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RESTRICTED TOTAL STABILITY AND TOTAL ATTRACTIVITY

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Abstract. In this paper the new concepts of restricted total stability and total attractivity is formulated. For this purpose the classical theory of Malkin with suitable changes and the theory of limiting equations, introduced by Sell developed by Artstein and Andreev, are used. Significant examples are presented.

1. Introduction

Consider the differential system

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{1.1}$$

where $f \in C[\mathbb{R}^+ \times R^n, R^n]$. Assume that

$$f(t, 0) \equiv 0.$$

Denote the perturbation of (1.1) by

$$\dot{x} = f(t, x) + F(t, x), \quad x(t_0) = x_0 \tag{1.2}$$

where $F \in C[\mathbb{R}^+ \times R^n, R^n]$ and denote by $x(t, t_0, x_0)$ a solution of (1.2) through (t_0, x_0).

Now, the zero solution, $x \equiv 0$, of (1.1) is called **uniformly totally stable** [13] if, for every $\epsilon > 0$ and for each $t_0 \geq 0$ there exist $\delta_1(\epsilon)$ and $\delta_2(\epsilon) > 0$ so that for all $x_0 \in R^n$ with $\|x_0\| < \delta_1$ and for all F with $\|F\| < \delta_2$ we obtain $\|x(t, t_0, x_0)\| < \epsilon$ for $t \geq t_0$ where $x(t, t_0, x_0)$ is a solution of (1.2).

Malkin [13] deduced, under appropriate hypotheses, the following two properties.

(1) The solution $x = 0$ of (A_1) is uniformly totally stable (1944).
(2) For all $\epsilon > 0$ there exist $\delta_1, \delta_2 > 0$ and for any $\eta \in]0, \epsilon]$ there exists $\delta_3 \in]0, \delta_2]$ with the property that for all $t_0 \geq 0$, for all x_0 satisfying $\|x_0\| < \delta_1$ and for all F with $\|F\| < \delta_3$ there exists $T > 0$ such that $\|x(t, t_0, x_0)\| < \eta$ for $t \geq T + t_0$ where $x(t, t_0, x_0)$ is a solution of the perturbed system (1952).

The second property brings to mind the strong stability under perturbations in generalized dynamical systems introduced by Seibert [15].

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In the classical total stability theory of Malkin, the norm $\|F(t,x)\|$ of the perturbing function must either be bounded by a constant [12,13] or by a suitable function depending upon ϵ [18]. Under either of these restrictions a solution $x(t,t_0,x_0)$ through (t_0,x_0) of the perturbed differential equation remains in some arbitrarily small neighborhood of the solution $x \equiv 0$ for all time $t \geq 0$ and x_0 small. Note that the limit of the solution $\lim_{t\to\infty} x(t,t_0,x_0)$ is uncertain.

In this paper we replace the bound for $\|F\|$ with a different criteria on F. Under this new criteria we can define and motivate a new concept called the restricted total stability of (1.1). We then use this concept to motivate the total attractivity of (1.1).

The Liapunov submethods employed in this work are the families of Liapunov functions used by Salvadori [14] and the theory of limiting equations by Sell [16], Artstein [5,6,7] and Andreev [1,2,3,4]. The topological limit theory by Cartan-Silov [17] is used to formulate several theorems. Finally, the theoretical results are supported by significant examples.

2. Preliminaries and basic concepts

Let $I = \mathbb{R}^+ = [0,+\infty[$ and $I' =]0,+\infty[$. Let $\mathbb{R}^n$ be a real n-dimensional vector space. Let $\|x\|$ be the Euclidean norm of x for any $x \in \mathbb{R}^n$. Following the convention of Hahn [13], let K denote the class of all strictly increasing functions $c : I \to I \in C$ which satisfy $c(0) = 0$. Throughout this paper let $b = b(u)$, $c = c(u)$, $m = m(u) \in K$, let $\chi = \chi(t,x)$, $U = U(t,x)$, $V = V(t,x)$, $W = W(t,x) \in C[R \times \mathbb{R}^n \to R]$ and $G = G(t,x)$, $\Gamma = \Gamma(t,x)$ be continuous vector valued functions mapping $I \times \mathbb{R}^n$ to $\mathbb{R}^n$. With $F \cdot G$ denote the scalar product of the vectors F and G in $\mathbb{R}^n$. Also, let $h = h(t,x) : I \times \mathbb{R}^n \to I$ and $h_0 = h_0(t,x) : I \times \mathbb{R}^n \to I$ be continuous functions having the property that

$$\inf_{x\in\mathbb{R}^n} h(t,x) = \inf_{x\in\mathbb{R}^n} h_0(t,x) = 0 \quad \text{for every } t \in I.$$

We shall select the functions h and h_0 as measures of stability. Also, assume that the measure h_0 is uniformly finer than h. This implies that there exists a constant $\lambda > 0$ and a function $m \in K$ so that when $h_0(t,x) < \lambda$ we have $h(t,x) \leq m[h_0(t,x)] < m(\lambda)$ [9].

2.1. **Topological limit theory.** Consider the sets

$$Q(s) = \{(t,x) \in I \times \mathbb{R}^n : 0 < h(t,x) \leq s\}.$$

Note that $Q(s_1) \subseteq Q(s)$ if $0 < s_1 \leq s$. Also the intersection $\bigcap_{s\in I} Q(s) = \emptyset$ where $\emptyset$ denotes the empty set. Let $Q = \{Q(s) : s \in I\}$. Then Q represents a Cartan-Silov direction or, simply stated, a direction. We say that

$$\lim_{h\to 0} V(t,x) = 0$$

if and only if for every direction Q satisfying $\lim\ h(t,x) = 0$ we have $\lim\ V(t,x) = 0$ [17].

2.2. **Differential systems.** Given a differential system

$$\dot{x} = f(t,x), \quad t \in I, \quad x \in \mathbb{R}^n, \quad x(t_0) = x_0 \tag{2.1}$$

and a continuous function $G = G(t,x)$ with $\|G\| > 0$. Consider only the perturbed systems

$$\dot{x} = \Gamma(t,x) = f(t,x) + F(t,x), \quad x(t_0) = x_0 \tag{2.2}$$

such that $F\cdot G\leq 0$ where f and F are measurable functions for $t\in I$ and continuous for $x\in\mathbb{R}^n$. Further assume that for every closed and bounded subset $B\subseteq\mathbb{R}^n$ there exists a locally integrable function $\sigma_B=\sigma_B(t)$ so that

$$\|f(t,x)\|\leq\sigma_B(t) \text{ and } \|F(t,x)\|\leq\sigma_B(t) \quad \text{when } x\in B. \tag{2.3}$$

These conditions of Caratheodory [8] ensure the existence and the general continuity of the solutions for (2.1) and (2.2).

Finally, let $[G]=\{F(t,x): F\cdot G\leq 0\}$ be the set of selected perturbations. And, as before, let $x(t)=x(t,t_0,x_0)$ be a solution of (2.2) through (t_0,x_0).

We introduce the following definitions of restricted total stability and attractivity.

Definition 2.1. System (2.1) is said to be **restrictedly totally** (h_0,h)**-stable** or t-stable, if, for every $t_0\in I$ and $\epsilon>0$ there exists $\delta=\delta(t_0,\epsilon)>0$ such that for all $x_0\in\mathbb{R}^n$ with $h_0(t_0,x_0)<\delta$ we have $h(t,x(t))<\epsilon$ for all $t\geq t_0$ where $x(t)=x(t,t_0,x_0)$. If $\delta=\delta(\epsilon)$ we have the uniformity, in short t,u-stability.

Definition 2.2. System (2.1) is called **totally** (h_0,h)**-attractive** or t-attractive, if there exists a function set $[G_1]\subseteq[G]$ with the property that for all $t_0\in I$ there exists $\delta_1=\delta_1(t_0)>0$ so that for all $x_0\in R^n$ satisfying $h_0(t_0,x_0)<\delta_1$ and for all $\eta>0$ and $F\in[G_1]$ there exists $T>0$ so that $h(t,x(t))<\eta$ for $t\geq t_0+T$. If δ is a constant and $T=T(\eta)$ the system (2.1) is t,u-attractive. If $\delta=+\infty$ we have global t-attractivity.

Definition 2.3. System (2.1) is said to be **restrictedly totally asymptotically stable** or t-asymptotically stable if it is t-stable and t-attractive.

Definition 2.4. The system (2.1) is said to be **restrictedly totally uniformly asymptotically stable** or t,u-asymptotically stable if it is t,u-stable and t,u-attractive.

2.3. **The derivatives.** Let $q=q(u)$ be a function from R to R. Denote by q_u the derivative of q with respect to u. For a function $V=V(t,x)$ we shall denote by $\dot V$ or $\dot V_1$ the derivative of V computed along the solutions of the differential system (2.1) and also denote $V_t=\frac{\partial V}{\partial t}$ and $\operatorname{grad} V=\frac{\partial V}{\partial x}$.

If we assume that $V=V(t,x)\in C^1$ is continuous with continuous derivative, then Malkin [13] shows that

$$\dot V_2(t,x)=\dot V_1(t,x)+F(t,x)\cdot\operatorname{grad} V(t,x) \tag{2.4}$$

where $\dot V_1=V_t+f\operatorname{grad} V$ and $\dot V_2$ is the derivative over (2.2).

Observe that if $\operatorname{grad} V=\chi G$ where $\chi=\chi(t,x)\geq 0$ and if $F\cdot G\leq 0$ we deduce that $\dot V_2=\dot V_1+\chi F\cdot G\leq\dot V_1$.

Throughout this paper, assume that $\lim(f_1,\dots,f_m)=(a_1,\dots,a_m)$ if and only if $\lim f_j=a_j$ for every $j=1,\dots,m$.

3. The restricted total stability

We use the Liapunov families of functions [14] in this section. The basic advantage of this method is that the single function needs to satisfy less rigid requirements.

Theorem 3.1. *Suppose that for every $s>0$ there exists two functions $V=V(t,x)\in C^1$ and $\chi=\chi(t,x)\in C$ and a constant l so that:*

(i) $h(t,x) = s$ *implies* $V(t,x) \geq l > 0$
(ii) $\lim_{h\to 0} V(t,x) = 0$
(iii) $\dot{V}(t,x) \leq 0$ *on* $Q(s)$
(iv) $\operatorname{grad} V(t,x) = (\chi G)(t,x)$ *on* $Q(s)$.

Then the system (2.1) *is* t*-stable.*

Proof. Given $t_0 \in I$, $\epsilon > 0$ and V, χ, l, by (ii) there exists $d > 0$ so that $h(t_0,x) < d$ implies $V(t_0,x) < l$. If we select $x_0 \in \mathbb{R}^n$ so that $h_0(t_0,x_0) < \delta = \min[\lambda, m^{-1}(d)]$ we obtain the results $h(t_0,x_0) \leq m[h_0(t_0,x_0)] < d$ and $V(t_0,x_0) < l$. This is due to the assumption throughout this paper that the measure of h_0 is finer than h.

From equation (2.4) and hypotheses (iii), (iv) we obtain $\dot{V}_2 \leq 0$ on $Q(\epsilon)$. Consider a solution $x(t) = x(t,t_0,x_0)$ and the corresponding function $V(t,x(t))$. Suppose that there exists $t' > t_0$ so that $h(t',x(t')) = \epsilon$ with $h(t,x(t)) < \epsilon$ for $t \in [t_0,t'[$ then we have $V(t',x(t')) \geq l$, hence there exists $\tau \in]t_0,t'[$ where $\dot{V}_2(\tau,x(\tau)) > 0$ which is a contradiction. □

The following proposition ensures the t,u-stability.

Theorem 3.2. *Suppose that for every* $s > 0$ *there exists two functions* $V = V(t,x) \in C^1$, $\chi = \chi(t,x) \in C$ *and two constants* $l, L \in \mathbb{R}$ *so that:*

(i) $\dot{V}(t,x) \leq 0$ *on* $Q(s)$
(ii) $h(t,x) = s$ *implies* $V(t,x) \geq l$
(iii) $0 < l \leq L$
(iv) $h(t,x) < s$ *implies* $V(t,x) \leq L$
(v) $\lim_{s\to 0}(l,L) = 0$
(vi) *if* $0 < s_1 < s$, *then* $h(t,x) < s_1$ *implies* $V(t,x) \leq V_1(t,x)$
(vii) $\operatorname{grad} V(t,x) = (\chi G)(t,x)$ *on* $Q(s)$.

Then system (2.1) *is* t,u*-stable.*

Proof. Given $\epsilon > 0$ with V, χ, l, L; from (v) we can select $\epsilon_1 \in]0,\epsilon[$ with V_1, χ_1, l_1, L_1 so that $L_1 < l$. For any $t_0 \in I$ we select $x_0 \in \mathbb{R}^n$ so that $h_0(t_0,x_0) < \delta(\epsilon_1) = \min[\lambda, m^{-1}(\epsilon_1)]$. Then the connection between the measures of h and h_0 implies $h(t_0,x_0) \leq m[h_0(t_0,x_0)] < \epsilon_1$ and hence $V(t_0,x_0) \leq V_1(t_0,x_0) \leq L_1 < l$. On the basis of the previous theorem we obtain the proof. □

We deduce the following theorem by using a single Liapunov function.

Theorem 3.3. *Suppose that there exists three functions* $V = V(t,x) \in C^1$, $\chi = \chi(t,x) \in C$, $b = b(u) \in K$ *and a constant* $s \geq m(\lambda)$ *so that:*

(i) $V(t,x) \geq b[h(t,x)]$ *on* $Q(s)$
(ii) $\lim_{h\to 0} V(t,x) = 0$
(iii) $\dot{V}(t,x) \leq 0$ *on* $Q(s)$
(iv) $\operatorname{grad} V(t,x) = (\chi G)(t,x)$ *on* $Q(s)$.

Then the system (2.1) *is* t*-stable. Also, if there exists* $c \in K$ *such that* $V(t,x) \leq c[h(t,x)]$ *on* $Q(s)$ *then the system* (2.1) *is* t,u*-stable.*

Proof. Given $\epsilon > 0$ we obtain all the hypotheses of Theorem 3.1. In the following formulation, fix $\epsilon > 0$. If $c[h(t,x)] < b(\epsilon)$, then $h(t,x) < c^{-1}[b(\epsilon)] = d$ and we obtain $V(t,x) < b(\epsilon)$. If we consider (t_0,x_0) so that $h_0(t_0,x_0) < \delta = \min[\lambda, m^{-1}(d)]$ we have $h(t_0,x_0) \leq m[h_0(t_0,x_0)] < d$, therefore $V(t_0,x_0) < b(\epsilon)$. By applying Theorem 3.1 we have the proof. □

Corollary 3.4. *If $V(t,x)$ ensures the stability of* (2.1)*, with $\dot V \le 0$, then we obtain the restricted total stability of* (2.1) *with respect to the systems* (2.2) *for which $F \cdot \operatorname{grad} V \le 0$.*

Corollary 3.4 is very useful in several applications.

Remark 3.5. The previous formulations of total stability theory require that $\dot V < 0$. Note also that, in restricted total stability, the modulus of perturbation can be unbounded.

4. Preparation for the total attractivity in the x-bounded domains

Assume that there exists $\lambda' > \lambda$ so that the sets where $h_0(t,x) < \lambda'$ and $h(t,x) < \lambda'$ are bounded in x. Further, assume that there exists a compact set $D \subset \mathbb{R}^n$ so that $I \times D$ contains the set $\{(t,x) : h(t,x) \le m(\lambda)\}$.

We combine the limiting equations theory formulated by Sell, the convergence of limiting equations by Artstein and the Liapunov's second method formulated by Andreev to study the asymptotic aspect of t-stability.

4.1. **Precompactness.** Let $X = X(t,x) : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector valued function which is continuous in x and measurable in t. Then the precompactness conditions for $X(t,x)$ are [5]: For every compact set $A \subset \mathbb{R}^n$ there exist two locally L_1 functions $M_A(t)$ and $N_A(t)$ so that if $x, y \in A$ and $t \in I$ then

(i) $\|X(t,x)\| \le M_A(t)$
(ii) $\|X(t,x) - X(t,y)\| \le N_A(t)\|x - y\|$

where the functions $M_A(t)$ and $N_A(t)$ satisfy the following two criteria:

(a) For every $t, \epsilon > 0$ there exists a $\mu_A = \mu_A(\epsilon) > 0$ so that if E is a measurable set in I, with measure less than μ_A, and E is contained in the interval $[t, t+1]$, then $\int_E M_A(\tau)d\tau \le \epsilon$.
(b) There exists a number $L_A > 0$ so that $\int_t^{t+1} N_A(\tau)d\tau \le L_A$ for all $t \in I$.

We assume that the functions f and F satisfy the precompactness conditions. Observe that: (i) we can select the compact sets A, B, D so that $A = B = D$; (ii) the above precompactness condition (ii) implies the uniqueness of solutions of (2.1). We denote with $\Im$ the family of functions from $I \times \mathbb{R}^n$ to $\mathbb{R}^n$ which satisfy the precompactness conditions.

4.2. **The translates of the first kind.** Given $T > 0, x \in D$, a sequence t_m diverging to $+\infty$ and a solution $x = x(t)$ of (2.2), the functions $\Gamma_m(t,x) = \Gamma(t + t_m, x)$, $x_m(t) = x(t + t_m)$, defined for $t \in [0,T]$ and $x \in D$, are called the translates of $\Gamma(t,x)$ and $x(t)$; obviously $\dot x_m(t) = \Gamma_m[t, x_m(t)]$ for $m \in \{1,2,\dots\}$ [16].

4.3. **The convergence and the limiting equations.** In this subsection, the mathematical problem, which we face, is basically the convergence of the translates previously considered and, particularly, to embed the functions Γ_m in a compact metric space [5]. According to Artstein, the convergence in $\Im$ can be induced by a suitable metric defined on equivalence classes of $\Im$. Therefore we do not distinguish between two elements that differ only on some t-set of measure zero [6].

Hence, under the previous hypotheses of precompactness, there exists, with respect to a suitable metric, a subsequence $\{\Gamma_r(t,x)\}$ of $\{\Gamma_m(t,x)\}$ convergent to $g(t,x)$. In other words, the sequence $\int_0^t \Gamma_r(u,x)du$ converges in $\mathbb{R}^n$ to $\int_0^t g(u,x)du$

when $t \in [0,T]$ and $x \in D$ [5]. This convergence concept is fairly weak and covers a wide family of functions.

More clearly; let $x \in A$, $t \in [0,T], m = 1,2,\dots$ and $G_m(t,x) = \int_0^t \Gamma_m(u,x)du$. Then, from the precompactness conditions, one deduces the following:

(i) $\|G_m(t,x)\| \leq \int_0^t \|\Gamma_m(u,x)\| \leq 2\int_0^T M_A(u)du < \rho(A)$

(ii) $\|G_m(t',x') - G_m(t'',x'')\| \leq 2\|x'-x''\| \int_0^T N_A(u)du + 2\| \int_{t'}^{t''} M_A(u)du\|$

where $\rho(A) > 0$ is a suitable constant. Hence we obtain, in order, the equiboundedness and the equicontinuity for $G_m(t,x)$.

From the Ascoli-Arzela' theorem we deduce the uniform convergence for a suitable subsequence G_r of G_m. Since, on every closed interval $[0,T]$, the functions $x_m(t) = x(t_m) + \int_0^t \Gamma_m[u, x_m(u)]du$ are equibounded and equicontinuous, there exists a subsequence $x_r(t)$ which converges uniformly to $y = y(t)$ where $\dot{y}(t) = g(t, y(t))$. The functions $g(t,x)$ and $y(t)$ are called limiting or limit functions with respect to t_m for $\Gamma(t,x)$ and $x(t)$; we similarly define the limiting equation of (2.2) the the following equation $\dot{x} = g(t,x)$. [6,16]. Hence the limiting equations of the nonautonomous ordinary differential equations are limit points of the translated equations. The limit is taken in a prespecified and suitable space. The general motivation for introducing the limiting equations is that there is a strong connection between the asymptotic behavior of the solutions of the original equation and the solutions of the limiting equations [6].

We suppose also that the perturbation $F(t,x)$ is **integrally convergent to zero**. Then

$$\lim_{m\to+\infty} \int_\alpha^\beta F(\tau + t_m, \phi_m(\tau))d\tau = 0 \tag{4.1}$$

whenever $\{\phi_m\}$ is a uniformly convergent sequence of functions from $[\alpha,\beta] \subset I$ to $\mathbb{R}^n$ and $\{t_m\}$ is a sequence diverging to $+\infty$ [7]. Consequently systems (2.1) and (2.2) have identical limiting equations [4]. This result implies that we can develop the theory of total attractvity according to the restricted total stability.

Consider the limit set of Andreev. This set is very important and is denoted by $V_\infty^{-1}(t,c)$ [2]; Now, $x \in V_\infty^{-1}(t,c)$ if there exists a sequence $\{t_r\}$ which diverges to $+\infty$ and a sequence $\{x_r\}$ which converges to x so that for $t \geq 0$ and $c \in \mathbb{R}$ we have

$$\lim_{r\to+\infty} V(t+t_r, x_r) = c.$$

Let $\Im_1$ denote the metric space of the scalar precompact functions U mapping $I \times \mathbb{R}^n$ to I. Then $U \in \Im_1$.

5. The restricted total asymptotic stability and applications

In this section we consider theorems and examples on restricted total asymptotic stability in bounded domains. According to Artstein [7], the Liapunov and La Salle [10] theoretical constructions are a very powerful tool to establish the asymptotic and the uniform asymptotic stability of ordinary differential equations. These two methods are direct methods. A major role in our technique is played by the limiting equations. This abstract characterization becomes practical in those cases where the structure of the limiting equations is relatively easy.

Theorem 5.1. *Under the first hypothises of Theorem 3.3 and under the hypotheses and results of section 4, suppose that there exists $U = U(t,x) \in \Im_1$ so that:*

(i) $\dot{V}(t,x) \le -U(t,x) \le 0$ *on* $I \times D$
(ii) $\lim_{V\to 0} h(t,x) = 0$
(iii) *for every limit pair* (g,W) *of* (f,U) *the set* $\{(W=0)\cap V_\infty^{-1}(t,c)\}$ *contains no solutions of* $\dot{x} = g(t,x)$*, the limiting equation of* (2.2)*, for* $c > 0$*.*

Then the system (2.1) *is* t*-asymptotically stable.*

Proof. Suppose that there exists $t_0 \in I$ with the property that for all $\delta > 0$ there exists $x_0 \in D$ satisfying $h_0(t_0,x_0) < \delta$ and there exists $\eta > 0$ and a sequence $\{t_m\}$ diverging to $+\infty$ so that we have $h(t'_m, x(t'_m)) \ge \eta$ where $t'_m = t_0 + t_m$ and $x(t) = x(t,t_0,x_0)$. Let $v(t) = V(t,x(t))$. Then by (i), we deduce

$$\lim_{t\to+\infty} v(t) = c \ge 0$$

If $c = 0$ we have the proof; otherwise, when $c > 0$, consider for every $T > 0$ the following three sequences of translates, of the first kind, and an obvious equality

$$\Gamma_m(t,x) = f(t+t'_m,x) + F(t+t'_m,x), \quad U_m(t,x) = U(t+t'_m,x) \tag{5.1}$$

$$x_m(t) = x(t+t'_m), \quad \dot{x}_m(t) = \Gamma_m(t,x_m(t)) \tag{5.2}$$

where $t \in [0,T]$ and $x \in D$. According to the convergence of limiting equations, we can select from $\Gamma_m(t,x)$ and $U_m(t,x)$ two weakly converging subsequences $\Gamma_r(t,x)$ and $U_r(t,x)$ which converge respectively to $g(t,x)$ and $W(t,x)$. We can also select a subsequence $x_r(t)$ which converges uniformly to $y(t)$ with $\dot{y}(t) = g(t,y(t))$.

From $\dot{v}(t) \le -U(t,x(t)) \le 0$ and for r diverging to $+\infty$ deduce successively:

$$v(t+t'_r) - v(t'_r) \le -\int_0^t U_r(u,x_r(u))du \le 0 \tag{5.3}$$

and

$$0 = c - c \le -\int_0^t W(u,y(u))du \le 0. \tag{5.4}$$

Hence $y(t) \in \{(W=0)\cap V_\infty^{-1}(t,c)\}$ for $c > 0$ which is an absurdity. Therefore

$$\lim_{t\to+\infty} h(t,x(t)) = 0.$$

□

The set $[G_1]$, relative to the definition of attractivity given in section 2, is the set of functions belonging to $[G]$ which are measurable, precompact and integrally convergent to zero.

Corollary 5.2. *If* $V(t,x)$ *and* $h(t,x)$ *satisfy the hypotheses of Theorem 5.1 and if there exists a function* $c \in K$ *so that* $V \le c(h)$ *then the system* (2.1) *is* t,u*-stable and* t*-attractive.*

We obtain the following theorem if we suppose also that the measure $h(t,x)$ is precompact.

Theorem 5.3. *Under the assumptions and the results of section 4, suppose that the system* (2.1) *is* t,u*-stable and* t*-attractive. Let* $h \in \Im_1$ *and assume that for every limit pair* (g,l) *of* (f,h) *the set* $h_\infty^{-1}(t,0)$ *contains only the solutions* $y = y(t)$ *of the limiting system* $\dot{x} = g(t,x)$ *so that* $l(0,y(0)) = 0$ *. Then the system* (2.1) *is* t,u*-asymptotically stable.*

Proof. It is sufficient to show that the system (2.1) is uniformly attractive. Assume the contrary. Then for all $\delta > 0$ there exists $\eta > 0$, there exists $t_m \in I$ and there exists $x_m \in D$ satisfying $h_0(t_m, x_m) < \delta$, so that for some divergent sequence $\{t'_m\}$ we have $h(t''_m, x_m(t''_m)) \geq \eta$ where $t''_m = t'_m + t_m$ and $x_m(t) = x(t, t_m, x_m)$. Consider the sequences of translates

$$h_m(t,x) = h(t + t''_m, x), \quad \Gamma_m(t,x) = \Gamma(t + t''_m, x), \quad z_m(t) = x_m(t + t''_m) \tag{5.5}$$

where $t \in [0, T(>0)]$, $x \in D$. We obtain

$$\dot{z}_m = \Gamma_m(t, z_m(t)), \quad z_m(0) = x_m(t''_m), \quad h_m(0, z_m(0)) \geq \eta. \tag{5.6}$$

Since the functions $z_m(t)$ are equibounded and equicontinuous we can select, under the conditions of precompactness, three converging subsequences $\Gamma_r(t,x)$ converging to $g(t,x)$, $h_r(t,x)$ converging to $l(t,x)$, $z_r(t)$ converging uniformly to $y(t)$ where $t \in [0,T]$ and $x \in D$. Obviously

$$\dot{y}(t) = g(t, y(t)), \quad l(0, y(0)) \geq \eta > 0. \tag{5.7}$$

Since the attractivity implies that

$$\lim_{r \to +\infty} h(t + t''_r, x(t + t''_r)) = 0 \tag{5.8}$$

we deduce $y(t) \in h_\infty^{-1}(t, 0)$ which is a contradiction. See Theorem 3.3 of [19]. $\square$

Remark 5.4. In this paper $t \in I$ and $x \in \mathbb{R}^n$, the restriction of the present theory on a suitable subset of $\mathbb{R}^n$ is obvious.

We conclude this section by studying the behavior of two differential systems. We establish the more general classes of perturbations corresponding to these two different types of stability.

5.1. **Example.** Consider the differential system from $I \times \mathbb{R}^3$ to $\mathbb{R}^3$

$$\begin{gathered} \dot{x} = a^2(x^2 - by^2 + z^2)x^3y^4 + a^2(x^2 + by^2 + z^2)xy^2 + a^3 z = f_1 \\ \dot{y} = -a(x^2 + by^2 + z^2)x^2 y + a(x^2 - by^2 + z^2)x^4y^3 - a^3 z = f_2 \\ \dot{z} = -a^2 x + 3a^3 y = f_3 \end{gathered} \tag{5.9}$$

where $a = a(t) : I \to I$ and $b = b(t) : I \to I$ are locally integrable functions.

Theorem 5.5. *Under the previous hypotheses suppose that*

(i) $0 \leq a(t) \leq 2$
(ii) $h = x^2 + 3ay^2 + az^2$ *and* $h_0 = x^2 + 6y^2 + 2z^2$
(iii) $2x^2y^2 \leq 1$
(iv) $\dot{a}(t) \leq 0$.

Then the system (5.9) *is* t,u*-stable with respect to the perturbations*

$$F[-2xM_1, -6ayM_2, -2azM_3]$$

where $M_\iota = M_\iota(t,x,y,z) \geq 0$ ($\iota = 1,2,3$) *are arbitrary functions that satisfy, locally, the Caratheodory conditions given in* (2.3).

Proof. Let $V = h$ and $G = \operatorname{grad} V = (2x, 6ay, 2az)$. Note that we obtain the conditions of Theorem 3.3. In fact we have

(i) $V \geq h$
(ii) for every direction $\lim h = \lim V$
(iii) on the set $I \times \mathbb{R}^3$ the derivative $\dot{V}$ satisfies

$$= 8a^2(x^2 - by^2 + z^2)x^4y^4 - 4a^2(x^2 + by^2 + z^2)x^2y^2 + \dot{a}(3y^2 + z^2).$$

Also, on the set $\{(x, y, z) : 2x^2y^2 \leq 1\}$, we obtain

$$\dot{V} \leq \dot{a}(3y^2 + z^2) = -U \leq 0. \tag{5.10}$$

Observing that $F \cdot \operatorname{grad} V \leq 0$ we have the proof. □

Theorem 5.6. *Under the hypotheses of Theorem 5.5, suppose that*

(i) *$a(t)$ and $b(t)$ are bounded*
(ii) *M_ι is integrally convergent to zero*
(iii) *$\lim_{t\to+\infty}\{a, \dot{a}, b\} = \{\alpha > 0, -\alpha_1 < 0, \beta > 0\}$*

Then the system (5.9) *is t, u-asymptotically stable.*

Proof. Since, for j=1,2,3, the derivatives of f_j with respect to x, y, z are bounded on every compact set $D \subset \mathbb{R}^3$, we deduce that system (5.9) is precompact. From (ii) and (iii) the limiting system of (5.9) and the limiting system of a perturbed system are identical and unique. Let $W = \lim U = \alpha_1(3y^2 + z^2)$ and $l(x, y, z) = \lim h(t, x, y, z) = x^2 + 3\alpha y^2 + \beta z^2$. Then on the set $\{W = 0\} = \{y = z = 0\}$ the limiting system of (5.9) assumes the form $\dot{x} = 0$, $0 = 0$ and $0 = x$. Therefore, only the solution $x = y = z = 0$ of the limiting system belongs to $\{W = 0\}$. Hence, according to Theorems 5.1 and 5.3, the result follows by observing that the set $\{(W = 0) \cap V_\infty^{-1}(t, c)\}$ contains no solutions of limiting equations for $c > 0$ and $h(t, 0, 0, 0) = l(0, 0, 0) = 0$. □

We next consider the following application.

5.2. **Application.** Suppose that the adimensionate motion equations of a particular rigid body, with a fixed point and variable mass, are given by

$$\begin{gathered}\dot{A}p + 2A\dot{p} + 2(C - A)qr = 2Pz\gamma_2 - 2f_1p - 2f_4qr \\ \dot{A}q + 2A\dot{q} + 2(A - C)pr = -2Pz\gamma_1 - 2f_2q - 2f_5pr \\ \dot{C}r + 2C\dot{r} = 2(f_4 + f_5)pq - 2f_3r \\ \dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1, \quad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2,\ \gamma^2 = 1 - \gamma_3.\end{gathered} \tag{5.11}$$

where

(i) $A = A(t), C = C(t)$ from I to I' are continuous functions
(ii) $\dot{A} = \dot{A}(t)$, $\dot{C} = \dot{C}(t)$, $P = P(t), z = z(t)$ from I to R are locally integrable functions;
(iii) for $j = 1, 2, \dots, 5$ the functions $f_j = f_j(t, p, q, r, \gamma_1, \gamma_2)$ from $I \times \mathbb{R}^5$ to R satisfy the conditions of Caratheodory given in (2.3)
(iv) $p, q, r, \gamma_1, \gamma_2, \gamma_3$ are the unknown variables.

We assume also that

(i) $Pz < 0$
(ii) There exists three constants A_0, C_0, P_0 so that $A \geq A_0 > 0$, $C \geq C_0 > 0$, and $P \geq P_0 > 0$.

Lemma 5.7. *If we select the auxiliary function of Matrosov's type [11]*

$$V = \frac{1}{2}[A(p^2 + q^2) + Cr^2] - \frac{1}{2}Pz(\gamma_1^2 + \gamma_2^2 + \gamma^4) \tag{5.12}$$

we deduce $\dot{V} = -(f_1p^2 + f_2q^2 + f_3r^2) = -U$.

Lemma 5.8. *If $f_1,\ f_2,\ f_3 > 0$ and $\min(A, C) > 0$ then V is positive definite with $\dot{V} = -U \le 0$.*

During the rest of this section, we will assume that $4h = 2V = h_0$. Then let $G = [Ap, Aq, Cr, -Pz\gamma_1, -Pz\gamma_2, Pz(1-\gamma_3)]$ and consider only the perturbations

$$F = [-ApM_1, -AqM_2, -CrM_3, pz\gamma_1 M_4, Pz\gamma_2 M_5, -Pz(1-\gamma_3)M_6]$$

where $M_\iota = M_\iota(t, p, \dots, \gamma_2) : I \times \mathbb{R}^5 \to I$ for $\iota = 1, 2, \dots, 6$ are arbitrary functions that satisfy Caratheodory's conditions.

Theorem 5.9. *Under the hyptheses of Lemma 5.8 and the above assumptions and the definition of the perturbation F, the system (5.11) is t, u-stable. In fact we have*

- (i) $V \ge h$
- (ii) *for every direction* $\lim 2h = \lim V$
- (iii) $\dot{V} \le 0$
- (iv) $\operatorname{grad} V = G$
- (v) $F \cdot \operatorname{grad} V \le 0$.

Theorem 5.10. *Under the previous hypotheses suppose that*

- (i) $\lim_{t\to+\infty}\{A, C, \dot{A}, \dot{C}, P, f_3\} = \{A_1, C_1, A_2, C_2, P_1, \Lambda > 0\} = d$ *for some constant d and where $A_2 = 2A_1\alpha > 0$*
- (ii) $\lim_{t\to+\infty}\{f_{(j=1,2,4,5)}, z\} = \{0, 0\}$
- (iii) *$A, C, \dot{A}, \dot{C}, Pz$ are bounded*
- (iv) *M_ι are integrally convergent to zero*
- (v) *the functions f_j and the derivatives of f_j with respect to $p, q, r, \gamma_1, \gamma_2$ are bounded on every compact set $D \subset \mathbb{R}^5$.*

Then the system (5.11) is t, u-asymptotically stable.

Proof. From the previous hypotheses and assumptions we have: (i) the system (5.11) is precompact; (ii) the limit of (5.11) and the limit of a perturbed system are identical and unique. The mutual limiting system is

$$\begin{gathered} A_2 p + 2A_1 \dot{p} + 2(C_1 - A_1)qr = 0 \\ A_2 q + 2A_1 \dot{q} + 2(A_1 - C_1)pr = 0 \\ C_2 r + 2C_1 \dot{r} = -2\Lambda r \end{gathered} \tag{5.13}$$

and the fourth line of (5.11) is unchanged; (iii) the limit of U is $W = \Lambda r^2$ hence $W = 0$ implies $r = 0$. Also, the limiting system on $\{W = 0\}$ assumes the form

$$A_2 p + 2A_1 \dot{p} = 0, \quad A_2 q + 2A_1 \dot{q} = 0\,, \quad 0 = 0. \tag{5.14}$$

From this we deduce $p = q = Ze^{-\alpha t}$ where Z is some arbitrary constant.

We complete the proof with the following considerations. Since over the solutions of system (5.14) we have

$$\lim_{t\to+\infty} V = 0$$

then the set $\{(W = 0) \cap V_\infty^{-1}(t, c)\}$ contains no solutions of limiting equations for $c > 0$. According to Theorem 5.3, since the limit of h is $l = \frac{1}{4}[A_1(p^2 + q^2) + C_1 r^2]$, the unique solution of (5.14) which belongs to $h^{-1}(t, 0)$ and satisfies $l = 0$, is $p = q = r = 0$. Therefore the system (5.11) is t, u -asymptotically stable with respect to the select measures. $\square$

5.3. **Partial stability of motion.** In system (5.11) suppose that:

(i) $f_j = f_j(t,p,q,\gamma_1,\gamma_2)$ for $j=1,2$
(ii) $f_3 = f_3(r)$
(iii) $f_4 = f_5 = 0$
(iv) the third equation $\dot{C}r + 2C\dot{r} = -2f_3 r$ determines, as a solution for every (t_0, r_0), a continuous function $r = r(t,t_0,r_0)$ defined for $t \geq t_0$.

Theorem 5.11. *Under the previous hypotheses, the system*

$$\begin{gathered}\dot{A}p + 2A\dot{p} + 2(C-A)qr = 2Pz\gamma_2 - 2f_1 p \\ \dot{A}q + 2A\dot{q} + 2(A-C)pr = -2Pz\gamma_1 - 2f_2 q \\ \dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1, \quad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2, \quad \gamma^2 = 1-\gamma_3\end{gathered} \tag{5.15}$$

is t,u-stable with respect to the measures $h = \frac{1}{4}[A(p^2+q^2) - Pz(\gamma_1^2+\gamma_2^2+\gamma^4)]$ and $h_0 = 4h$.

Proof. Consider the new positive definite auxiliary function

$$V = \frac{1}{2}[A(p^2+q^2) - Pz(\gamma_1^2+\gamma_2^2+\gamma^4)]$$

with the derivative $\dot{V} = -(f_1 p^2 + f_2 q^2) = -U_1 \leq 0$. Select

$$\begin{gathered}G = [Ap, Aq, -Pz\gamma_1, -Pz\gamma_2, Pz(1-\gamma_3)], \\ F = [-ApM_1, -AqM_2, pz\gamma_1 M_3, pz\gamma_2 M_4, -pz(1-\gamma_3)M_5]\end{gathered}$$

where M_ι satisfies the Caratheodory conditions. We deduce the proof using the previous theory. ☐

Observe that the Matrosov motion [11] $p=q=\gamma_1=\gamma_2=\gamma=0$ and $r=r(t)$ describing an irregular rotation of the body around the vertical axis of symmetry, is t,u-stable with respect to the variables $p,q,\gamma_1,\gamma_2,\gamma$ and to the select measures. Under the corresponding hypotheses of Theorem 5.10 we also obtain the partial t,u -attractivity with respect to $p,q,\gamma_1,\gamma_2,\gamma$ of the system (5.15).

6. Assumptions for the attractivity in x-unbounded domains

Definitions and theorems. In this section we consider the case where

$$\lim_{\|x\|\to+\infty} h(t,x) = 0.$$

Hence we can consider only the solutions $x = x(t)$ of (2.2) so that $h(t,x(t)) \leq m(\lambda)$ for $t \geq t_0$. Since the hypotheses of precompactness are not sufficient to obtain the previous convergences discussed in section 4, we assume more restrictive conditions that ensure the uniform convergence for suitable subsequences of translates. In this section we use the known translates of the first kind and the translates of the second kind which are defined later. We shall assume also that the perturbation F is integrally convergent to zero with respect to two different divergent sequences.

Throughout this section, assume that the set $\{x \in \mathbb{R}^n : h_0(t,x) < \lambda$ for some $t \in I\}$ is bounded and that the system (2.1) is t-stable. Let S be the set $S = \{x \in \mathbb{R}^n : h(t,x) < m(\lambda)$ for some $t \in I\}$ and assume that S is open and unbounded. We also assume that the function $f = f(t,x)$, satisfies the hypotheses:

$$\begin{gathered}\|f(t,x)\| < M, \\ \|f(t',x') - f(t'',x'')\| < L[|t'-t''| + \|x'-x''\|]\end{gathered} \tag{6.1}$$

where $L, M > 0$ are constants and where $F = F(t,x)$ and $U = U(t,x)$ satisfy the above two criteria as well. Observe that criteria (ii) implies the uniqueness of solutions. Let $t_m \in I$ and $z_m \in \mathbb{R}^n$ denote two sequences so that $\lim_{m\to\infty} t_m = \infty$ and $\lim_{m\to\infty} \|z_m\| = \infty$.

Definition 6.1. We say that $f_{2m}(t,x) = f(t+t_m, x+z_m)$, $F_{2m}(t,x) = F(t+t_m, x+z_m)$, $U_{2m}(t,x) = U(t+t_m, x+z_m)$ are the translates of second kind [3], respectively for f, F, U when $t \in I$ and $x \in \mathbb{R}^n$.

Assume that the perturbation $F(t,x)$ is **integrally bi-convergent to zero** with respect to the sequences t_m and z_m i.e: for every sequence of uniformly convergent functions $\phi_m(u) : [\alpha, \beta] \subset I \to \mathbb{R}^n$ we have

$$\lim_{m\to+\infty} \int_\alpha^\beta F(t_m+s, \phi_m(s)+z_m)ds = 0. \tag{6.2}$$

Hence, as in section 4, the limit of the original system (2.1) is the same as the limit of the perturbed system (2.2).

Theorem 6.2. *Given a differential system* (2.2)*, a constant $T > 0$ and a sequence $\{t_m\}$ suppose there exists a solution $x = x(t)$ so that*

$$\lim_{t\to+\infty} \|x(t)\| = +\infty.$$

Let $z_m = x(t_m)$, $z_m(t) = x(t+t_m)$ and $z_{2m}(t) = z_m(t) - z_m$. Then the translates of second kind, $z_{2m}(t)$, are equibounded and equicontinuous for $t \in [0,T]$.

Proof. Since $x(t) = x(0) + \int_0^t \Gamma(u, x(u))du$ for all $t \in [0,T]$ we have

$$z_{2m}(t) = \int_0^t \Gamma_{2m}(u, z_{2m}(u))du = z_m(t) - z_m \tag{6.3}$$

and therefore from criteria (i) given in equation (6.1) we deduce $\|z_{2m}(t)\| < 2MT$ and $\|z_{2m}(t') - z_{2m}(t'')\| < 2M|t'-t''|$. □

We now give a corollary to Theorem 6.2.

Corollary 6.3. *For all $T > 0$, there exists a compact set $E \subset \mathbb{R}^n$ so that $z_{2m}(t) \in E$ for $0 \le t \le T$ and there exists a subsequence $z_{2r}(t)$, defined on $[0,T]$ which converges uniformly to $\phi(t)$ [or $y_2(t)$]. This subsequence converges to the limit of the second kind of the solution $x(t)$.*

Theorem 6.4. *Suppose that $x(t)$ and E are respectively the solution and the compact set defined in Corollary 6.3. Then the sequence of functions $G_{2m}(t,x) = \int_0^t \Gamma_{2m}(u,x)du$, where $0 \le t \le T, x \in E, m = 1, 2, \dots$ is equibounded and equicontinuous.*

Proof. From the two criteria given in equation (6.1) we deduce that $\|G_{2m}(t,x)\| < 2MT$, and $\|G_{2m}(t',x') - G_{2m}(t'',x'')\| < \epsilon$ when $|t'-t''| < \frac{\epsilon}{4M}$ and $\|x'-x''\| < \frac{\epsilon}{4LT}$. Obviously the sequence of functions $W_{2m}(t,x) = \int_0^t U_{2m}(u,x)du$ are also equibounded and equicontinuous. Consequently there exists two subsequences $G_{2r}(t,x)$ and $W_{2r}(t,x)$ which are uniformly convergent respectively to $\int_0^t g_2(u,x)du$ and to $\int_0^t W_2(u,x)du$. This implies that the functions g_2 and W_2 are limiting functions of second kind for Γ and U.

Observe that from $\dot{x}(t) = f(t, x(t)) + F(t, x(t))$ for $0 \le t \le T$ we deduce

$$\dot{z}_{2m}(t) = f_{2m}(t, z_{2m}(t)) + F_{2m}(t, z_{2m}(t)) \tag{6.4}$$

and hence $\dot{\phi}(t) = g_2(t, \phi(t))$ for $0 \le t \le T$. □

Theorems 6.2 and 6.4 are, respectively, the analog of Lemmas 1 and 2 of [3]. We denote by $g_1(t, x)$, $y_1(t)$ and $W_1(t, x)$ respectively the limits of the first kind for $f(t, x)$, $x(t)$ and $U(t, x)$ with regard to t_m.

Definition 6.5. Given $V = V(t, x)$, $t \ge 0$ and a constant $c \in \mathbb{R}$ we define and denote by $V_\infty^{-2}(t, c)$ the set $\{x \in \mathbb{R}^n\}$ such that there exists two sequences $\{t_m\}$, $\{z_m\}$ for which

$$\lim_{m \to +\infty} V(t + t_m, x + z_m) = \lim_{m \to +\infty} V_{2m}(t, x) = c.$$

The previous definition recalls the definition of $N_2(t, c)$ given in [3].

Theorem 6.6. *Under the first hypotheses of Theorem 3.3, suppose that*

(i) $\dot{V}(t, x) \le -U(t, x) \le 0$ *on* S
(ii) $\lim_{V \to 0} h = 0$
(iii) $j = 1, 2$
(iv) *for every pair* (g_j, W_j) *limit of* (f, U) *the set* $\{(W_j = 0) \cap V_\infty^{-j}(t, c)\}$ *contains no solutions of the limiting system* $\dot{x} = g_j(t, x)$ *for every* $c > 0$.

Then the system (2.1) *is t-asymptotically stable.*

Proof. The system (2.1) is t-stable so suppose that $t_0 \in I$ exists so that for all $\delta > 0$ there exists an $\eta > 0$ and there exists $x_0 \in \mathbb{R}^n$ satisfying $h_0(t_0, x_0) < \delta$ and there exists a sequence t_m so that we have $h(t'_m, x(t'_m)) \ge \eta$ where $t'_m = t_0 + t_m$ and $x(t) = x(t, t_0, x_0)$. Let $v(t) = V(t, x(t))$. Then, since $\dot{V} \le 0$, we have

$$\lim_{t \to +\infty} v(t) = c \ge 0$$

and if $c = 0$ we have the proof. If $c > 0$ with $\|x(t)\|$ bounded we proceed as in Theorem 5.1. If $c > 0$ and $\sup\{x^2(t)\} = +\infty$ then consider the sequences $x(t'_m) = z_m$, $x(t + t'_m) = z_m(t)$, $z_{2m}(t) = z_m(t) - z_m$ and the corresponding translates of the second kind $\Gamma_{2m}(t, x)$, $U_{2m}(t, x)$ where $t \in [0, T]$ and $x \in E$. The set E is defined in Corollary 6.3. We have

$$\dot{z}_{2m}(t) = \Gamma_{2m}(t, z_{2m}(t)). \tag{6.5}$$

From the previous theorem and their assumptions there exist three subsequences $\Gamma_{2r}(t, x)$ converging to $g_2(t, x)$, $U_{2r}(t, x)$ converging to $W_2(t, x)$, and $z_{2r}(t)$ converging uniformly to $\phi(t)$ with $\dot{\phi}(t) = g_2(t, \phi(t))$ on $[0, T]$. From (i) we deduce

$$v(t + t'_m) - v(t'_m) \le -\int_{t'_m}^{t'_m + t} U(u, x(u))du \le 0. \tag{6.6}$$

With the known procedure we obtain $W_2(t, \phi(t)) = 0$ $\forall t \in [0, T]$ and also $\phi(t) \in V_\infty^{-2}(t, c)$ with $c > 0$, which is a contradiction. □

Theorem 6.7. *Under the hypotheses of Theorem 6.6, suppose that:*

(i) *h satisfies the assumptions of Section 6*
(ii) *the system* (2.1) *is* t, u*-stable*

(iii) *for every limiting pair* (g_j, l_j) *of* (f, h) *the set* $\{h_\infty^{-j}(t,0)\}$ *contains only the solutions* $y_j = y_j(t)$ *of the limiting system* $\dot{x} = g_i(t,x)$ *so that* $l_j(0, y_j(0)) = 0$ *when* $j = 1$ *and* 2.

Then the system (2.1) *is* t, u*-asymptotically stable.*

This theorem brings to mind Theorem 5.3. In fact, given $T > 0$ and $t \in [0, T]$, if we let

$$x_m(t) = x(t, t_m, x_m), \quad z_m(t) = x_m(t + t''_m),$$
$$z_m = z_m(0) = x_m(t''_m), \quad z_{2m}(t) = z_m(t) - z_m$$

we find that the functions $z_{2m}(t)$ are equibounded and equicontinuous in $[0, T]$, and so it is sufficient to establish the proof.

We proceed with an example.

6.1. **Example.** Consider the differential system from $I \times \mathbb{R}^3$ to $\mathbb{R}^3$

$$\begin{gathered}
\dot{u} = -qq_u + [(1-\mu)q - \frac{r}{p}]v \\
\dot{v} = [1 - (1-\mu)\frac{pq}{r}]qq_u + [\frac{1}{2}(\frac{\dot{p}}{p} - \frac{\dot{r}}{r}) - \frac{r}{p}]v - qrz \\
\dot{z} = \frac{r^2 q}{p} v - \nu^2 z
\end{gathered} \tag{6.7}$$

and the vector $G = [qq_u, \frac{r}{p}v, z]$. In system (6.7) u, v, z are the unknown variables. The function $q = q(u) : \mathbb{R} \to \mathbb{R}$ belongs to C^2; $\mu = \mu(t, u, v, z) : I \times \mathbb{R}^3 \to I$ satisfies the Caratheodory conditions given in equation (2.3); $p = p(t)$, $r = r(t) : I \to R$ are continuous with locally integrable derivatives $\dot{p}(t), \dot{r}(t)$; also $\nu = \nu(t) : I \to \mathbb{R}$ is locally integrable. Relatively to p and r assume $p_1 \geq p(t) \geq p_0 > 0, r_1 \geq r(t) \geq r_0 > 0$ with p_0, p_1 and r_0, r_1 as constants.

Theorem 6.8. *Suppose that*

(i) $q(u)q_u(u) \neq 0$ *for* $u \in \mathbb{R}$
(ii) $0 < \frac{r}{p} \leq 2$ *for* $t \geq 0$
(iii) $h = q^2 + \frac{r}{p}v^2 + z^2$, $h_0 = q^2 + 2v^2 + z^2$.

Then the system (6.7) *is* t*-stable with respect to the following perturbations:*

$$F = [-qq_u M_1, \quad -\frac{r}{p}vM_2, \quad -zM_3] \tag{6.8}$$

where $M_\iota = M_\iota(t, u, v, z) : I \times \mathbb{R}^3 \to I$ *are arbitrary functions for* $\iota = 1, 2, 3$ *that satisfy the conditions given in equation* (2.3). *If the ratio of* r *to* p *is constant then the system* (6.7) *is* (t, u)*-stable.*

Proof. If we select the auxiliary function

$$V = q^2 + \frac{r}{p}v^2 + z^2 = h$$

we obtain

$$\dot{V} = -2[(qq_u)^2 + (\frac{r}{p})^2 v^2 + \nu^2 z^2] \leq 0; \quad \operatorname{grad} V = 2G. \tag{6.9}$$

Now the proof follows by Theorem 3.3, since $\lim_{h \to 0} V = 0$ and vice versa and since $F \cdot \operatorname{grad} V \leq 0$. □

Observe that the sets where h and $\|f\| < \sigma$, for every $\sigma > 0$, are bounded in v, z and unbounded in u.

Theorem 6.9. *Under the hypothesis of Theorem 6.8 suppose that*

(i) $\lim_{u\to+\infty} q(u) = 0$ *and*
$$\lim_{t\to+\infty}\{(1-\mu), \frac{r}{p}, \nu, (\frac{\dot p}{p} - \frac{\dot r}{r})\} = \{0, r_2 \neq 0, \nu_2 \neq 0, 0\}$$
(ii) *the functions q, q_u and μ are bounded with their derivatives with respect to u, v, z on every compact set of $\mathbb{R}^3$*
(iii) *M_ι is integrally bi-convergent to zero.*

Then the system (6.7) *is t-asymptotically stable.*

Proof. System (6.7) is precompact. The limit of system (6.7) and the limit of a perturbed system are identical and unique. Letting
$$U = (qq_u)^2 + (\frac{r}{p}v)^2 + (\nu z)^2$$
we obtain
$$W_1 = (qq_u)^2 + (r_2 v)^2 + (\nu_2 z)^2$$
and so $\{W_1 = 0\} = \emptyset$. Since
$$W_2 = (r_2 v)^2 + (\nu_2 z)^2 \quad \text{and} \quad \{W_2 = 0\} = \{v = z = 0\}$$
we consider the limiting system of second kind given by
$$\dot u = -r_2 v, \quad \dot v = -r_2 v, \quad \dot z = -{\nu_2}^2 z. \tag{6.10}$$
On the set $\{W_2 = 0\}$ we have the system $\dot u = 0$, $0 = 0$, $0 = 0$ with the solutions $\{u = c, v = z = 0\}$ for some constant c. Also $V = q^2(u)$. Hence
$$\lim_{m\to\infty} V(t + t_m, u + x_m) = \lim_{m\to\infty} q^2(u + x_m) = 0.$$
Therefore, the set $\{(W_j = 0) \cap V_\infty^{-j}(t, 0)\}$ contains no solutions of first and second type for $c > 0$ and $j = 1, 2$. $\square$

Conclusion. The total stability is a property of some differential equations that has the power to influence the behavior of the solutions of some other differential equations. Since, in literature, there is not an analogous work relative to the attractivity, this paper is written to fill this gap.

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References

[1] A. S. Andreev, *On the asymptotic stability and instability of zeroth solution of an non-autonomous system*, PMM. Vol 48, 2 (1984).
[2] A. S. Andreev, *Sulla stabilità asintotica ed instabilità*, Rend. Sem. Mat. Univ. Padova. Vol. 75 (1986).
[3] A. S. Andreev, *Investigation of partial asymptotic stability and instability based on the limiting equations* PMM, Vol. 51, 2 (1987).
[4] A. S. Andreev, G. Zappala', *On stability for perturbed differential equations*, Le Matematiche Vol. LI (1996).
[5] Z. Artstein, *Topological dynamics of an ordinary differential equation*, J. of Differ. Equat. 23 (1977).

[6] Z. Artstein, *The limiting equations of non autonomous ordinary differential equations*, J. of Differ. Equat. 25 (1977).

[7] Z. Artstein, *Uniform asymptotic stability via limiting equations*, J. of Differ. Equat. 27 (1977).

[8] E. A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York (1955).

[9] V. Lakshmikantham, S. Leela, A. Martynyuk, *Stability Analysis of Nonlinear Systems.* Marcel Dekker, New York (1989).

[10] J. P. LaSalle, *Stability of nonautonomous systems*, J. Nonlinear Analysis, TMA, 1 (1976).

[11] V. M. Matrosov, *On the stability of motion*, PMM 26 (1962).

[12] A. s. Oziraner, *On stability of motion relative to a part of variables under constantly acting perturbations*, PMM, Vol. 45 (1982).

[13] N. Rouche, P. Habets, M. Laloy, *Stability theory by Liapunov Direct Method*, Springer Verlag N.Y. Berlin (1977).

[14] L. Salvadori, *Famiglie ad un parametro di funzioni di Liapunov nello studio della stabilità*, Symposia Math. Academic Press. London (1971).

[15] P. Seibert, *Stability under perturbations in generalized dynamical systems*, Proc. Intern. Symp. Non-lin. Diff. Eq. anal Non-lin. Mech, Academic Press (1963).

[16] G. Sell, *Non autonomous differential equations and topological dynamics*, I- II Trans. Amer. Math. Soc. 127 (1967).

[17] G. E. Silov, *Analisi Matematica, Funzioni di una variabile* , ed. MIR. Mosca (1978).

[18] G. Zappala', *On total functional stability with two measures*, Le Matematiche, Vol. LIII(1998)-Fasc. II, pp. 263-275.

[19] G. Zappala', *On relative asymptotic stability with two measures via limiting equations*, Procedings "WASCOM 2003".

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