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EXPONENTIAL DECAY FOR THE SEMILINEAR WAVE EQUATION WITH SOURCE TERMS

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Abstract. In this paper, we prove that for a semilinear wave equation with source terms, the energy decays exponentially as time approaches infinity. For this end we use the the multiplier method.

1. Introduction

Main results. Let Ω be a bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We are concerned with the mixed problems

$$u_{tt} - \Delta u + \delta u_t = |u|^{p-1}u, \quad x \in \Omega, \quad t \geq 0, \tag{1.1}$$

$$u(0,x) = u_0(x) \in H_0^1(\Omega), \quad u_t(0,x) = u_1(x) \in L^2(\Omega), \quad x \in \Omega, \tag{1.2}$$

$$u(t,x)|_{\partial\Omega} = 0, \quad \text{for } t \geq 0. \tag{1.3}$$

Here $\delta > 0$ and $1 < p \leq \frac{n}{n-2}$ $(n \geq 3)$, $1 < p$ $(n = 1, 2)$. Set

$$I(u) := \int_\Omega (|\nabla u|^2 - |u|^{p+1})dx, \tag{1.4}$$

$$J(u) := \int_\Omega (\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1})dx, \tag{1.5}$$

$$E(t) := \frac{1}{2}\int_\Omega |u_t|^2 dx + J(u) \quad . \tag{1.6}$$

Also let the Nehari manifold

$$N := \{u \in H_0^1(\Omega) : I(u) = 0, \ u \neq 0\}; \tag{1.7}$$

and the potential depth

$$d := \inf_{u \in N} J(u). \tag{1.8}$$

For problem (1.1)-(1.3), Ikehata and Suzuki [1] have shown the following results:

$$d > 0; \tag{1.9}$$

$$E(t) + \int_0^t \int_\Omega \delta |u_t|^2 dx\, dt = E(0); \tag{1.10}$$

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If $E(0) < d$ and $I(u(0,x)) > 0$ then we have

$$E(t) < d \quad \text{and} \quad I(u(t,x)) > 0, \quad \forall t \in [0,\infty); \tag{1.11}$$

$$\theta \int_\Omega |\nabla u|^2 dx \geq \int_\Omega |u|^{p+1} dx, \quad \theta \in (0,1),\ \forall t \in [0,\infty); \tag{1.12}$$

$$\lim_{t\to+\infty} \int_\Omega (|u_t|^2 + |\nabla u|^2) dx = 0; \tag{1.13}$$

$$\int_0^t \int_\Omega |\nabla u|^2 dx\, dt \leq C. \tag{1.14}$$

In this paper we will use the multiplier technique to prove the following result.

Theorem 1.1. *If $E(0) < d$ and $I(u(0,x)) > 0$, then there exists positive constant γ and $C > 1$ such that*

$$E(t) \leq Ce^{-\gamma t}, \quad \forall t \in [0,\infty). \tag{1.15}$$

Our results and their relationship to the literature. The Problem

$$\begin{gathered} u_{tt} - \Delta u + a(x)|u_t|^{m-1}u_t + |u|^{p-1}u = 0, \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad (u,u_t)\big|_{t=0} = (u_0,u_1) \end{gathered} \tag{1.16}$$

has been studied, among others, by Nakao [2, 3] and Zuazua [4]. In [2, 3, 4], the authors assumed that $a(x) \geq 0$ in Ω, $\inf a(x) > 0$ in $\Omega_0 \subset\subset \Omega$ and $m = 1$. The case $m > 1$ is still open [4].

The following problem, with $m > 1$ and $a(x) \geq a_0 > 0$ in $\bar\Omega$,

$$\begin{gathered} u_{tt} - \Delta u + a(x)|u_t|^{m-1}u_t = |u|^{p-1}u, \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad (u,u_t)\big|_{t=0} = (u_0,u_1) \end{gathered} \tag{1.17}$$

has been studied by many authors, Ball [5], Ikehata [6], Ikehata and Tanizawa[7], Levine [8, 9], Georgiev and Todorova [10], Georgiev and Milani [11], Todorova [12], Barbu, et al [13], Todorova and Vitillaro [14], Messaoudi [15], Serrin [16], Kawashima, et al [17]. Ball [5] proved the existence of a global attractor when $m = 1$. In [6, 7, 14, 17], the authors obtained a time-decay result when $\Omega = \mathbb{R}^N$. In [8, 9, 10, 11, 12, 13, 15, 16], the authors mainly concerned the existence or nonexistence of global weak (or strong) solutions.

By the multiplier method in [18], Benaissa and Mimouni [19] studied very recently the decay properties of the solutions to the wave equation of p-Laplacian type with a weak nonlinear dissipative.

Here it should be noted that our main result Theorem 1.1 is also true for the locally damping case i.e., $\delta = \delta(x) \geq 0$ in Ω and $\delta(x) \geq \delta_0 > 0$ in $\Omega_0 \subset\subset \Omega$. We did not find references for the case with boundary damping term.

2. Proof of the Main Result

Take $x_0 \in R^n$ and set $m(x) := x - x_0$. Let ν denote the outward normal vector to $\partial\Omega$. Set

$$\Gamma(x_0) := \{x \in \partial\Omega : (x - x_0)\cdot\nu > 0\},$$

$$\chi := \int_\Omega \big(u_t(m\cdot\nabla u) + \frac{n}{p+1}u(u_t + \frac{\delta}{2}u)\big)dx\big|_0^T.$$

Lemma 2.1. *There exists positive constant C depending only on n, p, δ, Ω such that*

$$\int_0^T E(t)dt \le C\Big\{\int_0^T \int_{\Gamma(x_0)} (m\cdot\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt + \int_0^T\int_\Omega |u_t|^2 dx\,dt + |\chi|\Big\}. \tag{2.1}$$

Proof. Multiplying (1.1) by $q(x)\cdot\nabla u$ and integrating by parts gives, [4, 20],

$$\begin{aligned}
&\Big(\int_\Omega u_t(q\cdot\nabla u)dx\Big)\Big|_0^T + \frac{1}{2}\int_0^T\int_\Omega (\mathrm{div} q)(|u_t|^2 - |\nabla u|^2)dx\,dt \\
&+ \int_0^T\int_\Omega (\sum_{k,j=1}^n \frac{\partial q_k}{\partial x_j}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_j})dx\,dt + \int_0^T\int_\Omega (\mathrm{div} q)\frac{|u|^{p+1}}{p+1}dx\,dt \\
&+ \int_0^T\int_\Omega \delta u_t(q\cdot\nabla u)dx\,dt \\
&= \frac{1}{2}\int_0^T\int_{\partial\Omega}(q\cdot\nu)|\frac{\partial u}{\partial\nu}|^2 d\Gamma\,dt.
\end{aligned} \tag{2.2}$$

Here $q(x)\in W^{1,\infty}(\Omega)$. Applying identity (2.2) with $q(x) = m(x)$, we deduce

$$\begin{aligned}
&\Big(\int_\Omega u_t(m\cdot\nabla u)dx\Big)\Big|_0^T + \frac{n}{2}\int_0^T\int_\Omega (|u_t|^2 - |\nabla u|^2)dx\,dt + \int_0^T\int_\Omega |\nabla u|^2 dx\,dt \\
&+ \frac{n}{p+1}\int_0^T\int_\Omega |u|^{p+1}dx\,dt + \int_0^T\int_\Omega \delta u_t(m\cdot\nabla u)dx\,dt \\
&= \frac{1}{2}\int_0^T\int_{\partial\Omega}(m\cdot\nu)|\frac{\partial u}{\partial\nu}|^2 d\Gamma dt \\
&\le \frac{1}{2}\int_0^T\int_{\Gamma(x_0)}(m\cdot\nu)|\frac{\partial u}{\partial\nu}|^2 d\Gamma dt\,.
\end{aligned} \tag{2.3}$$

We now multiply (1.1) by u and integrate by parts, then we have

$$\Big(\int_\Omega u(u_t + \frac{\delta}{2}u)dx\Big)\Big|_0^T = \int_0^T\int_\Omega (|u_t|^2 - |\nabla u|^2)dx\,dt + \int_0^T\int_\Omega |u|^{p+1}dx\,dt. \tag{2.4}$$

Combining (2.3) and (2.4) we obtain

$$\begin{aligned}
&\chi + (\frac{n}{2} - \frac{n}{p+1})\int_0^T\int_\Omega |u_t|^2 dx\,dt + (1 + \frac{n}{p+1} - \frac{n}{2})\int_0^T\int_\Omega |\nabla u|^2 dx\,dt \\
&+ \int_0^T\int_\Omega \delta u_t(m\cdot\nabla u)dx\,dt \\
&\le \frac{1}{2}\int_0^T\int_{\Gamma(x_0)}(m\cdot\nu)|\frac{\partial u}{\partial\nu}|^2 d\Gamma dt.
\end{aligned} \tag{2.5}$$

On the other hand, for any given $\varepsilon > 0$,

$$\begin{aligned}
&\Big|\int_0^T\int_\Omega \delta u_t(m\cdot\nabla u)dx\,dt\Big| \\
&\le \varepsilon\|m\|^2_{L^\infty(\Omega)}\int_0^T\int_\Omega |\nabla u|^2 dx\,dt + \frac{\delta^2}{2\varepsilon}\int_0^T\int_\Omega |u_t|^2 dx\,dt.
\end{aligned} \tag{2.6}$$

Taking ε sufficiently small in (2.6), then substituting (2.6) into (2.5) we obtain (2.1). ∎

Lemma 2.2. *With the above notation,*

$$E(t) \le C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) dx\, dt. \tag{2.7}$$

Proof. First, we construct a function $h(x) \in W^{1,\infty}(\Omega)$ such that $h(x) = \nu$ on $\Gamma(x_0)$; $h(x) \cdot \nu > 0$ a.e in $\partial\Omega$; see[4]. Applying (2.2) with $q(x) = h(x)$, we have

$$\begin{aligned} \int_0^T \int_{\Gamma(x_0)} |\frac{\partial u}{\partial \nu}|^2 d\Gamma dt &\le \int_0^T \int_{\partial\Omega} (h \cdot \nu) |\frac{\partial u}{\partial \nu}|^2 d\Gamma dt \\ &\le C \int_0^T \int_\Omega (|u_t|^2 + |\nabla u|^2) dx\, dt + 2\Big(\int_\Omega u_t (h \cdot \nabla u) dx \Big)\Big|_0^T. \end{aligned} \tag{2.8}$$

From (2.4), we see that

$$\int_0^T \int_{\Gamma(x_0)} |\frac{\partial u}{\partial \nu}|^2 d\Gamma dt \le C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) dx\, dt + Y, \tag{2.9}$$

where

$$Y = \Big(\int_\Omega u(u_t + \frac{\delta}{2} u) dx \Big)\Big|_0^T + 2\Big(\int_\Omega u_t (h \cdot \nabla u) dx \Big)\Big|_0^T.$$

Combining (2.1), (2.9) and (1.10) we obtain

$$\begin{aligned} TE(T) &\le \int_0^T E(t) dt \\ &\le C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) dx\, dt + |\chi| + |Y| \\ &\le C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) dx\, dt + C(E(0) + E(T)) \\ &\le C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) dx\, dt + C\Big(2E(T) + \delta \int_0^T \int_\Omega |u_t|^2 dx\, dt \Big). \end{aligned} \tag{2.10}$$

Taking T sufficiently large we get (2.7). □

Lemma 2.3.

$$\int_0^T \int_\Omega |u|^{p+1} dx\, dt \le C \int_0^T \int_\Omega |u_t|^2 dx\, dt. \tag{2.11}$$

Proof. We argue by contradiction. If (2.11) is not satisfied for some $C > 0$, then there exists a sequence of solutions $\{u_n\}$ of (1.1)-(1.3) with

$$\lim_{n\to\infty} \frac{\int_0^T \int_\Omega |u_n|^{p+1} dx\, dt}{\int_0^T \int_\Omega |u_{nt}|^2 dx\, dt} = \infty. \tag{2.12}$$

From (1.12) and (1.14) we have

$$\int_0^T \int_\Omega |u_n|^{p+1} dx\, dt \le \theta \int_0^T \int_\Omega |\nabla u_n|^2 dx\, dt \le C. \tag{2.13}$$

Thus we get

$$\lim_{n\to\infty} \int_0^T \int_\Omega |u_{nt}|^2 dx\, dt = 0. \tag{2.14}$$

We extract a subsequence (still denote by $\{u_n\}$) such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\Omega \times (0,T)), \tag{2.15}$$

$$u_n \to u \quad \text{strongly in } L^2(\Omega \times (0,T)), \tag{2.16}$$

$$u_n \to u \quad \text{a.e. in } \Omega \times (0,T), \tag{2.17}$$

$$|u_n|^{p-1}u_n \to |u|^{p-1}u \quad \text{strongly in } L^\infty(0,T;L^r(\Omega)) \tag{2.18}$$

where $r \in [1, \frac{2n}{p(n-2)})$ if $n \geq 3$ and $r \in [1,\infty)$ if $n = 1,2$. From (2.14) we know that

$$u_t = 0, \quad \text{a.e. in } \Omega \times (0,T) \tag{2.19}$$

and so we have

$$-\Delta u = |u|^{p-1}u, \quad \text{in } \Omega \times (0,T) \tag{2.20}$$

$$u = 0, \quad \text{on } \partial\Omega \times (0,T). \tag{2.21}$$

From (2.13) we get

$$\int_0^T \int_\Omega |u|^{p+1} dx\, dt \leq \theta \int_0^T \int_\Omega |\nabla u|^2 dx\, dt < \int_0^T \int_\Omega |\nabla u|^2 dx\, dt \tag{2.22}$$

which contradicts (2.20) and (2.21). This proves (2.11). □

By Lemmas 2.2 and 2.3, we obtain

$$E(T) \leq C \int_0^T \int_\Omega |u_t|^2 dx\, dt. \tag{2.23}$$

This inequality, (1.10), and semigroup properties complete the proof of Theorem 1.1. For properties of semigroups, we refer the reader to [21].

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