpha)}{2r} \int_{-r}^{r} dt \int_{-\infty}^{-r} (t-s)^{-\alpha} e^{-\gamma(t-s)} \|\zeta(s)\| ds.$$

Now

$$
\begin{aligned}
I &= \lim_{r\to\infty} \frac{M(\alpha)}{2r} \int_{-r}^{r} \|\zeta(t)\|\, dt \left(\int_{-r}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} ds \right) \\
&= \lim_{r\to\infty} \frac{M(\alpha)}{2r} \int_{-r}^{r} \|\zeta(t)\|\, dt \left(\int_{0}^{t+r} \sigma^{-\alpha} e^{-\gamma\sigma} d\sigma \right) \\
&\le \lim_{r\to\infty} \frac{M(\alpha)}{2r} \int_{-r}^{r} \|\zeta(t)\|\, dt \left(\int_{0}^{+\infty} \sigma^{-\alpha} e^{-\gamma\sigma} d\sigma \right) \\
&= \lim_{r\to\infty} \frac{M(\alpha)\Gamma(1-\alpha)}{2r\gamma^{1-\alpha}} \int_{-r}^{r} \|\zeta(t)\|\, dt \\
&= \frac{M(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}} \,.\, \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\zeta(t)\| dt = 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J &\le \lim_{r\to\infty} \frac{M(\alpha).\|\zeta\|_\infty}{2r} \int_{-r}^{r} dt \int_{t+r}^{+\infty} \sigma^{-\alpha} e^{-\gamma\sigma} d\sigma \\
&\le \lim_{r\to\infty} \frac{M(\alpha).\|\zeta\|_\infty}{2r} \int_{-r}^{r} dt \int_{2r}^{+\infty} \sigma^{-\alpha} e^{-\gamma\sigma} d\sigma \\
&\le \lim_{r\to\infty} \frac{M(\alpha).\|\zeta\|_\infty}{2r} \int_{-r}^{r} dt \int_{2r}^{+\infty} (2r)^{-\alpha} e^{-\gamma\sigma} d\sigma \\
&= \lim_{r\to\infty} \left[\frac{M(\alpha).\|\zeta\|_\infty .e^{-2\gamma r}}{(2r)^{\alpha}.\gamma} \right] = 0,
\end{aligned}
$$

and hence Ψ belongs to $AP_0(\mathcal{M}_\alpha)$.

The proof for $Qu(\cdot)$ is similar to that of $Pu(\cdot)$. However one makes use of (2.6) rather than (2.7). □

Definition 3.3. *A mild solution of* (1.4) *is a continuous function $u : \mathbb{R} \to \mathcal{M}_\alpha$ satisfying the integral equation*

$$
u(t) = T(t-s)u(s) + \int_{s}^{t} T(t-\sigma)\left[f(u(\sigma)) + F(\sigma) + Ku(\sigma) \right] d\sigma \tag{3.5}
$$

for all $t \ge s$ and all $s \in \mathbb{R}$.

To study (1.4) we require the following assumptions:

- (H1) The sectorial operator $-L$ is the generator of a hyperbolic analytic semigroup $(T(t))_{t\ge 0}$;
- (H2) The operator $K : \mathcal{M}_\alpha \mapsto \mathcal{M}_\alpha$ is bounded. We then set $\|K\|_{B(\mathcal{M}_\alpha)} = K_\alpha$.
- (H3) The function $f : PAP(\mathcal{M}_\alpha) \mapsto PAP(\mathcal{M}_\alpha)$ and f is Lipschitz as follows: there exists $R > 0$ such that
$$\|f(u(t)) - f(v(t))\| \le R \,.\, \|u(t) - v(t)\|_\alpha$$
for all $t \in \mathbb{R}$ and $u, v \in PAP(\mathcal{M}_\alpha)$.
- (H4) $F \in PAP(\mathcal{M}_\alpha)$.

Theorem 3.4. *Under the assumptions (H1)-(H2)-(H3)-(H4), the evolution equation* (1.4) *has a unique pseudo almost periodic mild solution whenever*

$$
\Theta_\alpha = (R + K_\alpha) \left[\frac{C(\alpha)}{\delta} + \frac{M(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}} \right] < 1.
$$

Proof. Using similar arguments as in the proof of Proposition 3.2, it can be easily seen that each mild solution u to (1.4) is given by

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} T(t-s)P[f(u(s)) + F(s) + Ku(s)]ds \\ &\quad - \int_{t}^{+\infty} T(t-s)Q[f(u(s)) + F(s) + Ku(s)]ds, \quad t \in \mathbb{R}. \end{aligned} \tag{3.6}$$

Now consider the nonlinear operator on $C(\mathbb{R}, \mathcal{M}_\alpha)$ given by

$$\begin{aligned} \mathbb{D}y(t) &= \int_{-\infty}^{t} T(t-s)P[f(y(s)) + F(s) + Ky(s)]ds \\ &\quad - \int_{t}^{+\infty} T(t-s)Q[f(y(s)) + F(s) + Ky(s)]ds, \quad t \in \mathbb{R}, \end{aligned} \tag{3.7}$$

for each $y \in C(\mathbb{R}, \mathcal{M}_\alpha)$.

From the boundedness of K as an operator of $\mathcal{M}_\alpha$ it is clear that $Ky(\cdot)$ is also pseudo almost periodic on $\mathcal{M}_\alpha$ whenever y does. Under (H.3), if $y \in PAP(\mathcal{M}_\alpha)$, then $f(y(\cdot)) \in PAP(\mathcal{M}_\alpha)$. Considering Proposition 3.2, for $h(s) = f(y(s)) + F(s) + Ky(s)$, it follows that the operator $\mathbb{D}$ maps $PAP(\mathcal{M}_\alpha)$ into itself.

Let $v, w \in PAP(\mathcal{M}_\alpha)$,

$$\begin{aligned} \|\mathbb{D}v(t) - \mathbb{D}w(t)\|_\alpha &\le \| \int_{-\infty}^{t} T(t-s)P\left[f(v(s)) - f(w(s))\right] ds\|_\alpha \\ &\quad + \| \int_{-\infty}^{t} T(t-s)P\left[Kv(s)) - Kw(s))\right] ds\|_\alpha \\ &\quad + \| \int_{t}^{+\infty} T(t-s)Q\left[f(v(s)) - f(w(s))\right] ds\|_\alpha \\ &\quad + \| \int_{t}^{+\infty} T(t-s)Q\left[Kv(s) - Kw(s)\right] ds\|_\alpha. \end{aligned}$$

Using (2.7) and (2.6) it follows that

$$\begin{aligned} \|\mathbb{D}v(t) - \mathbb{D}w(t)\|_\alpha &\le R.M(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\delta(t-s)} \|v(s) - w(s)\|_\alpha ds \\ &\quad + K_\alpha.M(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\delta(t-s)} \|v(s) - w(s)\|_\alpha ds \\ &\quad + R.C(\alpha) \int_{t}^{+\infty} e^{\delta(t-s)} \|v(s) - w(s)\|_\alpha ds \\ &\quad + K_\alpha.C(\alpha) \int_{t}^{+\infty} e^{\delta(t-s)} \|v(s) - w(s)\|_\alpha ds, \end{aligned}$$

and hence $\|\mathbb{D}v - \mathbb{D}w\|_{\infty,\alpha} \le \Theta_\alpha.\|v - w\|_{\infty,\alpha}$, where

$$\Theta_\alpha = (R + K_\alpha)\left[\frac{C(\alpha)}{\delta} + \frac{M(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}}\right].$$

Clearly, if $\Theta_\alpha < 1$, then (1.4) has a unique fixed-point by the Banach fixed point theorem, which obviously is the only pseudo almost periodic (mild) solution to (1.4). $\square$

Example 3.5. To deal with the system (1.2)-(1.3), take $\mathcal{M} := C[0,1]$, equipped with the sup norm. Define the operator $-L$ by

$$-L(\varphi) := \varphi'' + H.\varphi, \quad \forall \varphi \in D(-L),$$

where $D(-L) := \{\varphi \in C^2[0,1], \varphi(0) = \varphi(1) = 0\} \subset C[0,1]$ and $H \in \mathbb{R}$ is a constant.

Clearly $-L$ is sectorial, and hence is the generator of an analytic semigroup. In addition to the above, the resolvent and spectrum of $-L$ are respectively given by

$$\rho(-L) = \mathbb{C} - \{-n^2\pi^2 + H : n \in \mathbb{N}\} \quad \text{and} \quad \sigma(-L) = \{-n^2\pi^2 + H : n \in \mathbb{N}\}$$

so that $\sigma(-L) \cap i\mathbb{R} = \{\emptyset\}$ whenever $H \neq n^2\pi^2$. In particular, if $H = m\pi^2$ where $m \in \mathbb{N} - \{0\}$ and m is not a square, then $-L$ is hyperbolic.

Theorem 3.6. *Under assumptions (H2)–(H4), suppose that the constant $H \neq n^2\pi^2$ for $n \in \mathbb{N}$. Then the heat equation* (1.2)-(1.3) *has a unique $\mathcal{M}_\alpha$-valued pseudo almost periodic mild solution whenever $R + K_\alpha$ is small enough.*

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