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TRAVELLING WAVE FOR ABSORPTION-CONVECTION-DIFFUSION EQUATIONS

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Abstract. In this paper, we use the phase plane method for finding finite travelling waves solutions for the diffusion-absorption-convection equation

$$u_t = A(|u_x|^{p-2}u_x)_x + B(u^n)_x - Cu^q, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

We show that the existence of solutions which depends on the parameters p, q and n. Also we study the asymptotic behavior of these solutions.

1. Introduction

This work concerns the nonlinear parabolic equation

$$U_t = A\Delta_p U + B(U^n)_x - CU^q, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \tag{1.1}$$

with $\Delta_p U = (|U_x|^{p-2}U_x)_x$, $A, B, C, q > 0$, $p > 2$ and $1 \neq n > 0$. Equation (1.1) is usually called a convection absorption diffusion equation. It is a simple and widely used model for various physical, chemical problems involving diffusion with absorption and with convection. Note that the operator of diffusion Δ_p (the p-laplacian) arises from a variety of physical phenomena. It is used in Non-Newtonian fluids and also appears in nonlinear elasticity, glaciology and petroleum extraction (see [2, 3, 4, 5]).

Our main interest is finite travelling waves solutions of (1.1). For $p = 2$, the travelling waves have been widely studied since the paper [10] was published (see also [8]). It is worth mentioning that, if we change $\Delta_p U$ by $(U^m)_{xx}$, we get the porous medium equation, for which the travelling waves have been studied with a term of diffusion, or convection or both at the same time ([6, 7, 9]). We know that for $p > 2$ the operator Δ_p have a finite propagation property. The introduction of absorption term and convection term may have a deep influence on the qualitative behavior of the solutions of (1.1). To clear up this phenomena we will study a particular family of solutions. More precisely, we investigate the existence and the asymptotic behavior of finite travelling waves for variants values of $A > 0, B > 0$ and $C > 0$. By a simple rescaling we may put $A = 1$ and $B = \frac{1}{n}$. The formulae will be much simpler. In fact we take

$$V(x,t) = \alpha U(\beta x, \gamma t) \tag{1.2}$$

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with

$$\alpha = 1, \quad \beta = (nB)^{-\frac{1}{p-1}}, \quad \gamma = A^{-\frac{1}{p-1}}(nB)^{-\frac{p}{p-1}} \tag{1.3}$$

Hence V satisfies the diffusion absorption convection equation

$$V_t = (|V_x|^{p-2}V_x)_x + \frac{1}{n}(V^n)_x - dV^q, \tag{1.4}$$

where $d = \gamma C$. For simplicity of notation we take $d = 1$. By a local finite travelling wave solution with velocity $c \in \mathbb{R}$, we mean a solution $w(x,t)$ of (1.4) in $I \times \mathbb{R}^+$ (for some interval $I =]-\infty, \xi_\infty[, \xi_\infty > 0$) of the form

$$w(x,t) = \varphi(ct - x) = \varphi(\xi) \tag{1.5}$$

with $\varphi : I \to \mathbb{R}^+$ such that φ is strictly positive in $]0,\infty[\cap I$ and vanishing in $]-\infty, 0]$. If $I = \mathbb{R}$, we say that φ is a finite travelling wave solution (finite travelling wave).

Before stating the main results of this paper, let us introduce some useful notations. For any real r we set M_r (resp. A_r) the solution of the algebra equation $M^{-\frac{1}{p-1}} - M + r = 0$ (resp .$M^{-\frac{1}{p-1}} - rM + 1 = 0$). Define,

$$Q = \frac{(q+1)(p-1)}{p}, \tag{1.6}$$

$$L = \max\{n, Q\}, \quad l = \min\{n, Q\}, \tag{1.7}$$

$$M_0 = \begin{cases} M_c & \text{if } l = Q = 1, \\ \frac{1}{n}, & \text{if } l = n \neq Q, \\ A_n, & \text{if } n = Q = l, \\ Q^{(1-p)/p} & \text{if } l = Q \neq n, 1, \end{cases} \tag{1.8}$$

$$M_\infty = \begin{cases} M_c & \text{if } L = Q = 1, \\ \frac{1}{n}, & \text{if } L = n \neq Q, \\ A_n, & \text{if } L = n = Q, \\ Q^{(1-p)/p} & \text{if } L = Q \neq n, 1. \end{cases} \tag{1.9}$$

The main results are follows.

Theorem 1.1. *Let $c \in \mathbb{R}^\star$ $(c \neq 0)$. Equation* (1.4) *admits a finite travelling wave solution with velocity c if and only if one of the following conditions is satisfied :*

- (i) $c > 0$, $L \le p - 1$;
- (ii) $c < 0$, $l \le 1 \le L \le p - 1$;
- (iii) $c < 0$, $L < 1$, $q \le 1$;
- (iv) $c < 0$, $l > 1$, $q < 1$.

Theorem 1.2. *Let φ a finite travelling wave solution at some velocity $c \neq 0$ of* (1.4). *Then $\xi^{-\alpha}\varphi(\xi)$ converges to constant $K = \alpha^{-\alpha}M^{\alpha/p-1}$, when ξ approaches* 0*, with :*

- (i) *For $c > 0$ or $c(l-1) \ge 0$, $\alpha = \frac{p-1}{p - inf\{n,Q,1\} - 1}$ and $M = M_0$.*
- (ii) *For $c < 0$ and $l > 1$, $\alpha = \frac{1}{1-q}$ and $M = (-c)^{1-p}$.*

For the asymptotic behavior when ξ approaches ∞, there is the following result:

Theorem 1.3. *Let φ a finite travelling wave solution of* (1.4) *at some velocity $c \neq 0$.*

(i) *For $c > 0$ or $c\ (L-1) \geq 0$, we have: If $L < p-1$, $\xi^{-\alpha}\varphi(\xi)$ converges to $K = \alpha^{-\alpha} M^{\alpha/(p-1)}$, when ξ approaches ∞, with $\alpha = \frac{p-1}{p-sup\{n,Q,1\}-1}$ and $M = M_\infty$. If $L = p-1, \xi^{-1}\log(\varphi(\xi))$ converges to $K = M_\infty^{1/(p-1)}$, when ξ approaches ∞.*

(ii) *For $c < 0$ and $L < 1$, $\xi^{-1/(1-q)}\varphi(\xi)$ converges to $K = (\frac{1-q}{-c})^{1/(1-q)}$, when ξ approaches ∞.*

2. Preliminary results and the proof of the theorems 1.1, 1.2 and 1.3

We consider (1.4) with $p > 2$, $q > 0$ and $1 \neq n > 0$. Substituting $u(x,t) = \varphi(ct-x)$ in (1.4) we get

$$\big(|\varphi'|^{p-2}\varphi'\big)' - c\varphi' - \frac{1}{n}(\varphi^n)' - \varphi^q = 0 \quad \text{in } \mathbb{R}, \tag{2.1}$$

If φ vanishes for $\xi \leq \xi_0$, by translation we may put $\xi_0 = 0$. Integrating (2.1) we obtain $\varphi'(0) = 0$. Consequently, our problem can reformulated as finding a real c and a function φ such that

$$\big(|\varphi'|^{p-2}\varphi'\big)' - c\varphi' - \frac{1}{n}(\varphi^n)' - \varphi^q = 0 \quad \text{in } \mathbb{R}, \tag{2.2}$$

$$\varphi(0) = 0, \varphi'(0) = 0. \tag{2.3}$$

We start with the following lemma which gives us a monotonicity property of a finite travelling wave solutions .

Lemma 2.1. *Let $c \in \mathbb{R}^*$ and φ a local finite travelling wave solution of* (1.4) *with velocity c. Then φ is increasing on $]0,+\infty[$.*

Proof. The proof is divided into two steps.
Step 1. $c > 0$. Let E the energy function defined by

$$E(\xi) = \frac{p-1}{p}|\varphi'(\xi)|^p - \frac{1}{q+1}\varphi^{q+1}(\xi) \tag{2.4}$$

Then E satisfies

$$E'(\xi) = [c + \varphi^{n-1}(\xi)]\varphi'^2(\xi) \tag{2.5}$$

Hence E is increasing. Assume φ is not increasing on $]0,+\infty[$ and let ξ_1 the first zero of φ'. Therefore

$$E(\xi_1) = -\frac{1}{q+1}\varphi^{q+1}(\xi_1) \geq E(0) = 0 \tag{2.6}$$

which a contradiction and then $\varphi'(\xi) > 0$, for any $\xi > 0$.
Step 2. $c < 0$. From the definition of finite travelling wave, there exists some $\xi_0 > 0$ such that φ is strictly increasing on $[0,\xi_0[$. Assume that ξ_0 is a local maximum, then $(|\varphi'|^{p-2}\varphi')'(\xi_0) \leq 0$. But from the equation satisfied by φ we get $(|\varphi'|^{p-2}\varphi')'(\xi_0) > 0$, which a contradiction. Consequently for any $c \in \mathbb{R}^*, \varphi'(\xi) \geq 0$. □

In the sequel we analyze the corresponding phase portrait of the O.D.E system associated to problem (2.2)–(2.3). Hence we introduce the following change variables

$$X = \varphi \quad \text{and} \quad Y = (\varphi')^{p-1} \tag{2.7}$$

and then (2.2)-(2.3) is equivalent to the system of O.D.E

$$\begin{gathered} X' = Y^{\frac{1}{p-1}} \\ Y' = X^q + (c + X^{n-1})Y^{\frac{1}{p-1}} \\ (X(0), Y(0)) = (0,0) \end{gathered} \tag{2.8}$$

In order to solve the above, we write the first O.D.E for the trajectories:

$$\frac{dY}{dX} = c + X^{n-1} + X^q Y^{-\frac{1}{p-1}}. \tag{2.9}$$

and consequently, we consider the problem: Finding the non trivial trajectories (X, Y) solutions of

$$\begin{gathered} \frac{dY}{dX} = c + X^{n-1} + X^q Y^{-\frac{1}{p-1}} \\ Y(0) = 0\,. \end{gathered} \tag{2.10}$$

We start with the following result.

Proposition 2.2. *For any $c \in \mathbb{R}^*$, problem* (2.10) *has a unique global solution.*

Here the fixed-point does not work. So we use perturbation methods. We consider the following approximation problem

$$\begin{gathered} \frac{dY}{dX} = c + X^{n-1} + X^q Y^{-\frac{1}{p-1}} = F(X, Y) \\ Y(0) = \varepsilon. \end{gathered} \tag{2.11}$$

for any $\varepsilon > 0$.

Lemma 2.3. *For any $\varepsilon > 0$, problem* (2.11) *has a unique global solution.*

Proof. As the function $(X, Y) \to F(X, Y)$ is locally Lipschitzienne continuous function in $\mathbb{R}^+ \times [\varepsilon, +\infty[$, we deduce from the theory of O.D.E (see for example [1]) the existence of unique local solution Y_ε of (Q_ε). First, we remark that if $c > 0$, the function $X \to Y_\varepsilon(X)$ is strictly increasing and satisfies the following inequality

$$\frac{dY_\varepsilon}{dX} \le c + X^{n-1} + X^q \varepsilon^{-\frac{1}{p-1}} \tag{2.12}$$

and thereby Y_ε is global solution. On the other hand if $c < 0$, introduce the curve (C) solution of the equation $F(X, Y) = 0$. It is given explicitly by

$$\widetilde{Y}(X) = \Big(\frac{-X^q}{c + X^{n-1}}\Big)^{1/(p-1)}, \tag{2.13}$$

with $0 \le X \le (-c)^{\frac{1}{n-1}}\ if\ n > 1$ and $X > (-c)^{1/(n-1)} if n < 1$.

The curve (C) divide the plane into two regions: in the first R_1 we have $F(X, Y) < 0$ while in the second part (say R_2), $F(X, Y) > 0$; see figure 1).

For $n > 1$, Y_ε starts in the region R_1 and $Y_\varepsilon(0) = \varepsilon > 0$, then Y_ε must cross the curve (C) in some point with horizontal tangent and after Y_ε lies in the region R_2, where Y_ε is strictly increasing. Hence the minimum m_ε of Y_ε reaches on (C) and strictly positive. But, for $n < 1$, Y_ε starts in the region R_2 and $Y_\varepsilon(0) = \varepsilon > 0$, then Y_ε remains always increasing or it must cross the curve (C) in some point with horizontal tangent and after Y_ε lies in the region R_1, where Y_ε is strictly decreasing,

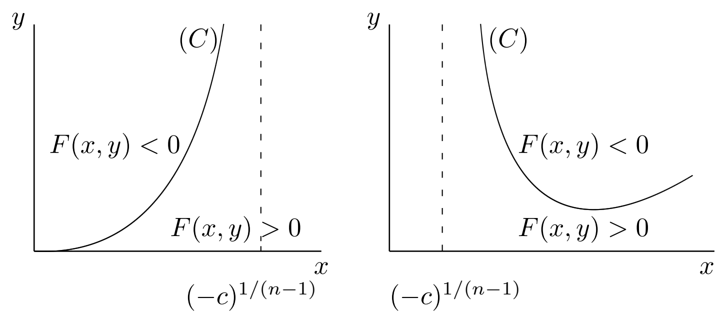

FIGURE 1. Curve C: Case $n > 1$ (left). Case $n < 1$ (right)

but if it leaves R_1, it becomes increasing. As $lim_{X\to\infty}\widetilde{Y}(X) = \infty$ then also in this case the minimum m_ε of Y_ε is strictly positive. So

$$\frac{dY_\varepsilon}{dX} \le c + X^{n-1} + X^q m_\varepsilon^{-\frac{1}{p-1}} \tag{2.14}$$

and thereby Y_ε is a global solution. This completes the proof of the lemma. □

Now we consider the Cauchy problem

$$\begin{gathered} \frac{dZ_\varepsilon}{dX} = c + X^{n-1} + X^q Z_\varepsilon^{-\frac{1}{p-1}} \\ Z_\varepsilon(\varepsilon) = 0\,. \end{gathered} \tag{2.15}$$

Lemma 2.4. *For any $\varepsilon > 0$, problem* (2.15) *has a global solution.*

Proof. We consider the Cauchy problem

$$\begin{gathered} \frac{du}{dt} = \frac{t^{\frac{1}{p-1}}}{u^q(t) + [c + u^{n-1}(t)]t^{\frac{1}{p-1}}} = \frac{1}{F(u,t)} = g(t,u) \\ u(0) = \varepsilon \end{gathered} \tag{2.16}$$

It is easy to see that the above problem has a unique local solution u_ε. In fact the local existence of u_ε follows easily from the theory of O.D.E [1]. In a first place, we suppose that $c > 0$ or $c < 0$ and $n > 1$. If $c > 0$, we have

$$0 \le \frac{du_\varepsilon}{dt} \le \frac{1}{c + u^{n-1}} \le \frac{1}{c + \varepsilon^{n-1}} \tag{2.17}$$

and thereby $u_\varepsilon$is global. But if $c < 0$ and $n > 1$, we note by C' the curve $F(u,t) = 0$, which is exactly symmetrical with C (introduce in the proof of lemma 2.3) compared to axis, $t = u$ (see Figure 2).

Then C' divide the plane into two parts: In the first part $\frac{du_\varepsilon}{dt}$ is strictly positive and approaches $+\infty$ when $F(t, u_\varepsilon)$ approaches 0; that it is $(t, u_\varepsilon(t))$ draw near to the curve C'. Consequently u_ε is strictly increasing and does never touch the curve C'. Therefore, u_ε is global. On another side, as u_ϵ is increasing then $\lim_{t\to+\infty} u_\varepsilon(t) = l$ exists in $]0, +\infty[$. If l is finite then $\lim_{t\to+\infty} \frac{du_\varepsilon}{dt} = 0$, but from (2.16), we get $\lim_{t\to+\infty} \frac{du_\varepsilon}{dt} = \frac{1}{l^{n-1}+c}(> 0)$, which a contradiction and thereby $\lim_{t\to+\infty} u_\varepsilon(t) = +\infty$. Consequently u_ε is a one to one from $[0, +\infty[$ to $[\varepsilon, +\infty[$. Now set $Z_\varepsilon(= u_\varepsilon^{-1})$

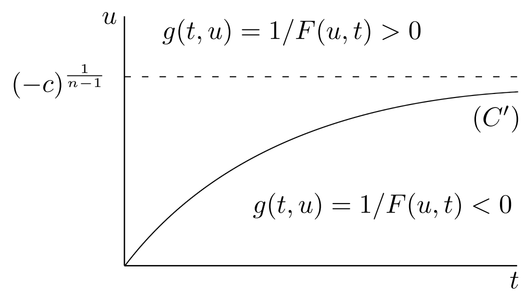

FIGURE 2. Curve C': Case $n > 1$

the inverse function of u_ε defined from $[\varepsilon, +\infty[$ to $[0, +\infty[$. By a simple computation we see that Z_ε satisfies the following Cauchy problem

$$\begin{gathered} \frac{dZ_\varepsilon}{dX} = c + X^{n-1} + X^q Z_\varepsilon^{-\frac{1}{p-1}} \\ Z_\varepsilon(\varepsilon) = 0 \end{gathered} \tag{2.18}$$

On the other hand, we suppose that $c < 0$ and $n < 1$. Since $\frac{du_\varepsilon}{dt} \simeq \varepsilon^{-q} t^{1/(p-1)}$, at neighborhood 0, then there is $t_\varepsilon > 0$ such as u_ε is a one to one from $[0, t_\varepsilon[$ to $[\varepsilon, u_\varepsilon(t_\varepsilon)[$. We take $Z_\varepsilon (= u_\varepsilon^{-1})$ the inverse function of u_ε defined from $[\varepsilon, u_\varepsilon(t_\varepsilon)[$ to $[0, t_\varepsilon[$. By a simple calculation, we obtain Z_ε satisfies, in $[\varepsilon, u_\varepsilon(t_\varepsilon)[$, the Cauchy problem

$$\begin{gathered} \frac{dZ_\varepsilon}{dX} = c + X^{n-1} + X^q Z_\varepsilon^{-\frac{1}{p-1}} \\ Z_\varepsilon(\varepsilon) = 0 \end{gathered} \tag{2.19}$$

With an aim of prolonging solution Z, one considers

$$\begin{gathered} \frac{dZ_\varepsilon}{dX} = c + X^{n-1} + X^q Z_\varepsilon^{-\frac{1}{p-1}} \\ Z_\varepsilon(u(t_\varepsilon)) = t_\varepsilon > 0 \end{gathered} \tag{2.20}$$

By employing the same technique that we used in lemma 2.3 one obtains that (2.20) admits a solution on $[u_\varepsilon(t_\varepsilon), \infty[$. It is deduced that the problem (D_ε) admits a global solution. □

Proof of proposition (2.2). The proof is divided into two steps
Step 1: Uniqueness. Assume that there exists two solutions Y and Z of (2.10) such that $Y \neq Z$. Define the real R by

$$R = \sup\{r > 0; Z(X) = Y(X), \text{ for } 0 \le X < r\} \tag{2.21}$$

and we take X_0 close to R, such that $X_0 > R$. Then without loss of generality we can assume

$$\begin{gathered} Y(X) = Z(X) \quad \text{in } [0, R[\\ Y(X_0) > Z(X_0) \end{gathered} \tag{2.22}$$

Set $f(X) = (Y - Z)(X)$, then there exists some real $\theta \in]R, X_0[$ such that

$$0 \le f(X_0) - f(R) = \theta^q [Y^{-\frac{1}{p-1}}(\theta) - Z^{-\frac{1}{p-1}}(\theta)] < 0 \tag{2.23}$$

which gives a contradiction. Consequently $Y = Z$.

Step 2: Existence. Let (Y_ε) the solution of (2.11). As $\varepsilon \to Y_\varepsilon$ is increasing and positive, then $Y_\varepsilon(x)$ converges to some function $Y(X) = lim_{\varepsilon\to 0} Y_\varepsilon(X) \geq 0, X \in]0,+\infty[$ with $Y_\varepsilon(0) = \varepsilon \to Y(0) = 0$. In order to prove that Y is the solution of (2.10), we start with the followings claims.

Claim 1. $Y(x)$ is strictly positive for any $x > 0$. In fact, Since Z_ε and Y_ε satisfy the same equation on $]\varepsilon,+\infty[$ and $Y_\varepsilon(\varepsilon) > Z_\varepsilon(\varepsilon) = 0$, $Y_\varepsilon(X) > Z_\varepsilon(X) > 0$ for any $X \in]\varepsilon,+\infty[$. Now, take some $X_0 \in]0,+\infty[$ and using the fact that $(Z_\varepsilon)_{\varepsilon>0}$ is a decreasing sequence we get

$$\lim_{\varepsilon\to 0} Y_\varepsilon(X_0) \geq \lim_{\varepsilon\to 0} Z_\varepsilon(X_0) \geq Z_{\frac{X_0}{2}}(X_0) > 0 \tag{2.24}$$

and consequently Y is strictly positive on $]0,+\infty[$.

Claim 2. The function Y is a solution of problem (2.10). In fact, since Y_ε is the solution of (2.11), then for any test function $\Phi \in D(]0,+\infty[)$,

$$\int_0^{+\infty} \Phi(X)\left\{c + X^{n-1} + X^q Y_\varepsilon^{-\frac{1}{p-1}}(X)\right\} dX + \int_0^{+\infty} Y_\varepsilon(X)\Phi'(X)dX = 0 \tag{2.25}$$

When ε approaches 0,

$$\frac{dY}{dX} = c + X^{n-1} + X^q Y^{-\frac{1}{p-1}} \text{ in } D'(]0,+\infty[) \tag{2.26}$$

Then for any $0 < a < b$, we deduce $\int_a^b |\frac{dY}{dX}|dX$ is finite (because $0 < Y(a) < Y(b)$ for any $x \in]a,b[$), therefore $Y \in W^{1,n}(]a,b[), \forall n \in \mathbb{N} - \{0\}$. So, Y and $\frac{dY}{dX}$ are continuous in $]a,b[$. Consequently, (2.26) holds in the usual sense in $]0,\infty[$. □

Lemma 2.5. *let Y the solution of problem* (2.10). *For any $A > 0$, the problem*

$$\begin{gathered} \frac{d\varphi}{d\xi}(\xi) = Y^{\frac{1}{p-1}}(\varphi(\xi)) \\ \varphi(0) = A \end{gathered} \tag{2.27}$$

has a unique maximal solution defined in $]-\infty,\beta[$, where $\beta \in \overline{\mathbb{R}}$. Moreover

$$\lim_{\xi\to-\infty} \varphi(\xi) = 0 \text{ and } \lim_{\xi\to\beta^-} \varphi(\xi) = +\infty \tag{2.28}$$

Proof. Since Y is regular and non-negative in $]0,+\infty[$, there exists a maximal solution φ on some interval $]\alpha,\beta[$. Moreover since φ is positive and increasing in $]\alpha,\beta[$, $\lim_{\xi\to\alpha^+} \varphi(\xi) = l$ exists and $l \geq 0$. If $l = 0$ we get $\lim_{\xi\to\alpha^+} \frac{d\varphi}{d\xi}(\xi) = 0$ and thereby if α is finite we can prolong solution φ by 0 on $]-\infty,\alpha]$, what contradicts the fact that $]\alpha,\beta[$ is a maximum interval, thereby $\alpha = -\infty$. While if $l > 0$ we remark that the Cauchy problem

$$\begin{gathered} \frac{d\varphi(\xi)}{d\xi} = Y^{\frac{1}{p-1}}(\varphi(\xi)) \\ \varphi(\alpha) = l > 0 \end{gathered} \tag{2.29}$$

has a unique local solution around β and then inevitably $\alpha = -\infty$. We put $l = lim_{\xi\to-\infty}\varphi(\xi)$, employing (2.27) we have $l = 0$. On the other hand, as φ is strictly increasing we obtain $\lim_{\xi\to\beta^-} \varphi(\xi) = +\infty$ if β is finite; while if $\beta = +\infty$, it is easy enough to use (2.27) to get also $\lim_{\xi\to\beta^-} \varphi(\xi) = +\infty$. □

Remark 2.6. Any solution φ of (2.27) defined in $]-\infty,\beta[$ satisfies

$$\left(|\varphi'|^{p-2}\varphi'\right)'(\xi) - c\varphi' - \frac{1}{n}(\varphi^n)' - \varphi^q(\xi) = 0 \quad \text{in }]-\infty,\beta[. \tag{2.30}$$

Among the solutions of (2.27) one will seek the global solutions which satisfied $\varphi(0) = 0$. For that one needs the study of the asymptotic behavior of solutions $Y(X)$ of the problem (2.10) checked by the vector field.

Proposition 2.7. *Assume $c \in \mathbb{R}^*, q > 0$ and $1 \neq n > 0$. Let Y the solution of* (2.10). *Then Y have the following behavior:*

(a) *When X approaches* 0*:*
(i) If $c > 0$ or $c(l-1) \geq 0$ then $Y(X) \approx MX^l$, with $M = M_0$.
(ii) If $c < 0$ and $l > 1$ then $Y(X) \approx (-c)^{1-p}X^{q(p-1)}$

(b) *When X approaches ∞:*
(i) If $c > 0$ or $c(L-1) \leq 0$ then $Y(X) \approx MX^L$, with $M = M_\infty$.
(ii) If $c < 0$ and $L < 1$ then $Y(X) \approx (-c)^{1-p}X^{q(p-1)}$.

Proof. We consider $H(X) = MX^\theta$, where $M > 0$ and $\theta > 0$. It is easy to see that H is a super-solution of (2.10) (resp. sub-solution) if and only if

$$\theta M \geq cX^{1-\theta} + X^{n-\theta} + M^{-1/(p-1)}X^{p(Q-\theta)/(p-1)}, \tag{2.31}$$

respectively

$$\theta M \leq cX^{1-\theta} + X^{n-\theta} + M^{\frac{-1}{p-1}}X^{p(Q-\theta)/(p-1)}, \tag{2.32}$$

We start with the asymptotic behavior at the neighborhood of 0, in fact we have two cases:
(a) $c > 0$ or $(l-1)c \geq 0$. In order to have H satisfied (2.31) (resp.(2.32)) at the neighborhood 0, we must have $l \geq \theta$ (resp. $l \leq \theta$). Let $\theta = l$. Then H is a super-solution of (2.10) (resp. sub-solution) for all $M > M_0$ (resp. $M < M_0$). Consequently $Y(X) \approx M_0X^l$.
(b) $c < 0$ and $l > 1$. We take in this case $\theta = q(p-1)$ then (2.31) (resp.(2.32)) becomes

$$\theta M \geq [c + M^{-1/(p-1)} + X^{n-1}]X^{1-q(p-1)}, \tag{2.33}$$

respectively

$$\theta M \leq [c + M^{-1/(p-1)} + X^{n-1}]X^{1-q(p-1)}, \tag{2.34}$$

Since $Q > 1$, we have $1-q(p-1) < 0$, this gives that (2.33) (resp. (2.34)) is checked for all $M \geq (-c)^{1-p}$ (resp. $M < (-c)^{1-p}$), from where $Y(X) \approx (-c)^{1-p}X^{q(p-1)}$.

Now, we pass to the behavior at neighborhood of ∞, we distinguish two cases:
(a) $c > 0$ or $(L-1)c < 0$. We take $\theta = L$ then H super-solution (resp. sub-solution) for all $M > M_\infty$ (resp. $M < M_\infty$), we deduce $Y(X) \approx M_\infty X^L$.
(b) $c < 0$ et $L < 1$. We take $\theta = q(p-1)$, (2.31) (resp. (2.32)) becomes (2.33) (resp. (2.34)). Since $1 - q(p-1) > 0$ (because $Q < 1$) then $Y(X) \approx (-c)^{1-p}X^{q(p-1)}$. □

Proof of Theorems 1.1, 1.2 and 1.3). That is to say φ a solution of the problem (2.27) whose maximum interval of existence is $]-\infty, \beta[$. Then, as long as $\varphi(\xi) \neq 0$ (consequently $Y(\varphi(\xi)) \neq 0$) one has

$$Y^{-1/(p-1)}(\varphi(\xi))\varphi'(\xi) = 1. \tag{2.35}$$

While integrating (2.35) on $(\xi, \xi_1) \subset]-\infty, \beta[$ one obtains

$$\xi_1 - \xi = \int_{\varphi(\xi)}^{\varphi(\xi_1)} Y^{-1/(p-1)}(s)ds, \tag{2.36}$$

for all $\xi \in]-\infty, \beta[$, such as $\varphi(\xi) > 0$. If φ never vanishes on $]-\infty, \beta[$, we can make tending ξ to $-\infty$ in the formula (2.36), thereby we have $\int_0^{\varphi(\xi_1)} Y^{-1/(p-1)}(s)ds = \infty$. Thus φ vanish in a point if and only if

$$\int_0 Y^{-1/(p-1)}(s)ds < \infty \tag{2.37}$$

In addition, by tending ξ_1 to β in the formula (2.36) we obtain $\beta = \infty$ if and only if

$$\int^{+\infty} Y^{-1/(p-1)}(s)ds = \infty. \tag{2.38}$$

Let us call the asymptotic behavior Y (solution of the problem (2.10) and remark 2.6, the theorem 1.1 rises immediately. One combines again the results of the behavior asymptotic of the solution Y and relations (2.28) and (2.31), one obtains the results concerning the asymptotic behavior (theorems 1.2 and 1.3). □

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