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EXISTENCE AND UNIQUENESS RESULTS OF POSITIVE SOLUTIONS FOR NONVARIATIONAL QUASILINEAR ELLIPTIC SYSTEMS

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Abstract. We provide conditions for the existence and uniqueness of positive solutions to the quasilinear elliptic system

$$\begin{aligned} -\Delta_p u &= f(x,u,v) \\ -\Delta_q v &= g(x,u,v) \end{aligned}$$

with Dirichlet boundary conditions on a bounded domain $\Omega \subseteq \mathbb{R}^N$.

1. Introduction

The aim of this paper is to provide existence and uniqueness results for positive solutions of the following weakly coupled quasilinear eliptic system with homogeneous Dirichlet data

$$\begin{gathered} -\Delta_p u = f(x,u,v) \quad \text{in } \Omega \\ -\Delta_q v = g(x,u,v) \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $f, g : \overline{\Omega} \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions. As usual, for $s > 1$,

$$-\Delta_s u := \operatorname{div}(|\nabla u|^{s-2}\nabla u)$$

denotes the s-Laplace operator.

Elliptic equations involving the s-Laplace operator arise in some physical models like the flow of non-Newtonian fluids: pseudo-plastic fluids correspond to $s \in (1,2)$ while dilatant fluids correspond to $s > 2$. The case $s = 2$ expresses Newtonian fluids [3]. On the other hand, quasilinear systems like (1.1) describe various nonlinear phenomena such as chemical reactions, pattern formation, population evolution where, for example, u and v represent the concentrations of two species in the process. As a consequence, positive solutions of (1.1) are of interest.

Several methods have been used to treat quasilinear equations and systems. In the scalar case, weak solutions can be obtained through variational methods which provide critical points of the corresponding energy functional, an approach which

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is also fruitful in the case of potential systems i.e, the nonlinearities on the right hand side are the gradient of a C^1-functional [2], [7], [10] . However, due to the loss of the variational structure, the treatment of nonvariational systems like (1.1) is more complicated and is based mostly on topological methods [1].

Recently, there have been significant studies of (1.1). Dalmasso [6] provided existence and uniqueness results for positive solutions in the semilinear case $p = q = 2$ with the assumption that f is a function of v and g is a function of u, that is, (1.1) is the Lane-Emden system. Existence results in the case f and g are monomials of u and v are also provided in [5], while the quasilinear Lane-Emden system was studied by Hai [9]. In this paper we adopt the method in [9] to complement and extend corresponding results in the aforementioned papers.

2. Main Results

We make the following assumptions:

(H1) $f, g : \overline{\Omega} \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions such that
(i) $u \to f(x,u,v)$ and $u \to g(x,u,v)$ are nondecreasing for every $x \in \overline{\Omega}$ and $v \geq 0$.
(ii) $v \to f(x,u,v)$ and $v \to g(x,u,v)$ are nondecreasing for every $x \in \overline{\Omega}$ and $u \geq 0$.

(H2) For each $a > 0$,
$$\limsup_{z\to 0^+} \frac{h^{\frac{1}{p-1}}(z, ak^{\frac{1}{q-1}}(z,z))}{z} = \infty,$$
where $h(u,v) := \min_{x\in\overline{\Omega}} f(x,u,v)$ and $k(u,v) := \min_{x\in\overline{\Omega}} g(x,u,v)$.

(H3) For each $b > 0$,
$$\liminf_{z\to +\infty} \frac{F^{\frac{1}{p-1}}(z, bG^{\frac{1}{q-1}}(z,z))}{z} = 0,$$
where $F(u,v) := \max_{x\in\overline{\Omega}} f(x,u,v)$ and $G(u,v) := \max_{x\in\overline{\Omega}} g(x,u,v)$.

(H4)
$$\liminf_{z\to +\infty} \frac{G^{\frac{1}{q-1}}(z,z)}{z} = 0 \quad \text{and} \quad \limsup_{z\to 0^+} \frac{k^{\frac{1}{q-1}}(z,z)}{z} = \infty.$$

Suppose now that D is a sub-domain of Ω with $\overline{D} \subset \Omega$. Let $\delta(.) := \chi_D(.)$, the characteristic function of D. The solutions $\widetilde{\varphi}$, $\widetilde{\psi}$ of the problems
$$\begin{gathered} -\Delta_p \widetilde{\varphi} = \delta \quad \text{in } \Omega \\ \widetilde{\varphi} = 0 \quad \text{on } \partial\Omega \end{gathered}$$
and
$$\begin{gathered} -\Delta_q \widetilde{\psi} = \delta \quad \text{in } \Omega \\ \widetilde{\psi} = 0 \quad \text{on } \partial\Omega \end{gathered}$$
will be useful in what follows. Let φ (respectively ψ) denote the torsion functions relative to Ω and to the operators $-\Delta_p$ (respectively $-\Delta_q$), that is,
$$\begin{gathered} -\Delta_p \varphi = 1 \quad \text{in } \Omega \\ \varphi = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.1}$$

and
$$\begin{aligned} -\Delta_q \psi &= 1 \quad \text{in } \Omega \\ \psi &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$
Note that, by the strong comparison principle [8], there exist positive numbers M and m such that $\widetilde{\varphi} \geq M\varphi$, $\widetilde{\psi} \geq M\psi$ in Ω and $\widetilde{\varphi}$, $\widetilde{\psi}, \varphi, \psi \geq m$ on $\overline{D}$.

Our existence and uniqueness results are the following.

Theorem 2.1. *Let f, g satisfy (H1)-(H4). Then* (1.1) *has a positive solution* (u, v).

Theorem 2.2. *Let f, g satisfy (H1) and assume that there exist positive constants r_1, r_2, s_1, s_2 such that*
$$\frac{f(x,s,t)}{s^{r_1}}, \frac{g(x,s,t)}{s^{s_1}}$$
are nonincreasing for $x \in \overline{\Omega}$ and $t \geq 0$, and
$$\frac{f(x,s,t)}{t^{r_2}}, \quad \frac{g(x,s,t)}{t^{s_2}}$$
are nonincreasing for $x \in \overline{\Omega}$ and $s \geq 0$. If one of the following conditions is satisfied:

(i) $\frac{r_1+r_2}{p-1} < 1$ *and* $\frac{(r_1+r_2)s_1+s_2(p-1)}{(p-1)(q-1)} < 1$,
(ii) $\frac{s_1+s_2}{q-1} < 1$ *and* $\frac{(s_1+s_2)r_2+r_1(q-1)}{(p-1)(q-1)} < 1$,
(iii) $\frac{s_1+s_2}{q-1} < 1$ *and* $\frac{r_1+r_2}{p-1} < 1$,
(iv) $\frac{r_1+r_2}{p-1} > 1$, $\frac{s_1+s_2}{q-1} < 1$ *and* $\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(p-1)(q-1)^2} < 1$,
(v) $\frac{r_1+r_2}{p-1} < 1$, $\frac{s_1+s_2}{q-1} > 1$ *and* $\frac{(r_1+r_2)(q-1)s_1+(s_1+s_2)(p-1)s_2}{(p-1)^2(q-1)} < 1$,

then (1.1) admits at most one positive solution.

Remark 2.3. (i) If $f(u,v) = u^\alpha + v^\beta$ and $g(u,v) = u^\gamma + v^\delta$, $\alpha, \beta, \gamma, \delta \geq 0$, then (H2)-H(4) require
$$\alpha < p-1, \quad \max\{\gamma,\delta\} < q-1, \quad \text{and} \quad \max\{\gamma,\delta\}\beta < (p-1)(q-1).$$
(ii) Let $f(u,v) = u^\alpha v^\beta$ and $g(u,v) = u^\gamma v^\delta$. Then (H2)-H(4) are satisfied if
$$\alpha + \frac{\gamma+\delta}{q-1}\beta < p-1 \quad \text{and} \quad \gamma+\delta < q-1.$$

Proof of Theorem 2.1. In view of (H2) and (H4), there exists $\varepsilon \in (0,1)$ such that
$$Mh^{\frac{1}{p-1}}(\varepsilon m, mg^{\frac{1}{q-1}}(\varepsilon m, \varepsilon m)) \geq \varepsilon$$
and
$$Mk^{\frac{1}{q-1}}(\varepsilon m, \varepsilon m) \geq \varepsilon. \tag{2.3}$$
For $(w_1, w_2) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, let $T(w_1,w_2) := (u,v)$ be the solution of
$$\begin{gathered} -\Delta_p u = f(x, \max(w_1,\varepsilon\varphi), v) \quad \text{in } \Omega \\ -\Delta_q v = g(x, \max(w_1,\varepsilon\varphi), \max(w_2,\varepsilon\psi)) \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \Omega \end{gathered} \tag{2.4}$$
By standard arguments we can show that $T : C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})$ is completely continuous. By (H3) and (H4) there exists a number $R > \max\{|\varphi|_\infty, |\psi|_\infty\}$

such that

$$F^{\frac{1}{p-1}}(R, |\psi|_\infty G^{\frac{1}{q-1}}(R,R))|\varphi|_\infty \le R,$$
$$G^{\frac{1}{q-1}}(R,R)|\psi|_\infty \le R.$$

We claim that $T(\overline{B}(0,R) \times \overline{B}(0,R)) \subseteq \overline{B}(0,R) \times \overline{B}(0,R)$, where $\overline{B}(0,R)$ denotes the closed ball centered at 0 with radius R in $C(\overline{\Omega})$. Indeed, let $w_1, w_2 \in C(\overline{\Omega})$, with $|w_1|_\infty \le R$ and $|w_2|_\infty \le R$. Then, in view of (2.2),

$$\begin{aligned} -\Delta_q v &= g(x, \max(w_{1,}\varepsilon\varphi), \max(w_{2,}\varepsilon\psi)) \\ &\le G(R,R)(-\Delta_q \psi) = -\Delta_q(G^{\frac{1}{q-1}}(R,R)\psi) \quad \text{in } \Omega, \end{aligned}$$

which implies by strong comparison principle [8] that

$$v \le G^{\frac{1}{q-1}}(R,R)\psi.$$

Consequently, $|v|_\infty \le R$. On the other hand,

$$\begin{aligned} -\Delta_p u &= f(x, \max(w_{1,}\varepsilon\varphi), v) \le F(R,v) \\ &\le F(R, G^{\frac{1}{q-1}}(R,R)\psi) \le F(R, G^{\frac{1}{q-1}}(R,R)|\psi|_\infty), \end{aligned}$$

which, by the strong comparison principle, implies

$$u \le F^{\frac{1}{p-1}}(R, g^{\frac{1}{q-1}}(R,R)|\psi|_\infty)|\varphi|_\infty \le R,$$

and so $|u|_\infty \le R$, proving the claim.

By the Schauder fixed point theorem, T has a fixed point (u,v) with $|u|_\infty \le R$ and $|v|_\infty \le R$. We will show next that $|u|_\infty \ge \varepsilon\varphi$ and $|v|_\infty \ge \varepsilon\psi$. Since

$$\begin{aligned} -\Delta_q v &= g(x, \max(u, \varepsilon\varphi), \max(v, \varepsilon\psi)) \ge g(x, \varepsilon\varphi, \varepsilon\psi) \\ &\ge \begin{cases} g(x, \varepsilon m, \varepsilon m) & \text{in } \overline{D} \\ 0 & \text{in } \Omega\backslash\overline{D} \end{cases} \\ &\ge \begin{cases} k(\varepsilon m, \varepsilon m) & \text{in } \overline{D} \\ 0 & \text{in } \Omega\backslash\overline{D}, \end{cases} \end{aligned}$$

it follows from the strong comparison principle and (2.3) that

$$v \ge k^{\frac{1}{q-1}}(\varepsilon m, \varepsilon m)\widetilde{\psi} \ge M k^{\frac{1}{q-1}}(\varepsilon m, \varepsilon m)\psi \ge \varepsilon\psi.$$

Consequently,

$$\begin{aligned} -\Delta_p u &= f(x, \max(u, \varepsilon\varphi), v) \\ &\ge f(x, \max(u, \varepsilon\varphi), k^{\frac{1}{q-1}}(\varepsilon m, \varepsilon m)\widetilde{\psi}) \\ &\ge \begin{cases} f(x, \max(u, \varepsilon\varphi), k^{\frac{1}{q-1}}(\varepsilon m, \varepsilon m)m) & \text{in } \overline{D} \\ 0 & \text{in } \Omega\backslash\overline{D} \end{cases} \\ &\ge \begin{cases} h(\varepsilon m, m k^{\frac{1}{q-1}}(\varepsilon m, \varepsilon m)m) & \text{in } \overline{D} \\ 0 & \text{in } \Omega\backslash\overline{D}, \end{cases} \end{aligned}$$

and so

$$u \ge h^{\frac{1}{p-1}}(\varepsilon m, k^{\frac{1}{q-1}}(\varepsilon m, \varepsilon m)m)\tilde{\varphi} \ge M h^{\frac{1}{p-1}}(\varepsilon m, m k^{\frac{1}{q-1}}(\varepsilon m, \varepsilon m)) \ge \varepsilon\varphi.$$

The proof is complete. ∎

Proof of Theorem 2.2. We will provide the proof only for the cases (i), (iii) and (iv). Let (u,v) and (u_1,v_1) be positive solutions of (1). As in [4], we define

$$\Delta = \big\{\delta_1 \in (0,1] : u \geq \varepsilon u_1 \text{ and } v \geq \varepsilon v_1 \text{ in } \overline{\Omega} \text{ for } \varepsilon \in [0,\delta_1]\big\}.$$

Clearly $\Delta \neq \emptyset$. Let $\delta = \sup \Delta$. We will show that $\delta = 1$. So assume that $\delta < 1$.

Let (i) hold. Then

$$-\Delta_p u \geq f(x,\delta u_1,v) \geq \delta^{r_1} f(x,u_1,v) \geq \delta^{r_1}\delta^{r_2} f(x,u_1,v_1) = \delta^{r_1+r_2} f(x,u_1,v_1),$$

and since

$$-\Delta_p(\delta^{\frac{r_1+r_2}{p-1}} u_1) = \delta^{r_1+r_2} f(x,u_1,v_1),$$

it follows that

$$u \geq \delta^{\frac{r_1+r_2}{p-1}} u_1. \tag{2.5}$$

Using (2.5) in the equation for v in (1.1), we get

$$-\Delta_q v \geq g(x,\delta^{\frac{r_1+r_2}{p-1}} u_1, v) \geq g(x,\delta^{\frac{r_1+r_2}{p-1}} u_1, \delta v_1) \geq \delta^{\frac{(r_1+r_2)s_1}{p-1}} g(x,u_1,\delta v_1).$$

Therefore,

$$-\Delta_q v \geq \delta^{\frac{(r_1+r_2)s_1}{p-1}} \delta^{s_2} g(x,u_1,v_1) \geq \delta^{\frac{(r_1+r_2)s_1+s_2(p-1)}{(p-1)}} g(x,u_1,v_1),$$

and so

$$v \geq \delta^{\frac{(r_1+r_2)s_1+s_2(p-1)}{(p-1)(q-1)}} v_1, \tag{2.6}$$

contradicting the definition of δ.

In the case (iii) we have

$$-\Delta_q v \geq g(x,\delta u_1,v) \geq \delta^{s_1} g(x,u_1,v) \geq \delta^{s_1}\delta^{s_2} g(x,u_1,v_1) = \delta^{s_1+s_2} g(x,u_1,v_1).$$

Since

$$-\Delta_q(\delta^{\frac{s_1+s_2}{q-1}} v_1) = \delta^{s_1+s_2} g(x,u_1,v_1),$$

it follows that

$$v \geq \delta^{\frac{s_1+s_2}{q-1}} v_1. \tag{2.7}$$

In view of inequalities (2.5) and (2.7) we have a contradiction with the definition of δ.

Assume now that (iv) holds. Working as in (2.5) we get

$$v \geq \delta^{\frac{s_1+s_2}{q-1}} v_1. \tag{2.8}$$

Using (2.5) and (2.8) in the equation for u yields

$$\begin{aligned}
-\Delta_p u &\geq f(x,\delta^{\frac{r_1+r_2}{p-1}} u_1, \delta^{\frac{s_1+s_2}{q-1}} v_1)\\
&\geq \delta^{\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(q-1)(p-1)}} f(x,u_1,v_1)\\
&= -\delta^{\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(q-1)(p-1)}} \Delta_p u_1.
\end{aligned}$$

Thus,

$$u \geq \delta^{\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(p-1)^2(q-1)}} u_1.$$

On the other hand,

$$\begin{aligned}
-\Delta_q v &\geq g(x, \delta^{\frac{r_1+r_2}{p-1}} u_1, \delta^{\frac{s_1+s_2}{q-1}} v_1) \\
&\geq \delta^{\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(q-1)(p-1)}} g(x, u_1, v_1) \\
&= -\delta^{\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(q-1)(p-1)}} \Delta_q v_1.
\end{aligned}$$

Consequently,

$$v \geq \delta^{\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(p-1)(q-1)^2}} v_1,$$

contradicting the definition of δ.

Thus $\delta = 1$, i.e., $v \geq v_1$ and $u \geq u_1$. Similarly, $v \leq v_1$ and $u \leq u_1$. Consequently, $u = u_1$ and $v = v_1$. $\square$

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