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A TOPOLOGY ON INEQUALITIES

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Abstract. We consider sets of inequalities in Real Analysis and construct a topology such that inequalities usually called "limit cases" of certain sequences of inequalities are in fact limits - in the precise topological sense - of such sequences. To show the generality of the results, several examples are given for the notions introduced, and three main examples are considered: Sequences of inequalities relating real numbers, sequences of classical Hardy's inequalities, and sequences of embedding inequalities for fractional Sobolev spaces. All examples are considered along with their limit cases, and it is shown how they can be considered as sequences of one "big" space of inequalities. As a byproduct, we show how an abstract process to derive inequalities among homogeneous operators can be a tool for proving inequalities. Finally, we give some tools to compute limits of sequences of inequalities in the topology introduced, and we exhibit new applications.

1. Introduction

In Analysis it is frequent that authors consider inequalities that are limiting cases of sequences of inequalities, or, more generally, of a parametrized set of inequalities. The goal of this paper is to construct a topology such that inequalities usually called "limit cases" of certain sequences of inequalities are in fact limits - in the precise topological sense - of such sequences of inequalities. Such kind of problem can be studied from several points of view, because, for instance, it is possible - in a quite general, abstract setting - to speak about inequalities in ordered sets; moreover, even confining ourselves for instance to inequalities involving real functions, several notions of convergence can be considered. Our point of view has to be considered only as a first approach to the problem, which seems new.

2. The main question through examples

To give an idea of the general setting of our results, we will examine three examples of sequences of inequalities. The first one is the case of elementary numerical inequalities. The second one will be the classical integral inequality, known as *Hardy's inequality* and, finally, we conclude with a recent version of the Sobolev inequality for fractional Sobolev spaces.

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2.1. **Inequalities relating real numbers, part I.** Let (a_n), (b_n) be sequences of positive real numbers such that $a_n \to a > 0$, $b_n \to b > 0$, and

$$a_n \le b_n \quad \forall n \in \mathbb{N} \tag{2.1}$$

From elementary Analysis we know that one can "pass to the limit" in (2.1), obtaining

$$a \le b \tag{2.2}$$

The question here is to identify the inequalities in (2.1) with a sequence of "Inequalities" in a suitable topological space, let's call it $(\mathcal{I},\tau)$, and to show that the limit of such sequence, in the space $(\mathcal{I},\tau)$, is the "Inequality" identified with (2.2).

2.2. **Hardy's inequality, part I.** Let $p > 1$, f be a nonnegative (Lebesgue) measurable function on $(0,1)$. The classical Hardy's integral inequality states that ([17])

$$\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^1 f^p(x)dx \tag{2.3}$$

When $p \to 1+$ the constant (which is the best one such that (2.3) holds) $\big(\frac{p}{p-1}\big)^p$ blows up, and this leads immediately to conjecture that it cannot exist a constant $c > 0$ such that the inequality

$$\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx \le c\int_0^1 f(x)dx$$

holds for any f. This conjecture is in fact true: it is sufficient to consider the sequence $f_n(x) = x^{-1+1/n}$. Nevertheless, the "limiting" case of (2.3), when $p \to 1+$, can be expressed through the norm of the Zygmund space $LlogL(0,1)$:

$$\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx \le C\|f\|_{LlogL(0,1)}, \tag{2.4}$$

where

$$\|f\|_{LlogL(0,1)} = \inf\Big\{\lambda > 0 : \int_0^1 |\frac{f(x)}{\lambda}|\log\big(e + |\frac{f(x)}{\lambda}|\big)dx \le 1\Big\}. \tag{2.5}$$

For a recent digression on equivalent norms in $LlogL$ see e.g. [12].

As in the previous example, setting e.g. $p = 1 + 1/n$ in (2.3), it is natural to ask whether in some topological space it is really true that the sequence of Hardy's inequalities *converges* to the inequality (2.4).

2.3. **Sobolev inequalities for fractional Sobolev spaces, part I.** Let $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ be a bounded smooth open set, and let $p \ge 1$. Consider the classical Sobolev space $W_0^{1,p}(\Omega)$, defined as the completion of $\mathcal{C}_0^\infty(\Omega)$ - the set of all functions defined on Ω which have derivatives of any order on Ω, whose supports are compact sets - in the norm $\|\nabla f\|_{L^p(\Omega)}$. Let $0 < s < 1$ and consider the fractional Sobolev space $W_0^{s,p}(\Omega)$, defined (see e.g. [19, 22]) as the completion of $\mathcal{C}_0^\infty(\Omega)$ in the norm

$$\|f\|_{W^{s,p}(\Omega)} = \Big(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^{1/p}$$

The following version of Sobolev inequality for fractional Sobolev spaces holds (see e.g. [1, 26, 3]):

$$\|f\|_{L^q(\Omega)} \le c(s,p,N)\|f\|_{W_0^{s,p}(\Omega)} \tag{2.6}$$

where $sp < N$, $q = Np/(N - sp)$. Putting formally $s = 1$ in (2.6) we get

$$\|f\|_{L^{Np/(N-p)}(\Omega)} \leq c(p, N)\|f\|_{W_0^{1,p}(\Omega)} \tag{2.7}$$

It would be natural to think the inequality (2.7) as limit, in the topological sense, of (2.6), as $s \to 1$ (notice that, differently from the case of Hardy's inequality previously discussed, inequality (2.7) is true!). The main point here is that when $s \to 1$ the norm in $W_0^{s,p}(\Omega)$ blows up. This problem has been studied in [3, 4, 22], and solved by proving a version of (2.6) where the right dependence of the constant $c(s, p, N)$ with respect to s has been found. We will examine the problem of the convergence (in the topological sense) of (2.6) to (2.7) in Section 5.3.

3. Preliminary considerations for the well-posedness of the problem

Some preliminary considerations are due because the risk is that the problem is trivial or not well-posed.

Let us analyze, for the moment, the first example (Example 2.1). Let c_n be a sequence of positive real numbers such that $c_n \to 0$, say, $c_n = 1/n$. Then for each $n \in N$ the inequality $a_n \leq b_n$ is *equivalent* to $a_n c_n \leq b_n c_n$, and our topology should be chosen in such a way that both sequences of inequalities converge to the same inequality. But while it is natural that the first one converges to $a \leq b$, the second one (for analogous reason) should converge to the trivial $0 \leq 0$. This conclusion shows a first difficulty for our construction.

To avoid phenomena like the previous one, we should look carefully to the phenomenon described before. Suppose that $c_n \to c$, where c is *positive*. In this case no contradiction arises, because the limit of "$a_n \leq b_n$", i.e. "$a \leq b$", is *equivalent* to the limit of "$a_n c_n \leq b_n c_n$", i.e. "$ac \leq bc$". Therefore, in order to overcome the difficulty, we need always to consider inequalities in which both the left hand side and the right hand side are not zero, but positive (we will give in the sequel a precise meaning to this sentence). Of course, starting from $a_n \leq b_n$, one can always consider the equivalent inequality $a_n c_n \leq b_n c_n$, but the limit inequality, in order to be called *limit inequality*, must have each side different from zero. In the most general setting, this means that we have to introduce a class of possible left hand sides and right hand sides (the *admissible operators*) in which the trivial zero must be excluded.

The comment above suggests also that if one inequality in a sequence of inequalities is changed by another *equivalent* inequality, the limit should no change. This means that, for a given inequality, we must consider all the equivalent inequalities, and treat all of them in the same way. It will be natural, therefore, to consider classes of equivalence of inequalities, that we will call "Inequalities" with capital "I", and speak about limits of "Inequalities".

Consider now the second example (Example 2.2). If we start our limiting process from inequality (2.3)

$$\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx \leq \Big(\frac{p}{p-1}\Big)^p \int_0^1 f^p(x)dx$$

or from the same inequality raised to $1/p$:

$$\Big(\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx\Big)^{1/p} \leq \Big(\Big(\frac{p}{p-1}\Big)^p \int_0^1 f^p(x)dx\Big)^{1/p} \tag{3.1}$$

or, say, from the same inequality raised to 2:

$$\Big(\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx\Big)^2 \le \Big(\Big(\frac{p}{p-1}\Big)^p \int_0^1 f^p(x)dx\Big)^2 \tag{3.2}$$

we should expect to get the same limit, in the sense that the inequalities (possibly different) obtained as limits (respectively) of (2.3), (3.1), (3.2) should be equivalent each other. This gives an idea of the properties that we must require, to a couple of inequalities, in order to be defined "equivalent".

The plan of the rest of the paper is the following: in the next section we define the admissible operators and we introduce a method to *homogenize* an inequality. Besides being of interest in itself, this method will play a key role for one of our main examples (Hardy's inequality), which will be discussed later. In Section 5 we introduce a topology in inequalities, and describe the notion of convergence. Such notion is applied to three main examples: sequences of inequalities relating real numbers, sequences of classical Hardy's inequalities, and sequences of embedding inequalities for fractional Sobolev spaces. In Section 6 we introduce a notion of equivalent inequalities, and we construct an abstract setting, starting from the notions already introduced, which seems more suitable for inequalities relating admissible operators. Finally, in Section 7, we compute explicitly some limits and we see how the notion of convergence introduced in the paper can be a tool to derive new results.

4. Homogenizing inequalities

We begin by introducing some notation. If not differently specified, $\Omega \subset \mathbb{R}^N$ ($N \ge 1$) will denote a bounded smooth open set. $\mathcal{C}^\infty_{0,+}(\Omega)$ will be the set of all nonnegative functions defined on Ω which have derivatives of any order, and whose supports are compact sets. We will denote by $\mathbf{0}$ the function whose value is zero on all Ω.

Let $\mathcal{O}$ be the set of all operators (we will call them admissible)

$$T : \mathcal{C}^\infty_{0,+}(\Omega) \to [0, +\infty[$$

such that, setting

$$F_{T,f} : \lambda \in [0, +\infty[\; \to \; F_{T,f}(\lambda) = T(\lambda f) \in [0, +\infty[\quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

it is

$$F_{T,f} \text{ is continuous for all } f \in \mathcal{C}^\infty_{0,+}(\Omega) \tag{4.1}$$

$$F_{T,f} \text{ is strictly increasing for all } f \in \mathcal{C}^\infty_{0,+}(\Omega),\ f \ne \mathbf{0} \tag{4.2}$$

$$\lim_{\lambda\to\infty} F_{T,f}(\lambda) = +\infty \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega),\ f \ne \mathbf{0} \tag{4.3}$$

$$\inf_{\lambda>0} T\Big(\frac{f}{\lambda}\Big) = T(\mathbf{0}) := m_T \in [0, +\infty[\quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega) \tag{4.4}$$

For T, S in $\mathcal{O}$ we shall often write

$$d = d(T, S),$$

instead of

$$Tf \le Sf \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

We observe that the operators that we consider are very common in Analysis, because many known inequalities have both sides enjoying the properties listed above.

Before listing some examples, let us recall some definitions which will be useful in the sequel.

A function $A : [0, +\infty[\to [0, +\infty[$ is called *N-function* if it is continuous, strictly increasing, convex and such that

$$\lim_{t\to 0} \frac{A(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{A(t)}{t} = \infty.$$

Typical examples of N-functions are powers, with exponent greater than 1. Starting from the notion of N-function it is possible to consider the norm

$$\|f\|_A = \|f\|_{L^A(0,1)} = \inf\big\{\lambda > 0 : \int_0^1 A\big(\big|\frac{f(x)}{\lambda}\big|\big)dx \le 1\big\}$$

which defines the *Orlicz* space $L^A(0,1)$. If $A(t) = t\log(e+t)$, we get the norm considered in (2.5); in this case the Orlicz space is called *Zygmund* space. We refer to [12] for expressions for the norm in such spaces. For properties and further examples of N-functions and Orlicz spaces see e.g. [1].

Let f be a (Lebesgue) measurable function defined on $(0,1)$, a.e. finite, and for any Lebesgue measurable set $E \subset (0,1)$ let $|E|$ be its measure. The *decreasing rearrangement* of f is the function, denoted by f^*, defined by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in (0,1) : |f(x)| > \lambda\}| \le t\} \quad t \in (0,1)$$

This definition is usually given for a much more general class of functions, but it is not in our purposes to give details here. For interested readers we refer to [2].

Example 4.1. Let X be a Banach space whose elements are measurable functions. Suppose that $\mathcal{C}_0^\infty(\Omega) \subset X$. An example of operator in $\mathcal{O}$ is

$$T_1 f = \|f\|_X \quad \forall f \in \mathcal{C}_{0,+}^\infty(\Omega).$$

Example 4.2. Let p be a measurable function on Ω, whose values are in $[1,\infty]$, and set $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$. Then an example of operator in $\mathcal{O}$ is

$$T_2 f = \inf\big\{\lambda > 0 : \int_{\Omega\setminus\Omega_\infty} \big|\frac{f(x)}{\lambda}\big|^{p(x)} dx + \operatorname*{ess\,sup}_{\Omega_\infty} \big|\frac{f(x)}{\lambda}\big| \le 1\big\}$$

This operator is a special case of the previous example, in fact it is the norm in the space $L^{p(\cdot)}(\Omega)$. For details see [20].

Example 4.3. Let $A : [0, +\infty[\to [0, +\infty[$ be an N-function. Then an example of operator in $\mathcal{O}$ is

$$T_3 f = \int_\Omega A(f)dx \quad \forall f \in \mathcal{C}_{0,+}^\infty(\Omega).$$

Example 4.4. Let $A_1, A_2, A_3 : [0, +\infty[\to [0, +\infty[$ be continuous and strictly increasing functions such that $A_i(+\infty) = +\infty$, $i = 1,2,3$, and let w_1, w_2 be non-negative, locally integrable functions defined respectively in $\Omega \times \Omega$ and Ω, such that

$$w_1(x,\cdot) \not\equiv 0 \quad \forall x \in \Omega, \quad w_2 > 0 \quad \text{a.e. in } \Omega.$$

Then an example of operator in $\mathcal{O}$ is

$$T_4 f = A_1\Big(\int_\Omega A_2\Big(\int_\Omega A_3(f(y))w_1(x,y)dy\Big)w_2(x)dx\Big) \quad \forall f \in \mathcal{C}_{0,+}^\infty(\Omega)$$

Next we consider the particular case: $\Omega =]0,1[\subset \mathbb{R}$, $A_1(t) = A_2(t) = A_3(t) = t$, $w_1(x,y) = \chi_{(0,x)}(y)$, $w_2(x) = 1/x$, which gives the operator

$$T_5 = \int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

Example 4.5. Let $A_1, A_2 : [0,+\infty[\to [0,+\infty[$ be N-functions and let X be a Banach Function Space. Then an example of operator in $\mathcal{O}$ is

$$T_6 f = A_1(\|A_2(f)\|_X) \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega).$$

Example 4.6. Let us denote by f^* the decreasing rearrangement of f, defined in the interval $]0, |\Omega|]$. An example of operator in $\mathcal{O}$ is

$$T_7 f = f^*\big(\frac{|\operatorname{supp}(f)|}{2}\big) \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

where $|\operatorname{supp}(f)|$ denotes the Lebesgue measure of the support of f.

Example 4.7. Let us denote by f^* the decreasing rearrangement of f, defined in the interval $]0, |\Omega|]$, let A_1 be a strictly increasing, continuous function on $[0, |\Omega|]$ such that $A_1(0) = 0$, $A_1(+\infty) = +\infty$, let $A_2 : [0,+\infty[\to [0,+\infty[$ be an N-function, and let w_1, w_2 be positive, locally integrable functions defined in $]0, |\Omega|[$. Then an example of operator in $\mathcal{O}$ is

$$T_8 f = \int_0^{|\Omega|} A_1\Big(\int_0^t A_2(f^*(s))w_1(s)ds\Big)w_2(t)dt \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

Operators of this type occur in Function Space Theory, see e.g. [11].

Let us now introduce a notion which will play a key role in the sequel. For each $T \in \mathcal{O}$ we define the *associate family* $\{T^{(\mu)}\}$ *of homogeneous operators* as the class of operators $T^{(\mu)} : \mathcal{C}^\infty_{0,+}(\Omega) \to [0,+\infty[$ defined by

$$T^{(\mu)} f = \inf\big\{\lambda > 0 : T\big(\frac{f}{\lambda}\big) \le \mu\big\} \quad \forall \mu > m_T \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

We remark that since $T \in \mathcal{O}$ the set

$$\big\{\lambda > 0 : T\big(\frac{f}{\lambda}\big) \le \mu\big\}$$

is nonempty for all $f \in \mathcal{C}^\infty_{0,+}(\Omega)$ and all $\mu > m_T$, so that the definition of $T^{(\mu)} f$ is well posed for all $\mu > m_T$. The homogeneity of the operator $T^{(\mu)}$ is proved by the following result.

Proposition 4.8. *If T is admissible, then for all $\mu > m_T$ the operator $T^{(\mu)}$ is admissible and has the further property to be homogeneous (of degree 1):*

$$T^{(\mu)}(kf) = kT^{(\mu)} f \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)\ \forall k \ge 0.$$

Proof. First we prove the homogeneity. If $k = 0$ it is sufficient to see that

$$T^{(\mu)}\mathbf{0} = 0 \quad \forall \mu > m_T, \tag{4.5}$$

and this is trivial, because for all $\lambda > 0$ it is

$$T\big(\frac{\mathbf{0}}{\lambda}\big) = T\mathbf{0} \le \mu \quad \forall \mu > m_T.$$

For $k > 0$ we have

$$\begin{aligned} T^{(\mu)}(kf) &= \inf\big\{\lambda > 0 : T\big(\frac{kf}{\lambda}\big) \le \mu\big\} \\ &= \inf\big\{k\lambda > 0 : T\big(\frac{f}{\lambda}\big) \le \mu\big\} \\ &= k\inf\big\{\lambda > 0 : T\big(\frac{f}{\lambda}\big) \le \mu\big\} \\ &= kT^{(\mu)}(f) \end{aligned}$$

Now we show that the operator $T^{(\mu)}$ is admissible. Property (4.1) is trivial, because the homogeneity of $T^{(\mu)}$ implies that

$$\lambda \in [0, +\infty[\to T^{(\mu)}(\lambda f) \in [0, +\infty[$$

is linear. In order to show properties (4.2) and (4.3), because of the homogeneity of $T^{(\mu)}$, it is sufficient to see that

$$f \neq \mathbf{0} \Longrightarrow T^{(\mu)}(f) > 0$$

Since T is admissible, property (4.3) is true for T, therefore there exists $\lambda > 0$ such that $T\big(\frac{f}{\lambda}\big) > \mu$. The conclusion is that $T^{(\mu)}(f) > 0$.

Finally, we observe that both sides of (4.4) are equal to zero: the left hand side because of the homogeneity of $T^{(\mu)}$, the right hand side because of (4.5). □

Remark 4.9. If T is homogeneous, then it is easy to show that $T^{(\mu)} = (1/\mu)T$ for all $\mu > 0$.

The main result of this Section is the following.

Theorem 4.10. *Let $T, S \in \mathcal{O}$. The following equivalence holds:*

$$Tf \le Sf \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

if and only if

$$T^{(\mu)}f \le S^{(\mu)}f \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega) \quad \forall \mu > \max(m_T, m_S)$$

Proof. Let us assume first that

$$Tf \le Sf \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega) \tag{4.6}$$

and let $\mu > \max(m_T, m_S)$. Fix $f \in \mathcal{C}^\infty_{0,+}(\Omega)$ and let $\lambda > 0$ be such that $S\big(\frac{f}{\lambda}\big) \le \mu$. By (4.6) the number λ is also such that $T\big(\frac{f}{\lambda}\big) \le \mu$; therefore,

$$\big\{\lambda > 0 : T\big(\frac{f}{\lambda}\big) \le \mu\big\} \supseteq \big\{\lambda > 0 : S\big(\frac{f}{\lambda}\big) \le \mu\big\}$$

from which the first part of the assertion follows.

On the other hand, by contradiction, let us assume that there exists $\overline{f}$ such that

$$T\overline{f} > S\overline{f}$$

We observe that $\overline{f}$ can be always chosen different from $\mathbf{0}$. In fact, if

$$T\mathbf{0} > S\mathbf{0}$$

then, fixing any $f \neq \mathbf{0}$, by property (4.1) for S in $\lambda = 0$, there exists λ sufficiently small such that

$$T\mathbf{0} > S(\lambda f) > S\mathbf{0}$$

and therefore, by property (4.2) for T,

$$T(\lambda f) > T\mathbf{0} > S(\lambda f).$$

Setting $\overline{f} = \lambda f$ we get the existence of $\mu > \max(m_T, m_S)$ such that

$$T\overline{f} > \mu > S\overline{f} \tag{4.7}$$

Now from the inequality $S\overline{f} < \mu$, applying property (4.1) for S in $\lambda = 1$, we can consider $\epsilon > 0$ such that

$$1 - \epsilon \in \{\lambda > 0 : S\big(\frac{\overline{f}}{\lambda}\big) \le \mu\}$$

By our assumption

$$\inf\big\{\lambda > 0 : T\big(\frac{\overline{f}}{\lambda}\big) \le \mu\big\} = T^{(\mu)}\overline{f} \le S^{(\mu)}\overline{f} \le 1 - \epsilon$$

from which $T\overline{f} \le \mu$; this conclusion is in contrast with (4.7). □

In the sequel we will use the first implication proved above, which, starting from a generic inequality, leads to a family of inequalities relating homogeneous operators. We will say that such family of inequalities is obtained *homogenizing* the original inequality.

Application 1. Let $\varphi : [0, +\infty[\to [0, +\infty[$ be increasing and such that $\varphi(0) = 0$, and suppose to know that

$$\varphi(\|f\|_X) \le \|f\|_Y \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega), \tag{4.8}$$

where X and Y are Banach Function Spaces (see [2, Def 1.3 p. 3]), such that $\mathcal{C}^\infty_0(\Omega)$ functions are dense in X and Y. Applying [2, Theorem 1.8 p. 7], one can deduce that Y is continuously embedded into X; therefore, there exists a constant $k > 0$ such that

$$\|f\|_X \le k\|f\|_Y$$

We observe that it is possible to get the same conclusion homogenizing the inequality (4.8), getting also an estimate of the constant k.

In fact, by Theorem 4.10, from (4.8) we get, for all $\mu > 0$ and $f \in \mathcal{C}^\infty_{0,+}(\Omega)$,

$$\begin{gathered}
\inf\big\{\lambda > 0 : \varphi\big(\big\|\frac{f}{\lambda}\big\|_X\big) \le \mu\big\} \le \inf\big\{\lambda > 0 : \big\|\frac{f}{\lambda}\big\|_Y \le \mu\big\}, \\
\inf\big\{\lambda > 0 : \frac{1}{\lambda}\|f\|_X \le \sup\{\xi : \varphi(\xi) \le \mu\}\big\} \le \frac{1}{\mu}\|f\|_Y, \\
\|f\|_X \le \frac{\sup\{\xi : \varphi(\xi) \le \mu\}}{\mu}\|f\|_Y\,.
\end{gathered}$$

In conclusion:

$$\|f\|_X \le \inf_{\mu>0} \frac{\sup\{\xi : \varphi(\xi) \le \mu\}}{\mu}\|f\|_Y \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

and therefore the same inequality is true for all f, due to the assumed density of the $\mathcal{C}^\infty_0(\Omega)$ functions.

Application 2. Let Φ be an N-function. By the well-known Jensen inequality it is

$$\Phi\big(\int_0^1 f(x)dx\big) \le \int_0^1 \Phi(f)dx \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

Let us homogenize this inequality. For all $\mu > 0$ and $f \in \mathcal{C}^\infty_{0,+}(\Omega)$ we get

$$\begin{gathered}
\inf\{\lambda > 0 : \Phi\big(\int_0^1 \frac{f(x)}{\lambda}dx\big) \le \mu\} \le \inf\big\{\lambda > 0 : \int_0^1 \Phi\big(\frac{f(x)}{\lambda}\big)dx \le \mu\big\},\\
\inf\big\{\lambda > 0 : \int_0^1 \frac{f(x)}{\lambda}dx \le \Phi^{-1}(\mu)\big\} \le \inf\big\{\lambda > 0 : \int_0^1 \big(\frac{\Phi}{\mu}\big)\big(\frac{f(x)}{\lambda}\big)dx \le 1\big\},\\
\frac{1}{\Phi^{-1}(\mu)}\int_0^1 f(x)dx \le \|f\|_{\Phi/\mu},\\
\int_0^1 f(x)dx \le \Phi^{-1}(\mu)\|f\|_{\Phi/\mu}
\end{gathered}$$

For $\mu = 1$ such inequality reduces to

$$\int_0^1 f(x)dx \le \Phi^{-1}(1)\|f\|_\Phi$$

which is the inequality which shows that the Orlicz space $L^\Phi(0,1)$ is embedded in $L^1(0,1)$. Notice that the constant $\Phi^{-1}(1)$ on the right hand side is optimal (the inequality becomes equality for $f \equiv 1$).

Application 3. Let us consider - as usual, we will consider functions f in $\mathcal{C}^\infty_{0,+}$ - a well known inequality by Hardy and Littlewood (see [17, 241 (i) page 169], or [27, vol.1, p.32, Theorem 13.15(iii)]):

$$\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx \le c_1\int_0^1 f(x)\log^+ f(x)dx + c_2 \tag{4.9}$$

which is true for some $c_1, c_2 > 0$. Here $\log^+ t = \max(\log t, 0)$. We now compute the family of the homogenized inequalities. Since the left hand side is already homogeneous, by Remark 4.9 it is immediate to compute the associated family of homogeneous operators. Let us consider the right hand side:

$$Tf = c_1\int_0^1 f(x)\log^+ f(x)dx + c_2 = \int_0^1 [c_1 f(x)\log^+ f(x) + c_2]dx$$

Let $c_3 > 0$ be such that

$$c_1 t\log^+ t + c_2 \ge c_3 t\log(e+t) \quad \forall t > 0\,.$$

For all $\mu > 0$ we have,

$$\begin{aligned}
T^{(\mu)}(f) &= \inf\big\{\lambda > 0 : T\big(\frac{f}{\lambda}\big) \le \mu\big\} \\
&= \inf\big\{\lambda > 0 : \int_0^1 \big[c_1 \frac{f(x)}{\lambda}\log^+ \frac{f(x)}{\lambda} + c_2\big]dx \le \mu\big\} \\
&\ge \inf\big\{\lambda > 0 : \int_0^1 c_3 \frac{f(x)}{\lambda}\log\big(e + \frac{f(x)}{\lambda}\big)dx \le \mu\big\} \\
&= \inf\big\{\lambda > 0 : \int_0^1 \frac{c_3}{\mu}\frac{f(x)}{\lambda}\log\big(e + \frac{f(x)}{\lambda}\big)dx \le 1\big\} \\
&\ge \min\big(1, \frac{c_3}{\mu}\big)\|f\|_{LlogL(0,1)}
\end{aligned}$$

where $\|f\|_{LlogL(0,1)}$ is defined in (2.5). Last inequality is easily obtained considering separately the cases $c_3 \ge \mu$, $c_3 < \mu$. On the other hand, for all $\mu > c_2$ we have

$$\begin{aligned}
T^{(\mu)}(f) &= \inf\big\{\lambda > 0 : T\big(\frac{f}{\lambda}\big) \le \mu\big\} \\
&= \inf\big\{\lambda > 0 : \int_0^1 \big[c_1 \frac{f(x)}{\lambda}\log^+ \frac{f(x)}{\lambda} + c_2\big]dx \le \mu\big\} \\
&= \inf\big\{\lambda > 0 : \int_0^1 c_1 \frac{f(x)}{\lambda}\log^+ \frac{f(x)}{\lambda}dx \le \mu - c_2\big\} \\
&\le \inf\big\{\lambda > 0 : \int_0^1 \frac{c_1}{\mu - c_2}\frac{f(x)}{\lambda}\log\big(e + \frac{f(x)}{\lambda}\big)dx \le 1\big\} \\
&\le \max\big(1, \frac{c_1}{\mu - c_2}\big)\|f\|_{LlogL(0,1)}
\end{aligned}$$

In conclusion, the homogenized family of inequalities of (4.9) can be written as

$$\int_0^1 \big(\frac{1}{x}\int_0^x f(t)dt\big)dx \le C(\mu)\|f\|_{LlogL(0,1)} \quad \forall \mu > c_2.$$

We observe that such inequalities are of the type (2.4).

We conclude this Section answering to a natural question: What happens if we homogenize one of the inequalities obtained after a homogenization?

Let $T, S \in \mathcal{O}$ and fix a number $\mu > \max(m_T, m_S)$. If

$$Tf \le Sf \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

by Theorem 4.10 we know that

$$T^{(\mu)}f \le S^{(\mu)}f \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

Let us now apply again Theorem 4.10 to this inequality, and consider

$$T^{(\mu)(\sigma)}f \le S^{(\mu)(\sigma)}f \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega) \quad \forall \sigma > 0$$

By Remark 4.9 we get

$$\frac{1}{\sigma}T^{(\mu)}f \le \frac{1}{\sigma}S^{(\mu)}f \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega) \quad \forall \sigma > 0$$

i.e.

$$T^{(\mu)}f \le S^{(\mu)}f \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega)$$

The conclusion is that for each fixed $\mu > 0$ the inequality $T^{(\mu)}f \le S^{(\mu)}f$ has a trivial family of associated homogeneous inequalities, pairwise identical to the original one.

5. A topology on inequalities

Let us consider the set of inequalities

$$\mathcal{I}_0 = \{d(T,S) : T,S \in \mathcal{O}\}$$

We construct a topology in $\mathcal{I}_0$, taking inspiration from the classical procedure used to define the topology of pointwise convergence.

Fix $d = d(T,S) \in \mathcal{I}_0$. For any $n \in \mathbb{N}$, for any finite subset $F \subset \mathcal{C}^\infty_{0,+}(\Omega)$, let us set

$$\mathcal{U}_{n,F}(d) = \{d' = d'(T',S') : |T'f - Tf| < \frac{1}{n}\ \forall f \in F,\ |S'f - Sf| < \frac{1}{n}\ \forall f \in F\}$$

Let $\mathcal{N}(d)$ be the family of subsets of $\mathcal{I}_0$ whose elements contain some set of the type $\mathcal{U}_{n,F}(d)$:

$$A \in \mathcal{N}(d) \Leftrightarrow \exists n \in \mathbb{N},\ \exists F \subset \mathcal{C}^\infty_{0,+}(\Omega) \text{ finite: } A \supseteq \mathcal{U}_{n,F}(d).$$

We will prove the following result.

Proposition 5.1. *The family $\{\mathcal{N}(d)\}_{d\in\mathcal{I}_0}$ satisfies the following Hausdorff's axioms:*

(i) $d \in A$ *for all* $A \in \mathcal{N}(d)$
(ii) $A \in \mathcal{N}(d)$ *and* $B \in \mathcal{N}(d)$ *implies* $A \bigcap B \in \mathcal{N}(d)$
(iii) $A \in \mathcal{N}(d)$ *and* $B \supseteq A$ *implies* $B \in \mathcal{N}(d)$
(iv) *for all* $A \in \mathcal{N}(d)$ *there exists* $B \in \mathcal{N}(d)$ *such that* $A \in \mathcal{N}(d')$ *for all* $d' \in B$

After this proposition we know (see e.g. [9, Theorem 3.2, p. 67]) that there exists one and only one topology τ for $\mathcal{I}_0$ such that $\mathcal{N}(d)$ is, for any $d \in \mathcal{I}_0$, the set of the nbds of d.

Proof. (i) Let $A \in \mathcal{N}(d)$ and let $\mathcal{U}_{n,F}(d) \subseteq A$. Since $d \in \mathcal{U}_{n,F}(d)$, it is $d \in A$.
(ii) Let $\mathcal{U}_{n,F}(d) \subseteq A$, $\mathcal{U}_{m,G}(d) \subseteq B$. Then $\mathcal{U}_{\max(n,m),F\bigcup G}(d) \subseteq A \bigcap B$.
(iii) Let $A \in \mathcal{N}(d)$ and let $\mathcal{U}_{n,F}(d) \subseteq A$. Since $B \supseteq A$, then $B \supseteq \mathcal{U}_{n,F}(d)$; therefore $B \in \mathcal{N}(d)$
(iv) Let $A \in \mathcal{N}(d)$ and let $\mathcal{U}_{n,F}(d) \subseteq A$. We show that (iv) is true with $B = \mathcal{U}_{n,F}(d)$:

$$d' \in \mathcal{U}_{n,F}(d) \quad \Rightarrow \quad A \in \mathcal{N}(d')$$

To this goal we need to find some $\mathcal{U}_{\nu,F}(d')$ such that $\mathcal{U}_{\nu,F}(d') \subset A$. Since $d' \in \mathcal{U}_{n,F}(d)$,

$$|T'f - Tf| < \frac{1}{n} \quad \forall f \in F,$$
$$|S'f - Sf| < \frac{1}{n} \quad \forall f \in F\,.$$

Since F is finite, we may consider $\nu \in \mathbb{N}$ such that

$$\frac{1}{\nu} < \min\big\{\min_F\{\frac{1}{n} - |T'f - Tf|\}, \min_F\{\frac{1}{n} - |S'f - Sf|\}\big\}.$$

Let $d'' \in \mathcal{U}_{\nu,F}(d')$. Then

$$|T''f - T'f| < \frac{1}{\nu} \quad \forall f \in F,$$
$$|S''f - S'f| < \frac{1}{\nu} \quad \forall f \in F\,.$$

We have

$$|T''f - Tf| \le |T''f - T'f| + |T'f - Tf| < \frac{1}{\nu} + |T'f - Tf| < \frac{1}{n} \quad \forall f \in F$$

and similarly $|S''f - Sf| < 1/n$ for all $f \in F$. Therefore, $d'' \in \mathcal{U}_{n,F}(d) \subseteq A$. $\square$

At this point we have a topology τ in $\mathcal{I}_0$. The notion of convergence in this topology is, as usual,

$$d_n \to d \text{ in } \tau \quad \Leftrightarrow \quad \forall \mathcal{U}_{\nu,F}(d)\, \exists\, n_0 \in \mathbb{N} : d_n \in \mathcal{U}_{\nu,F}(d)\; \forall n > n_0$$

Set $d_n = d_n(T_n, S_n)$, $d = d(T, S)$. It is readily seen that $d_n \to d$ if and only if

$$\forall\, f \in \mathcal{C}^\infty_{0,+}(\Omega) \quad T_n f \to Tf \quad \text{and} \quad S_n f \to Sf \tag{5.1}$$

Now, we go back to the examples considered in Section 2, and we show how a suitable choice of the operators T_n, S_n, T, S gives that the considered inequalities converge to their respective limits in the sense of (5.1).

5.1. **Inequalities relating real numbers, part II.** Let (a_n), (b_n) be sequences of positive real numbers such that $a_n \to a > 0$, $b_n \to b > 0$, and

$$a_n \le b_n \quad \forall n \in \mathbb{N} \tag{5.2}$$

We cannot consider the most natural operators

$$T_n f \equiv a_n \quad S_n f \equiv b_n \quad \forall f \in \mathcal{C}^\infty_{0,+}(0,1)$$

because they are not admissible (notice that property (4.2) does not hold). Let us set

$$T_n f \equiv a_n \sup_{(0,1)} f(x) \quad S_n f \equiv b_n \sup_{(0,1)} f(x) \quad \forall f \in \mathcal{C}^\infty_{0,+}(0,1)$$

Observe that now the operators T_n, S_n are admissible. The limit (in the sense of (5.1)) of

$$T_n f \le S_n f \quad \forall f \in \mathcal{C}^\infty_{0,+}(0,1)$$

is

$$Tf \le Sf \quad \forall f \in \mathcal{C}^\infty_{0,+}(0,1),$$

where

$$Tf \equiv a \sup_{(0,1)} f(x) \quad Sf \equiv b \sup_{(0,1)} f(x) \quad \forall f \in \mathcal{C}^\infty_{0,+}(0,1).$$

5.2. **Hardy's inequality, part II.** We start with the homogenized version of inequality (2.3):

$$\Big(\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx\Big)^{1/p} \le \frac{p}{p-1}\Big(\int_0^1 f^p(x)dx\Big)^{1/p} \tag{5.3}$$

As a first step, fix $n \in \mathbb{N}$ and set $p = 1 + 1/n$. Here, again, the most natural operators T_n, S_n, respectively equal to the left hand side and the right hand side of (5.3) do not work, due to the blowup of the right hand side. In this case T_n and S_n would be admissible and it seems that there cannot be a better choice.

The important consideration to be made at this point is to understand *which* inequality we wish to use before passing to the limit. In fact the standard proof of Hardy's inequality leads to a better version of (5.3), containing one more term, which is usually dropped. Taking into consideration such term, we are able to prove that the limit of Hardy's inequality when $p \to 1+$ is inequality (2.4) (actually, even a better one).

We stress that inequality (2.4) has an independent, classical, simple proof in [27, vol.1, p.32, Theorem 13.15(iii)]. However, our main intention here is to prove a limiting process and to obtain, as a byproduct, a new tool for proving inequalities. We believe that the following procedure has an independent interest.

For completeness, we start here with the simple proof of (5.3) (as usual, we are assuming here to deal only with $\mathcal{C}^\infty_{0,+}$ functions, not identically zero). We have

$$\begin{aligned}
\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx &= \int_0^1 \Big(\int_0^x f(t)dt\Big)^p x^{-p} dx \\
&= \int_0^1 \Big(\int_0^x f(t)dt\Big)^p d\big(\frac{x^{1-p}}{1-p}\big) \\
&= \Big[-\frac{x^{1-p}\big(\int_0^x f(t)dt\big)^p}{p-1}\Big]_{x=0}^{x=1} - \int_0^1 \big(\frac{x^{1-p}}{1-p}\big) d\Big(\int_0^x f(t)dt\Big)^p \\
&= -\frac{\big(\int_0^1 f(t)dt\big)^p}{p-1} + \frac{p}{p-1}\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^{p-1} f(x)dx
\end{aligned}$$

Applying Holder's inequality,

$$\begin{aligned}
&\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx \\
&\le -\frac{\big(\int_0^1 f(t)dt\big)^p}{p-1} + \frac{p}{p-1}\Big[\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx\Big]^{1-1/p}\Big(\int_0^1 f(t)^p dt\Big)^{1/p}
\end{aligned}$$

At this point Hardy's inequality (5.3) is readily obtained dropping the first added and multiplying each side by $\Big[\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx\Big]^{1/p-1}$. We now *do not* drop the first added, which plays a key role when passing to the limit for $p \to 1+$, and we raise both sides to the power $1/p$, getting an inequality relating two homogeneous operators, which we call respectively $T_n f$ and $S_n f$.

We now compute the limit of $T_n f$ and $S_n f$. It is immediate to see that the limit of $T_n f$ is exactly the left hand side of inequality (2.4). As for $S_n f$, we have

$$\begin{aligned}
(S_n f)^p &= -\frac{\big(\int_0^1 f(t)dt\big)^p}{p-1} + \frac{p}{p-1}\Big[\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx\Big]^{1-1/p}\Big(\int_0^1 f(t)^p dt\Big)^{1/p} \\
&= \frac{\big(\int_0^1 f(t)^p dt\big)^{1/p} - \big(\int_0^1 f(t)dt\big)^p}{p-1} \\
&\quad + \frac{p\big[\int_0^1 \big(\frac{1}{x}\int_0^x f(t)dt\big)^p dx\big]^{1-1/p} - 1}{p-1}\Big(\int_0^1 f(t)^p dt\Big)^{1/p}
\end{aligned}$$

We can now compute the limit as $p \to 1+$, taking into account that the quotients are in fact difference-quotients (therefore it suffices to compute derivatives in p and set $p = 1$). We have:

$$\begin{aligned}
&\lim_{p\to 1+} \frac{\big(\int_0^1 f(t)^p dt\big)^{1/p} - \big(\int_0^1 f(t)dt\big)^p}{p-1} \\
&= \int_0^1 f(x)\log f(x)dx - 2\Big(\int_0^1 f(t)dt\Big)\log\Big(\int_0^1 f(t)dt\Big)
\end{aligned}$$

and

$$\lim_{p\to 1+} \frac{p\Big[\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx\Big]^{1-1/p} - 1}{p-1} = 1 + \log\Big(\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx\Big).$$

Hence the limit as $p \to 1+$ is the inequality

$$\begin{aligned}
&\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx \\
&\le \int_0^1 f(x)\log f(x)dx - 2\Big(\int_0^1 f(t)dt\Big)\log\Big(\int_0^1 f(t)dt\Big) + \int_0^1 f(t)dt \\
&\quad + \Big(\int_0^1 f(t)dt\Big)\log\Big(\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx\Big)
\end{aligned}$$

We will prove that this inequality is finer than inequality (4.9), which leads to inequality (2.4)(see Application 3 in Section 4). We consider two cases.

First case: $\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx \le 1$ so that the last term of our inequality is non-positive:

$$\Big(\int_0^1 f(t)dt\Big)\log\Big(\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx\Big) \le 0$$

We can drop it and observe that since

$$\sup_{t>0} -2t\log t + t = M_1 < \infty,$$

we get

$$\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx \le \int_0^1 f(x)\log f(x)dx + M_1$$

which is an inequality of the type (4.9).

Second case: $\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx > 1$. By Young's inequality

$$ab \le (a + \frac{1}{2})\log(1+2a) - a + \frac{1}{2}(e^b - b - 1) \quad \forall a, b \ge 0$$

and therefore, setting

$$a = \int_0^1 f(t)dt \quad b = \log\Big(\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx\Big)$$

the following inequality holds

$$\begin{aligned}&\Big(\int_0^1 f(t)dt\Big)\log\Big(\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx\Big)\\ &\le \Big(\int_0^1 f(t)dt+\frac{1}{2}\Big)\log\Big(1+2\int_0^1 f(t)dt\Big)-\int_0^1 f(t)dt+\frac{1}{2}\int_0^1\Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx.\end{aligned}$$

Substituting this into our limit inequality we get

$$\begin{aligned}\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx &\le \int_0^1 f(x)\log f(x)dx - 2\Big(\int_0^1 f(t)dt\Big)\log\Big(\int_0^1 f(t)dt\Big)\\ &\quad+\int_0^1 f(t)dt+\Big(\int_0^1 f(t)dt+\frac{1}{2}\Big)\log\Big(1+2\int_0^1 f(t)dt\Big)\\ &\quad-\int_0^1 f(t)dt+\frac{1}{2}\int_0^1\Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx\end{aligned}$$

i.e.,

$$\begin{aligned}\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx &\le 2\int_0^1 f(x)\log f(x)dx - 4\Big(\int_0^1 f(t)dt\Big)\log\Big(\int_0^1 f(t)dt\Big)\\ &\quad+\Big(2\int_0^1 f(t)dt+1\Big)\log\Big(1+2\int_0^1 f(t)dt\Big).\end{aligned}$$

Finally, since $\sup_{t>0} -4t\log t+(2t+1)\log(1+2t) = M_2 < \infty$, we get

$$\int_0^1 \Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx \le 2\int_0^1 f(x)\log f(x)dx + M_2$$

which is exactly an inequality of the type (4.9).

Remark 5.2. The passages to the limit in p, similar to that one made above, are made in [17, 6.8, p. 139], where a logarithm appears after the limiting process. Much more recently, a similar passage to the limit has been fruitful when studying maximal functions and related weight classes, see [24]. In both cases the limit of the Lebesgue quasinorm in L^r has been studied when $r \to 0$. The appearance of the logarithm in a limit for $r \to 1$, like in our case, has been noted and used in [23]. Finally, let us recall that the same procedure has been used to derive the *LlogL* integrability of the Jacobian (see [18, (8.44) p. 186)].

5.3. **Sobolev inequalities for fractional Sobolev spaces, part II.** The problem of the "not natural" blowup of the norm of $W_0^{s,p}$, $p < N$, when $s \uparrow 1$ has been studied in [3, 4, 22]. The following relation, established in [4] for any $p \in [1,\infty[$, shows that the factor $(1-s)$ permits to compute exactly the limit

$$\lim_{s\uparrow 1}(1-s)\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy = c(p)\|\nabla f\|^p_{L^p(\Omega)}.$$

Here Ω stands for the cube $\Omega = \{x \in \mathbb{R}^N : |x_j| < 1/2, 1 \le i \le N\}$, and, as usual, we restrict our attention to the functions f in $\mathcal{C}^\infty_{0,+}(\Omega)$. Therefore, when $s \uparrow 1$, the limit of the embedding inequality, established in [3]

$$\|f\|_{L^q(\Omega)} \le c_1(N)^{1/p} \frac{(1-s)^{1/p}}{(N-sp)^{(p-1)/p}} \|f\|_{W_0^{s,p}(\Omega)} \tag{5.4}$$

where $p < N$, $0 < s < 1$, $q = Np/(N-sp)$, is

$$\|f\|_{L^{Np/(N-p)}(\Omega)} \le c_2(p,N)\|f\|_{W_0^{1,p}(\Omega)} \tag{5.5}$$

Let us set now $s = 1 - 1/n$ ($n > 2$) in (5.4) and consider

$$\begin{gathered} T_n f = \|f\|_{L^{nNp/(nN-p)}(\Omega)}, \\ S_n f = c_1(N)^{1/p} \frac{(1/n)^{1/p}}{(N-p/n)^{(p-1)/p}} \|f\|_{W_0^{1-1/n,p}(\Omega)}, \\ Tf = \|f\|_{L^{Np/(N-p)}(\Omega)}, \\ Sf = c_2(p,N)\|f\|_{W_0^{1,p}(\Omega)} \end{gathered}$$

The limit (in the sense of (5.1)) of (5.4) is (5.5).

6. A topology on Inequalities

The topology on $\mathcal{I}_0$ is not satisfactory for our purposes, since, for instance, in $\mathcal{I}_0$ the inequalities

$$\begin{gathered} Tf \le Sf \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega), \\ 2Tf \le 2Sf \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega) \end{gathered}$$

are *different* objects. We are going to build up an abstract setting in which we identify all equivalent inequalities, and we will consider a topology on these new objects. Of course the convergence proved in the previous Section will be preserved in this new setting.

6.1. **Equivalence of inequalities.**

Definition 6.1. Let $T_1, S_1, T_2, S_2 \in \mathcal{O}$ and let the inequalities

$$\begin{aligned} d_1 : &\quad T_1 f \le S_1 f \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega), \\ d_2 : &\quad T_2 f \le S_2 f \quad \forall f \in \mathcal{C}^\infty_{0,+}(\Omega) \end{aligned}$$

be given. We will say that d_1 is equivalent to d_2, and we will write

$$d_1 \sim d_2$$

if it is possible to deduce d_2 from d_1 or d_1 from d_2 by combining a finite number of the following two operations:

- There exists $W \in \mathcal{O}$ such that $T_2 f = T_1 f + Wf$ and $S_2 f = S_1 f + Wf$ for all $f \in \mathcal{C}^\infty_{0,+}(\Omega)$
- There exists Φ nonnegative, strictly increasing function on $[0,\infty[$ such that $T_2 f = \Phi(T_1 f)$, $S_2 f = \Phi(S_1 f)$ for all $f \in \mathcal{C}^\infty_{0,+}(\Omega)$

We immediately observe that such definition is well-posed, in fact it is trivial to prove the following

Proposition 6.2. *The notion of equivalence introduced in Definition 6.1 is reflexive, symmetric, transitive.*

6.2. **The quotient space.** Definition 6.1 leads naturally to consider classes of equivalent inequalities. Let us consider the set of inequalities

$$\mathcal{I}_0 = \{d(T,S) : T, S \in \mathcal{O}\}$$

and in such set we consider the classes of equivalence given by $\sim$:

$$\mathcal{I} = \frac{\mathcal{I}_0}{\sim}$$

An element of $\mathcal{I}$ will be denoted by $[d]$, which is the class of all inequalities $d_1 \in \mathcal{I}_0$ equivalent to the inequality $d \in \mathcal{I}_0$:

$$[d] = \{d_1 \in \mathcal{I}_0 : d \sim d_1\}$$

In order to remember that $[d]$ is not an inequality, but a class of inequalities, we will refer to it, in the sequel, as "Inequality". The notion of convergence of inequalities and the introduction of a topology for inequalities have much more sense when dealing with *Inequalities* rather than *inequalities*.

We introduce in $\mathcal{I}$ the topology of the quotient space $\mathcal{I}_0/\sim$ (see e.g. [9, p. 125]): if we call P the projection

$$P : \mathcal{I}_0 \ \rightarrow \ \mathcal{I} = \frac{\mathcal{I}_0}{\sim}$$

then

$$\mathcal{A} \subseteq \mathcal{I} \text{ is open in } \mathcal{I}$$

if and only if

$$P^{-1}[\mathcal{A}] = \cup\{A : A \in \mathcal{A}\} \text{ is open in } \mathcal{I}_0.$$

It is well-known that

$$d_n \rightarrow d \quad \text{in } \mathcal{I}_0 \ \Rightarrow \ [d_n] \rightarrow [d] \quad \text{in } \mathcal{I}$$

therefore the convergence already shown in the previous Section still hold in $\mathcal{I}$.

We conclude writing explicitly, in terms of the admissible operators in $\mathcal{O}$, what does it mean that $[d_n] \rightarrow [d]$ in $\mathcal{I}$. The following notion of convergence represents the answer we wanted to find to our original question, settled in the Introduction.

> Let $[d_n]$, $[d]$ in $\mathcal{I}$. Write $d = d(T,S)$, $d_n = d_n(T_n, S_n)$ $\forall n \in \mathbb{N}$. It is $[d_n] \rightarrow [d]$ in $\mathcal{I}$ if for any $d' = d'(T', S')$, $d' \sim d$, for any $n' = n'(T', S') \in \mathbb{N}$, for any $F' = F'(T', S') \subset \mathcal{C}^\infty_{0,+}(\Omega)$ finite, there exists $\nu \in \mathbb{N}$ such that for $n > \nu$ the following holds:

$$\forall\, d'_n(T'_n, S'_n) \sim d_n(T_n, S_n)\ \exists d'(T', S') \sim d(T,S) : d'_n(T'_n, S'_n) \in \mathcal{U}_{n', F'}(d'(T', S'))$$

6.3. **The three main examples.** We now make a few comments on the examples discussed in Sections 5.1, 5.2, and 5.3. In the first case, the sequence of inequalities was a sequence of relations between norms, therefore, the step made in this last Section has no a relevant meaning. In the other two cases, we changed, for convenience, the sequences of inequalities: in the case of Hardy's inequality, we preferred to deal with (5.3) rather than (2.3); in the case of the Sobolev inequalities for fractional Sobolev spaces, we dealt with inequality (5.4), and we used a relation of limit involving the right norms, up to the factor $(1-s)$. It is trivial that such transformations (to raise the inequalities to a certain power, and to multiply the inequalities by a constant) can be done, giving of course *equivalent* inequalities. But without this last step (the construction of a topology on Inequalities, made in this Section 6), the trivial transformations would lead to $different$ inequalities, and this would be not natural for the problem we wished to study.

7. Computing limits

In this last Section we wish to provide some tools to compute explicitly the limits of some Inequalities. Some of them have been implicitly proved or used in the previous Sections, others come as a byproduct from Function Space Theory. We stress here that the novelty of the limits we are going to show is not in the difficulty of the computations, but in the new light given by our construction: more or less common "passages to the limit" are in fact concrete limits in a suitable topology. We will conclude the Section giving two applications, which show how the construction of the topology leads to the proof of new results.

7.1. **Some basic tools.** Given a sequence of true inequalities, it is evident that the explicit computation of the limit must be carried out by passing through equivalent inequalities (namely, the operations described in the Definition 6.1), and by computing the limits of the left hand side and the right hand side. Therefore the basic tools rely upon the study of sequences of admissible operators, rather than the inequalities themselves. Moreover, we observe that since the admissible operators have real values, the standard theorems on operation of limits (for instance, the limit of a sum, of a product, the composition with a continuous nonnegative real function) can be applied. We are going to show some first "bricks" that can be used in applications. For our purposes it will be sufficient to confine ourselves to functions defined in domains having measure 1.

The first tool, which has been already used in Sections 5.2 and 5.3, is completely standard, and it can be found in [17], n. 194, p. 143.

Proposition 7.1. *Let $0 < p$, $p_0 < \infty$. Then*

$$\Big(\int_0^1 f(t)^p dt\Big)^{1/p} \to \Big(\int_0^1 f(t)^{p_0} dt\Big)^{1/p_0} \quad \text{as } p \to p_0 \quad \forall f \in \mathcal{C}^\infty_{0,+}(0,1)$$

For completeness, we state here the case when $p_0 = 0$ (we refer to [17, n. 187, p. 139], and [17, Section 6.7] for the exact meaning of the limit expression).

Proposition 7.2. *Let $0 < p < \infty$. Then*

$$\Big(\int_0^1 f(t)^p dt\Big)^{1/p} \to \exp\Big(\int_0^1 \log f(t) dt\Big) \quad \text{as } p \to 0 \quad \forall f \in \mathcal{C}^\infty_{0,+}(0,1)$$

Now let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain and let $1 \le p < \infty$. Since by the Poincaré's inequality the expression $\||\nabla f|\|_{L^p(\Omega)}$ is equivalent to the norm $\|f\|_{W_0^{1,p}(\Omega)}$ of the Sobolev space $W_0^{1,p}(\Omega)$, from the Proposition 7.1 we get immediately that

Proposition 7.3. *Let $1 \le p$, $p_0 < \infty$. Then*

$$\|f\|_{W_0^{1,p}(\Omega)} \to \|f\|_{W_0^{1,p_0}(\Omega)} \quad \text{as } p \to p_0 \quad \forall f \in \mathcal{C}^\infty_{0,+}(0,1)$$

We consider now other two convergences, for which the deduction of the limit is less trivial. Since our goal is just to provide some tools for explicit computations, we omit the corresponding statements for Sobolev functions. Both of them can be deduced by a standard computation (it suffices to take into account that the limit to be computed is of a difference-quotient). The first one is used in [18, Section 8.6], and [17, Section 6.8, p.139].

Proposition 7.4. *Let $0 < p < \infty$. Then*

$$\frac{1}{p}\Big(\int_0^1 f(t)^p dt - 1\Big) \to \int_0^1 \log f(t)dt \quad \text{as } p \to 0 \quad \forall f \in \mathcal{C}^\infty_{0,+}(0,1)$$

The second convergence result will be stated for the parameter p approaching p_0 from the right, because the expression involved, if $p > p_0$, is an admissible operator in the sense of Section 4, by virtue of the classical Hölder's inequality.

Proposition 7.5. *Let $1 \le p_0 < p < \infty$. Then*

$$\frac{\|f\|_{L^p(\Omega)} - \|f\|_{L^{p_0}(\Omega)}}{p - p_0} \to \frac{1}{p_0}\Big(\frac{\|f^{p_0}\log f\|_{L^1(\Omega)}}{\|f\|^{p_0-1}_{L^{p_0}(\Omega)}} - \|f\|_{L^{p_0}(\Omega)}\log\|f\|_{L^{p_0}(\Omega)}\Big)$$

as $p \to p_0+$, for all $f \in \mathcal{C}^\infty_{0,+}(\Omega)$.

We conclude this subsection stating explicitly the following result, proved and used in Section 5.2.

Proposition 7.6. *Let $1 < p < \infty$. Then*

$$\frac{\|f\|_{L^p(\Omega)} - \|f\|^p_{L^1(\Omega)}}{p-1} \to \|f\log f\|_{L^1(\Omega)} - 2\|f\|_{L^1(\Omega)}\log\|f\|_{L^1(\Omega)}$$

as $p \to 1$ for all $f \in \mathcal{C}^\infty_{0,+}(\Omega)$.

7.2. **Application 1: A refinement of the endpoint Hardy's inequality.** In this section we highlight the following result, obtained in Section 5.2. It is an inequality, involved in the limit of the Hardy's inequalities when the exponent goes to 1, which has been shown to be sharper than the classical one. In order to give a meaning to the right hand side, we recall that the expression $f(x)\log f(x)$ (which is an integrand in the right hand side) has to be understood equal to zero whenever $f(x) = 0$.

Proposition 7.7. *The following inequality holds for every nonnegative, measurable function f on $(0,1)$:*

$$\begin{aligned}\int_0^1\Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx &\le \int_0^1 f(x)\log f(x)dx - 2\Big(\int_0^1 f(t)dt\Big)\log\Big(\int_0^1 f(t)dt\Big)\\ &\quad + \int_0^1 f(t)dt + \Big(\int_0^1 f(t)dt\Big)\log\Big(\int_0^1\Big(\frac{1}{x}\int_0^x f(t)dt\Big)dx\Big).\end{aligned}$$

7.3. **Application 2: Misproving inequalities without counterexamples.** We present here a new application of our limits of inequalities, in which the so-called small Lebesgue spaces are involved. Such spaces, introduced by the second author in [10] and then studied in [6], turned out to be useful in Sobolev-type estimates in borderline cases and in questions of regularity for quasilinear equations (see [13]). Our main goal now is to discuss an inequality, close to the classical multiplicative embedding inequality for Sobolev functions (see [8], chap. IX, n. 1, p. 423), which is in a class of inequalities useful for variational problems with critical exponent (see the recent paper [14]). A complete picture of this class of inequalities is in the paper [15], announced in [14].

Let Ω be a bounded, open, connected, smooth set in $\mathbb{R}^n$, $n > 1$, and let $u \in \mathcal{C}^\infty_{0,+}(\Omega)$. Let q, r, p be such that

$$1 \le r < n \quad 1 < p \le q < \frac{nr}{n-r}$$

and set

$$a = \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{n} - \frac{1}{r} + \frac{1}{p}}$$

Let us consider the inequality

$$\|u\|_{(q} \le \beta \||\nabla u|\|_{(r}^{a} \|u\|_{p}^{1-a} \tag{I_a}$$

where the symbol $\|u\|_p$ denotes the usual norm in the Lebesgue space $L^p(\Omega)$ and $\|\cdot\|_{(q}$ stands for the norm of the small Lebesgue space $L^{(q}(\Omega)$ (see e.g. [6] and references therein).

The problem is to know whether such inequality, when $0 \le a \le 1$, is true or not, in the sense that there exists a constant β, independent of $u \in \mathcal{C}^{\infty}_{0,+}(\Omega)$, possibly depending on q, r, p, n, Ω, such that (I_a) holds. We begin our discussion observing that the inequality (I_a) is false when $a = 0$, true when $a = 1$: in fact in the first case it is $p = q$, and the inequality reduces to

$$\|u\|_{(p} \le \beta \|u\|_p \tag{7.1}$$

and this is false, because the embedding $L^p(\Omega) \subset L^{(p}(\Omega)$ does not hold (see e.g. [10]); on the other hand, when $a = 1$, the inequality reduces to

$$\|u\|_{(q} \le \beta \||\nabla u|\|_{(r}$$

and this is true in view of the classical Sobolev embedding theorem:

$$\|u\|_{(q} \le \beta_1 \|u\|_{nr/n-r} \le \beta_2 \||\nabla u|\|_r \le \beta_3 \||\nabla u|\|_{(r}$$

(here β_i, $i = 1, 2, 3$ are independent of u).

The question in the cases $0 < a < 1$ will be analyzed, also in a much more general framework, in [15], where other applications involving PDEs will be given. Here we will show - by using a simple argument based on our limit processes - that for a sufficiently small the inequality (I_a) cannot hold with a uniform bound β. The importance of this fact is immediately understood after noticing that in the classical case

$$\|u\|_q \le \beta \||\nabla u|\|_r^a \|u\|_p^{1-a}$$

a uniform bound for β exists (with respect to a; see e.g. [21]).

We are now ready to prove the following result.

Proposition 7.8. *There exists a_0, $0 < a_0 < 1$, such that (I_a) cannot hold with a bound β uniform with respect to $0 \le a < a_0$.*

Proof. We make an argument by contradiction. Suppose, on the contrary, that there exists a sequence $(a_n)_{n\in\mathbb{N}}$, $0 < a_n < 1$, $a_n \to 0$, such that (I_{a_n}) is true for every n, for some $\beta > 0$. Fix

$$1 \le r < n \quad 1 < q < \frac{nr}{n-r}$$

and define p_n by

$$a_n = \frac{\frac{1}{p_n} - \frac{1}{q}}{\frac{1}{n} - \frac{1}{r} + \frac{1}{p_n}}$$

Set $T_n u = \|u\|_{(q}$, $S_n u = \||\nabla u|\|_{(r}^{a_n} \|u\|_p^{1-a_n}$. We observe that both T_n, S_n are admissible operators in the sense of Section 4. The limit of $d_n = d(T_n, S_n)$ is, in the sense of (5.1), the inequality (7.1), which is false. The proposition is therefore proven. $\square$

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