ON THE SOLVABILITY OF NONLINEAR FIRST-ORDER BOUNDARY-VALUE PROBLEMS

CHRISTOPHER C. TISDELL

Abstract. This article investigates the existence of solutions to first-order nonlinear boundary-value problems (BVPs) involving systems of ordinary differential equations and two-point boundary conditions. Some sufficient conditions are presented that will ensure solvability. The main tools employed are novel differential inequalities and fixed-point methods.

1. Introduction

This paper considers the existence of solutions to the first-order differential equation

$$x' = f(t, x), \quad t \in [a, c], \tag{1.1}$$

subject to the boundary conditions

$$Mx(a) + Rx(c) = 0, \tag{1.2}$$

where $f : [a, c] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous, nonlinear function; $a < c$ are given constants in $\mathbb{R}$; and M, R are given constants in $\mathbb{R}$.

Equation (1.1) subject to (1.2) is known as a first-order, two-point boundary-value problem (BVP). Two-point BVPs have been studied by many authors, both in recent times [2, 4, 7, 9, 10, 11, 12, 13, 14], and also in a more classical setting [5], [6, Chap VI], [8, Chap V]. A motivating factor for the study of these equations is their application to the areas of science, engineering and technology. The interested reader is referred to [1, Chap.1] for some nice examples. These applications naturally lead to a deeper theoretical analysis of the subject, including: solvability; uniqueness and multiplicity of solutions etc.

In [5], [6, Chap VI], [8, Chap V] Gaines and Mawhin formulated existence results for first-order BVPs by introducing the concepts of curvature bound sets and guiding functions through the formulation of appropriate differential inequalities. These methods ensured *a priori* bounds on possible solutions to a certain family of BVPs. They then coupled these *a priori* bound techniques with coincidence topological degree theory to guarantee the existence of solutions.

In this article an alternate approach is proposed. In particular, the ideas of curvature bound sets and guiding functions are not used. Instead, a general growth

---

condition on f is introduced. Novel inequalities involving f, M and R are formulated so all possible solutions to a certain family of BVPs are bounded *a priori*. Fixed-point methods are applied, ensuring the existence of solutions to (1.1), (1.2). Examples to illustrate the new results are presented throughout the paper. The results contained herein are new for both systems of BVPs and also for scalar BVPs.

A solution to (1.1), (1.2) is a continuously differentiable function $x : [a, c] \to \mathbb{R}^n$ (denoted by $x \in C^1([a, c]; \mathbb{R}^n)$) that satisfies (1.1) and (1.2).

A topological tool used in this paper is the following simplified version of the Nonlinear Alternative [3, Chap.4]. We say a map is compact if it is continuous with relatively compact range. Also, let J denote a convex subset of a normed, linear space E and let B_P denote an open ball in J with radius $P > 0$ and centre 0.

Theorem 1.1. *Let* $T : \overline{B_P} \to J$ *be a compact map and let* $\lambda \in [0, 1]$. *If*
$$u \neq \lambda T u, \quad \text{for all } u \in \partial B_P \text{ and all } \lambda \in (0, 1)$$
then there exists at least one $u \in B_P$ *such that* $u = Tu$.

2. Solvability

In this main section, the solvability of (1.1), (1.2) is established.

In what follows, if $y, z \in \mathbb{R}^n$ then $\langle y, z \rangle$ denotes the usual inner product and $\|z\|$ denotes the Euclidean norm of z on $\mathbb{R}^n$.

Throughout this work, assume
$$M + R \neq 0. \tag{2.1}$$

Lemma 2.1. *Suppose* (2.1) *holds. The BVP* (1.1), (1.2) *is equivalent to the integral equation*
$$x(t) = \int_a^t f(s, x(s)) \, ds - (M + R)^{-1} R \int_a^c f(s, x(s)) \, ds, \quad t \in [a, c]. \tag{2.2}$$

Proof. Let $x : [a, c] \to \mathbb{R}^n$ satisfy (1.1) and (1.2). It is easy to see that
$$x(t) = x(a) + \int_a^t f(s, x(s)) \, ds, \quad t \in [a, c], \tag{2.3}$$
and
$$x(c) = x(a) + \int_a^c f(s, x(s)) \, ds. \tag{2.4}$$
So (1.2) gives
$$0 = Mx(a) + R\Big(x(a) + \int_a^c f(s, x(s)) \, ds \Big) \tag{2.5}$$
and rearranging (2.5) we obtain
$$x(a) = -(M + R)^{-1} R \int_a^c f(s, x(s)) \, ds. \tag{2.6}$$
So substituting (2.6) into (2.3) we obtain, for $t \in [a, c]$,
$$x(t) = -(M + R)^{-1} R \int_a^c f(s, x(s)) \, ds + \int_a^t f(s, x(s)) \, ds. \tag{2.7}$$
If x is a solution to (2.2) then is it easy to show that (1.1) and (1.2) hold by direct calculation. $\square$

The two following two existence theorems are the main results of the paper.

Theorem 2.2. *Suppose* (2.1) *holds and* $f \in C([a,c] \times \mathbb{R}^n; \mathbb{R}^n)$. *If there exist non-negative constants* α *and* K *such that*

$$\|f(t,q)\| \leq 2\alpha\langle q, f(t,q)\rangle + K, \quad \text{for all } (t,q) \in [a,c] \times \mathbb{R}^n, \tag{2.8}$$

$$\text{and } |M/R| \leq 1, \tag{2.9}$$

then the BVP (1.1), (1.2) *has at least one solution.*

Proof. In view of Lemma 2.1, we want to show that there exists at least one solution to (2.2), which is equivalent to showing that (1.1), (1.2) has at least one solution. To do this, we use the Nonlinear Alternative.

Consider the map $T : C([a,c]; \mathbb{R}^n) \to C([a,c]; \mathbb{R}^n)$ defined for all $t \in [a,c]$ by

$$Tx(t) = -(M+R)^{-1} R \int_a^c f(s, x(s))\, ds + \int_a^t f(s, x(s))\, ds. \tag{2.10}$$

Thus our problem is reduced to proving the existence of at least one v such that

$$v = Tv. \tag{2.11}$$

Since f is continuous, see that T is also a continuous map. Next we show that $T : \overline{B_P} \to C([a,c]; \mathbb{R}^n)$ satisfies

$$x \neq \lambda Tx, \quad \text{for all } x \in \partial B_P \text{ and all } \lambda \in (0,1) \tag{2.12}$$

for some suitable ball $B_P \subset C([a,c]; \mathbb{R}^n)$ with radius $P > 0$. Let

$$B_P = \big\{ x \in C([a,c]; \mathbb{R}^n) \mid \max_{t \in [a,c]} \|x(t)\| < P \big\}$$

$$P = \big[1 + |(M+R)^{-1} R| \big] K(c-a) + 1.$$

See that the family $x = \lambda Tx$ is equivalent to the family of BVPs

$$x' = \lambda f(t,x), \quad t \in [a,c],\ \lambda \in [0,1], \tag{2.13}$$

$$Mx(a) + Rx(c) = 0. \tag{2.14}$$

Let x be a solution to the family of BVPs (2.13), (2.14). Consider $r(t) := \|x(t)\|^2$ for all $t \in [a,c]$. By the product rule we have

$$\begin{aligned} r'(t) &= 2\langle x(t), x'(t)\rangle, \quad t \in [a,c], \\ &= 2\langle x(t), \lambda f(t, x(t))\rangle. \end{aligned} \tag{2.15}$$

Multiplying both sides of (2.8) by $\lambda \in [0,1]$, with $q = x(t)$, we obtain

$$\|\lambda f(t,q)\| \leq 2\alpha\langle q, \lambda f(t,q)\rangle + \lambda K \tag{2.16}$$

$$\leq 2\alpha\langle q, \lambda f(t,q)\rangle + K \tag{2.17}$$

$$= \alpha r'(t) + K. \tag{2.18}$$

Also, (2.9) implies

$$r(c) \leq r(a) \tag{2.19}$$

since (2.14) gives

$$\|x(c)\| \leq |M/R|\, \|x(a)\| \leq \|x(a)\|.$$

Let $H := 1 + |(M + R)^{-1}R|$. All solutions to $x = \lambda T x$ must satisfy:

$$\|x(t)\| = \|\lambda T x(t)\|$$

$$= \| - (M + R)^{-1}R \int_a^c \lambda f(s, x(s))\, ds + \int_a^t \lambda f(s, x(s))\, ds\|$$

$$\leq H \int_a^c \|\lambda f(s, x(s))\|\, ds$$

$$\leq H \int_a^c \left(2\alpha\langle x(s), \lambda f(s, x(s))\rangle + K\right)\, ds, \quad \text{by (2.17)}$$

$$\leq H \int_a^c [\alpha r'(s) + K]\, ds, \quad \text{from (2.18)}$$

$$= H\left[\alpha(r(c) - r(a)) + K(c - a)\right]$$

$$\leq H\left[K(c - a)\right], \quad \text{from (2.19)}$$

$$< P.$$

Thus, (2.12) holds.

The operator $T : \overline{B_P} \to C([a, c]; \mathbb{R}^n)$ is compact by the Arzela-Ascoli theorem (because it is a completely continuous map restricted to a closed ball).

The Nonlinear Alternative ensures the existence of at least one solution in B_P to (2.2) and hence (1.1), (1.2) admits at least once solution. By an elementary compactness argument involving a suitable sequence of solutions, this solution is also in $C^1([a, c]; \mathbb{R}^n)$. □

Theorem 2.3. *Suppose* (2.1) *holds and* $f \in C([a, c] \times \mathbb{R}^n; \mathbb{R}^n)$. *If there exist non-negative constants* α *and* K *such that*

$$\|f(t, q)\| \leq -2\alpha\langle q, f(t, q)\rangle + K, \quad \text{for all } (t, q) \in [a, c] \times \mathbb{R}^n, \tag{2.20}$$

$$\text{and } |R/M| \leq 1, \tag{2.21}$$

then the BVP (1.1), (1.2) *has at least one solution.*

Proof. The proof follows similar steps to that of Theorem 2.2 and so only the essential points are mentioned. Follow the proof of Theorem 2.2, just replace "α" with "$-\alpha$" and use (2.21) to show $-\alpha(r(c) - r(a)) \leq 0$. □

Although (2.8) and (2.20) appear to be similar, the significant difference between Theorems 2.2 and 2.3 lie in their varied applicability to BVPs. Theorem 2.2 may apply to certain BVPs where Theorem 2.3 may not, and vice-versa. For example, it is not difficult to show that

$$f(t, p) = -p^3 - te^{-p}, \quad t \in [0, 1], \tag{2.22}$$

satisfies (2.20) (the choices $\alpha = 1/2$ and $K = 10$ will suffice), but no non-negative α and K can be found such that (2.22) satisfies (2.8).

If $M = 1 = N$, then the boundary conditions (1.2) become the so-called antiperiodic boundary conditions

$$x(a) = -x(c) \tag{2.23}$$

and the following corollaries to Theorems 2.2 and 2.3 follow.

Corollary 2.4. *Let $f \in C([a,c] \times \mathbb{R}^n; \mathbb{R}^n)$. If there exist non-negative constants α and K such that (2.8) holds then the anti-periodic BVP (1.1), (2.23) has at least one solution.*

Proof. It is easy to see that for $M = 1 = R$, all of the conditions of Theorem 2.2 hold. Thus the result follows from Theorem 2.2. $\qquad\square$

Corollary 2.5. *Let $f \in C([a,c] \times \mathbb{R}^n; \mathbb{R}^n)$. If there exist non-negative constants α and K such that (2.20) holds then the anti-periodic BVP (1.1), (2.23) has at least one solution.*

Proof. The result follows from Theorem 2.3. $\qquad\square$

Now consider (1.1), (1.2) with $n = 1$. For this case, the following new corollaries to Theorem 2.2 and 2.3 are obtained.

Corollary 2.6. *Let $M + R \neq 0$, let $f \in C([a,c] \times \mathbb{R}; \mathbb{R})$ and let α, K be non-negative constants such that*

$$|f(t,q)| \leq 2\alpha q f(t,q) + K, \quad \text{for all } (t,q) \in [a,c] \times \mathbb{R}, \tag{2.24}$$
$$\text{and } |M/R| \leq 1. \tag{2.25}$$

Then, for $n = 1$, the BVP (1.1), (1.2) has at least one solution.

Proof. It is easy to see that for $n = 1$: (2.8) becomes (2.24); and the result follows from Theorem 2.2. $\qquad\square$

Corollary 2.7. *Let $M + R \neq 0$, let $f \in C([a,c] \times \mathbb{R}; \mathbb{R})$ and let α, K be non-negative constants such that*

$$|f(t,q)| \leq -2\alpha q f(t,q) + K, \quad \text{for all } (t,q) \in [a,c] \times \mathbb{R}, \tag{2.26}$$
$$\text{and } |R/M| \leq 1. \tag{2.27}$$

Then, for $n = 1$, the BVP (1.1), (1.2) has at least one solution.

Proof. The result follows from Theorem 2.3. $\qquad\square$

Some examples are now presented to highlight the newly established theory.

Example 2.8. Consider the scalar BVP ($n = 1$) given by

$$x' = t\left([x(t)]^3 + 1\right), \quad t \in [0,1], \tag{2.28}$$
$$x(0) + x(1) = 0. \tag{2.29}$$

Let $f(t,q) = t[q^3 + 1]$. For α and K to be chosen below, see that

$$2\alpha q f(t,q) + K = 2\alpha t(q^4 + q) + K$$
$$= t(q^4 + q) + 3, \quad \text{for the choice } \alpha = 1/2, K = 3$$
$$\geq t|q^3 + 1| = |f(t,q)|, \quad \text{for all } (t,q) \in [0,1] \times \mathbb{R}$$

and thus (2.24) holds for the choices $\alpha = 1/2$ and $K = 3$. Since $M = 1 = R$, it is easy to see that $M + R \neq 0$ and (2.25) holds. Thus, all of the conditions of Corollary 2.6 hold and so the BVP (2.28), (2.29) has at least one solution.

Attention now turns to systems in the following example.

Example 2.9. Consider (1.1), (1.2) with $n = 2$ and f given by

$$f(t, p) = (h(t, y, z), j(t, y, z)), \quad t \in [0, 1],$$
$$= ((t + 1)y^3 + ye^{-z^2} + 1, (t + 1)z^3 + ze^{-y^2}). \tag{2.30}$$

The above function f satisfies the conditions of Theorem 2.2. Note that for all $(t, p) \in [0, 1] \times \mathbb{R}^2$ we have

$$\|f(t, p)\| \leq |h(t, y, z)| + |j(t, y, z)|$$
$$\leq 2|y|^3 + |y|e^{-z^2} + 2|z|^3 + |z|e^{-y^2} + 1.$$

Below, we will need the following simple inequalities:

$$w^4 \geq |w|^3 - 1, \quad w^4 + w \geq |w|^3 - 10, \quad \text{for all } w \in \mathbb{R};$$
$$b^2 e^{-a^2} \geq |b|e^{-a^2} - 1, \quad \text{for all } (a, b) \in \mathbb{R}^2.$$

For $\alpha \geq 0$ and $K \geq 0$ to be chosen below, consider for $(t, p) \in [0, 1] \times \mathbb{R}^2$,

$$2\alpha \langle p, f(t, p) \rangle + K \geq 2\alpha \left[y^4 + y + y^2 e^{-z^2} + z^4 + z^2 e^{-y^2} \right] + K$$
$$\geq 2\alpha \left[|y|^3 - 10 + |y|e^{-z^2} - 1 + |z|^3 - 1 + |z|e^{-y^2} - 1 \right] + K$$
$$\geq 2|y|^3 + |y|e^{-z^2} + 2|z|^3 + |z|e^{-y^2} + 1$$
$$\geq \|f(t, p)\|$$

choosing $\alpha = 1$, $K = 27$. Thus f satisfies the conditions of Theorem 2.2 for the choices $\alpha = 1$ and $K = 27$ and the solvability of the BVP associated with this example may be obtained.

The conditions of Theorems 2.2 and 2.3 are suitably generalised in the following new theorems. For differentiable functions $V : \mathbb{R}^n \to \mathbb{R}$, let

$$\text{grad } V(x) := (\partial V / \partial x_1, \ldots, \partial V / \partial x_n).$$

Theorem 2.10. *Suppose (2.1) holds and $f \in C([a, c] \times \mathbb{R}^n; \mathbb{R}^n)$. If there exists a C^1 function $V : \mathbb{R}^n \to [0, \infty)$ and non-negative constants α, K such that*

$$\|f(t, q)\| \leq \alpha \langle \text{grad } V(q), f(t, q) \rangle + K, \quad \text{for all } (t, q) \in [a, c] \times \mathbb{R}^n, \tag{2.31}$$

and the boundary conditions (1.2) are such that

$$V(x(a)) \geq V(x(c)) \tag{2.32}$$

then the BVP (1.1), (1.2) has at least one solution.

Proof. The proof is similar to that of Theorem 2.2 and so is only briefly discussed.

Consider the family of BVPs (2.13), (2.14). Let x be a solution and consider $r_1(t) := V(x(t))$ for all $t \in [a, c]$. Multiplying both sides of (2.31) by $\lambda \in [0, 1]$ obtain

$$\|\lambda f(t, q)\| \leq \alpha \langle \text{grad } V(q), \lambda f(t, q) \rangle + \lambda K$$
$$\leq \alpha \langle \text{grad } V(q), \lambda f(t, q) \rangle + K \tag{2.33}$$
$$= \alpha r_1'(t) + K.$$

Let the ball B_P, operator T and constant H all be defined as in the proof of Theorem 2.2. If x is a solution to $x = \lambda T x$ then for all $t \in [a, c]$ we have:

$$\|x(t)\| = \|\lambda T x(t)\|$$

$$\leq H \int_a^c \|\lambda f(s, x(s))\| \, ds$$

$$\leq H \int_a^c [\alpha r_1'(s) + K] \, ds, \quad \text{by (2.33)}$$

$$= H [\alpha(V(x(c)) - V(x(a))) + K(c - a)]$$

$$\leq H [K(c - a)], \quad \text{from (2.32)}$$

$$< P.$$

and thus (2.12) holds. The Nonlinear Alternative applies to (2.10), yielding the existence of a least one solution to (1.1), (1.2). $\square$

Theorem 2.11. *Suppose* (2.1) *holds and* $f \in C([a, c] \times \mathbb{R}^n; \mathbb{R}^n)$. *If there exists a* C^1 *function* $V : \mathbb{R}^n \to [0, \infty)$ *and non-negative constants* α, K *such that*

$$\|f(t, q)\| \leq -\alpha \langle \operatorname{grad} V(q), f(t, q) \rangle + K, \quad \text{for all } (t, q) \in [a, c] \times \mathbb{R}^n, \qquad (2.34)$$

and the boundary conditions (1.2) *are such that*

$$V(x(a)) \leq V(x(c)) \qquad (2.35)$$

then the BVP (1.1), (1.2) *has at least one solution.*

Proof. The steps of the proof are very similar to those of the proof of Theorem 2.10 and so are omitted. $\square$

Remark 2.12. For $V(q) = \|q\|^2$, conditions (2.31) and (2.32) in Theorem 2.10 reduce to (2.8) and (2.9), respectively, in Theorem 2.2. Theorem 2.2 may be viewed as more concrete than Theorem 2.10 as (2.8) and (2.9) are easily verifiable in practice. On the other hand, Theorem 2.10 is certainly more general, in a abstract sense, than Theorem 2.2. A possible candidate for V in Theorem 2.10 is $V(q) = e^q$, which leads to new differential inequalities and existence results for scalar BVPs.

Remark 2.13. It is noted that the theorems of this paper easily generalize to the case where M and R are $n \times n$ constant matrices, rather than real-valued constants in (1.2). Simply replace (2.1) throughout with "$\det(M + R) \neq 0$" and replace, respectively, (2.9) and (2.21) with "$\|R^{-1}M\| \leq 1$" and "$\|M^{-1}R\| \leq 1$" where we have imposed the natural norm of a matrix, rather than absolute values.

References

[1] U. M. Ascher, R. M. M. Mattheij and R. D. Russell; *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, SIAM Classics in Applied Mathematics, 13, 1995.

[2] J. Chen, *On the existence of solutions for PBVP for first order differential equations*, Acta Math. Sci. Ser. A Chin. Ed., 23 (2003), 129–134.

[3] J. Dugundji and A. Granas, *Fixed point theory, Springer Monographs in Mathematics.* Springer-Verlag, New York, 2003.

[4] D. Franco, J. J. Nieto and D. O'Regan; *Anti-periodic boundary value problem for nonlinear first-order differential equations*, Math. Inequal. Appl., 6 (2003), 477–485.

[5] R. E. Gaines, J. L. Mawhin; *Coincidence degree, and nonlinear differential equations.* Lecture Notes in Mathematics, Vol. 568. Springer-Verlag, Berlin-New York, 1977.

[6] R. E. Gaines, J. Mawhin; *Ordinary differential equations with nonlinear boundary conditions.* J. Differential Equations, 26 (1977), no. 2, 200–222.

[7] S Hong, *Boundary-value problems for first and second order functional differential inclusions.* Electron. J. Differential Equations 2003 (2003), No. 32, 10 pp. (electronic).

[8] J. Mawhin, *Topological degree methods in nonlinear boundary value problems.* Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977. CBMS Regional Conference Series in Mathematics, 40. American Mathematical Society, Providence, R.I., 1979.

[9] F. Obersnel and P. Omari, *Old and new results for first order periodic ODEs without uniqueness: a comprehensive study by lower and upper solutions*, Adv. Nonlinear Stud., 4 (2004), 323–376.

[10] S. Peng, *Positive solutions for first order periodic boundary value problem*, Appl. Math. Comput., 158 (2004), 345–351.

[11] C. C. Tisdell, *Existence of solutions to first-order periodic boundary value problems*, J. Math. Anal. Appl., (in press).

[12] Z Wan, Y Chen and J Chen; *Remarks on the periodic boundary value problems for first-order differential equations*, Comput. Math. Appl., 37 (1999), no. 8, 49–55.

[13] M. N. Yakovlev, *Solvability of a periodic boundary value problem for a system of first-order ordinary differential equations with (β, γ, δ)-comparison pairs*, J. Math. Sci. (New York), 101 (2000), 3365–3371.

[14] F. Zhang and Z. Ma, *Nonlinear boundary value problems for first order differential equations with piecewise constant arguments*, Ann. Differential Equations, 19 (2003), 431–438.

Christopher C. Tisdell

School of Mathematics, The University of New South Wales, UNSW Sydney 2052, Australia

E-mail address: cct@maths.unsw.edu.au