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EXISTENCE AND REGULARITY OF LOCAL SOLUTIONS TO PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this paper, we establish results concerning, existence, uniqueness, global continuation, and regularity of integral solutions to some partial neutral functional differential equations with infinite delay. These equations find their origin in the description of heat flow models, viscoelastic and thermoviscoelastic materials, and lossless transmission lines models; see for example [15] and [38].

1. Introduction

In this article, we consider the following nonlinear partial neutral functional differential equations with infinite delay

$$\begin{gathered} \frac{\partial}{\partial t}\mathcal{D}u_t = A\mathcal{D}u_t + F(t,u_t), \quad t \ge 0, \\ u_0 = \phi \in \mathcal{B}, \end{gathered} \tag{1.1}$$

where $A : D(A) \subseteq E \to E$ is a linear operator on a Banach space $(E, |\cdot|)$, $\mathcal{B}$ is the phase space of functions mapping $(-\infty, 0]$ into E, which will be specified later, $\mathcal{D}$ is a bounded linear operator from $\mathcal{B}$ to E defined by

$$\mathcal{D}\varphi = \varphi(0) - \mathcal{D}_0\varphi \quad \text{for } \varphi \in \mathcal{B}.$$

The operator $\mathcal{D}_0$ is a bounded and linear from $\mathcal{B}$ to E and for each $u : (-\infty, b] \to E$, $b > 0$, and $t \in [0, b]$, u_t represents, as usual, the mapping defined from $(-\infty, 0]$ to E by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in (-\infty, 0].$$

The operator F is an E-valued nonlinear continuous mapping on $\mathbb{R}_+ \times \mathcal{B}$.

Throughout this paper, we suppose that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a (semi)normed abstract linear space of functions mapping $(-\infty, 0]$ to E, and satisfies the following fundamental axioms which were introduced in [20] and widely discussed in [25].

(A1) There exist a positive constant H and functions $K(.)$, $M(.)$ form $\mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and

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$a > 0$, if $x : (-\infty, \sigma + a] \to E$, $x_\sigma \in \mathcal{B}$ and $x(.)$ is continuous on $[\sigma, \sigma + a]$, then for every t in $[\sigma, \sigma + a]$ the following conditions hold:
 (i) $x_t \in \mathcal{B}$,
 (ii) $|x(t)| \leq H \|x_t\|_{\mathcal{B}}$, which is equivalent to
 (ii') $|\varphi(0)| \leq H\|\varphi\|_{\mathcal{B}}$, for every $\varphi \in \mathcal{B}$
 (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} |x(s)| + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$.

(A2) For the function $x(.)$ in (A1), $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.

(B1) The space $\mathcal{B}$ is complete.

Example. Define for a constant γ the following standard space

$$C_\gamma := \{\phi : (-\infty, 0] \to E \text{ continuous such that } \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\}.$$

It is known from [25] that C_γ with the norm $\|\phi\|_\gamma = \sup_{\theta \leq 0} e^{\gamma\theta}|\phi(\theta)|$, $\phi \in C_\gamma$, satisfies the axioms $(A1)$, $(A2)$ and (B) with $H = 1$, $K(t) = \max(1, e^{-\gamma t})$ and $M(t) = e^{-\gamma t}$ for all $t \geq 0$.

Throughout, we also assume that the operator A satisfies the Hille-Yosida condition:

(H1) There exist $\bar{M} \geq 0$ and $\overline{\omega} \in \mathbb{N}$ such that $]\overline{\omega}, +\infty[\subset \rho(A)$ and

$$\sup\{(\lambda - \overline{\omega})^n \|(\lambda I - A)^{-n}\| : n \in \mathbb{N},\, \lambda > \overline{\omega}\} \leq \bar{M}. \tag{1.2}$$

Let A_0 be the part of the operator A in $\overline{D(A)}$, which is defined by

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\},$$
$$A_0 x = Ax, \text{ for } x \in D(A_0).$$

It is well known that $\overline{D(A_0)} = \overline{D(A)}$ and the operator A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.

From [33], we recall that for all $x \in \overline{D(A)}$ and $t \geq 0$, one has $\int_0^t T_0(s)x \in D(A_0)$ and

$$(A \int_0^t T_0(s)x ds) + x = T_0(t)x. \tag{1.3}$$

We also recall that $(T_0(t))_{t\geq 0}$ coincides on $\overline{D(A_0)}$ with the derivative of the locally Lipschitz integrated semigroup $(S(t))_{t\geq 0}$ generated by A on E. Which is, according to [8] and [27], a family of bounded linear operators on E, that satisfies
 (i) $S(0) = 0$,
 (ii) for any $y \in E$, $t \to S(t)y$ is strongly continuous with values in E,
 (iii) $S(s)S(t) = \int_0^s (S(t + r) - S(r))dr$ for all $t, s \geq 0$, and for any $\tau > 0$ there exists a constant $l(\tau) > 0$ such that

$$\|S(t) - S(s)\| \leq l(\tau)|t - s| \text{ for all } t, s \in [0, \tau].$$

This integrated semigroup is exponentially bounded, that is, there exist two constants $\bar{M}$ and $\overline{\omega}$ such that $\|S(t)\| \leq \bar{M}e^{\overline{\omega}t}$ for all $t \geq 0$.

As stated in Hale [17], Hale and Lunel [21] and the references therein, very much attention has been given to differential difference equations of neutral type. The reason was applications on lossless transmission lines. The development has concerned the general theory of partial neutral functional differential equations. The origin of the special form (1.1) is the description of heat flow models and of the viscoelastic and thermoviscoelastic materials dynamics; see [15] and the references therein. The recent study of (1.1) has been initiated in the case of finite delay by Hale in [18] and [19]. The motivation was a model for a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions introduced by Wu and Xia ([39], [40]). In addition, Magal and Ruan have stated in [29] that (1.1) is also a special case of age structured populations model.

Chen [12] proved some results concerning the existence, uniqueness, and asymptotic behavior of (local and global) solutions of (1.1) in the case where the delay is finite and A generates a compact C_0-semigroup on E. Based mainly on a detailed discussion in the book by Wu [38], Adimy and Ezzinbi have published some other interesting results about (1.1) but also with finite delay (cf. [4]-[7]).

This work (such as [1], [2] and [30]) contributes to the construction of a complete theory about the infinite delay case. It can be seen as an extension to the case of neutral type of some earlier results about functional differential equations with infinite delay in [3]. We do not suppose a global Lipschitz condition as in [1] or [30] nor a compact condition as in [2]. Under a local Lipschitz condition on F, we state the local existence, uniqueness, continuation and regularity.

We recall that in general, neutral functional differential equations with infinite delay are functional differential equations depending on all past and present values, which involve derivatives with infinite delay as well as the unknown function itself. In [23] and [24], existence and regularity of solutions were established to the following neutral functional differential equations with infinite delay

$$\begin{gathered} \frac{d}{dt}[x(t) - G(t, x_t)] = Ax(t) + F(t, x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{gathered} \tag{1.4}$$

where A generates a strongly continuous semigroup on E. G and F are appropriate continuous functions from $[0, +\infty) \times \mathcal{B}$ to E. The authors have essentially used the analytic semigroup theory. More recently, in [22] the same theory was used to prove existence of mild solutions for the so-called partial neutral functional integrodifferential equations with infinite delay using the Leray-Schauder alternative. Finally, more discussion about the comparison between the study of (1.1) and of (1.4) can be found in [1, 4, 9].

2. Preliminaries

Consider the system

$$\begin{gathered} \frac{\partial}{\partial t}\mathcal{D}u_t = A\mathcal{D}u_t \quad \text{if } t \geq 0, \\ u(\theta) = \varphi(\theta) \quad \text{if } \theta \in (-\infty, 0] \text{ with } \varphi \in \mathcal{B}. \end{gathered} \tag{2.1}$$

Using (1.3), we can see that a necessary condition for $u : (-\infty, b) \to E$, $b > 0$, to be a solution of (2.1) is that it verifies the following integrated equation on $(-\infty, b)$

$$\begin{gathered} \mathcal{D}u_t = T_0(t)\mathcal{D}\varphi, \quad t \geq 0, \\ u_0 = \varphi, \end{gathered} \tag{2.2}$$

where $\varphi \in \mathcal{Y} := \{\varphi \in \mathcal{B} : \mathcal{D}\varphi \in \overline{D(A)}\}$.

The following result is only the combination of [2, Lemma 3] and [1, Proposition 11] which are proved in a general framework. Precisely, here it suffices to take $h(t) := T_0(t)\mathcal{D}\varphi$.

Proposition 2.1. *Assume that Condition (H1) is satisfied and $\|\mathcal{D}_0\|K(0) < 1$. Then, for given $\varphi \in \mathcal{Y}$ there exists a unique function u which is continuous on $[0,T)$ and solves (2.2) on $(-\infty,T)$. Moreover, the family of operators $(\mathcal{T}(t))_{t\geq 0}$ defined on $\mathcal{Y}$ by $\mathcal{T}(t)\varphi = u_t(.,\varphi)$ is a C_0-semigroup on $\mathcal{Y}$.*

We now define a fundamental integral solution $Z(t)$ associated to (1.1). Consider for given $c \in E$ the following equation

$$\begin{gathered} \mathcal{D}z_t = S(t)c \quad \text{if } t \geq 0, \\ z(t) = 0 \quad \text{if } t \in (-\infty,0]. \end{gathered} \tag{2.3}$$

To our purpose, we make the following condition

(H2) There exists a continuous nondecreasing function $\delta : [0,+\infty) \to [0,+\infty[$, $\delta(0) = 0$ and a family of continuous linear operators $W_\varepsilon : \mathcal{B} \to E$, $\varepsilon \in [0,+\infty)$, such that

$$|\mathcal{D}_0\varphi - \mathcal{D}_\varepsilon\varphi| \leq \delta(\varepsilon)\|\varphi\|_{\mathcal{B}} \quad \text{for } \varepsilon \in [0,+\infty) \text{ and } \varphi \in \mathcal{B},$$

where the linear operator $\mathcal{D}_\varepsilon : \mathcal{B} \to E$ is defined, for $\varepsilon \in [0,+\infty)$, by

$$\begin{gathered} \mathcal{D}_\varepsilon = W_\varepsilon \circ \tau_\varepsilon, \\ \tau_\varepsilon(\varphi)(\theta) = \varphi(\theta-\varepsilon) \quad \text{for } \varphi \in \mathcal{B} \text{ and } \theta \in (-\infty,0]. \end{gathered}$$

Note that Assumption (H2) implies that the operator $\mathcal{D}_0$ does not depend very strongly upon $\varphi(0)$. It is the infinite delay version of the one introduced in [6, 7].

Proposition 2.2. *Assume that Conditions (H1) and (H2) are satisfied such that $K(0)\|\mathcal{D}_0\| < 1$. Then, for given $c \in E$, (2.3) has a unique integral solution $z := z(.)c : (-\infty,+\infty) \to E$. Moreover, the operator $Z(t) : E \to \mathcal{B}$ defined by*

$$Z(t)c = z_t(.)c$$

satisfies, for any continuous function $f : [0,+\infty) \to E$, the following properties

(i) *For each $T > 0$, there exists a function $\alpha(.) \in L^\infty([0,T],\mathbb{R}^+)$ and $\beta \in \mathbb{R}$, such that $\|Z(t)\| \leq \alpha(t)e^{t\beta}$ for all $t \in [0,T]$;*

(ii) *$Z(t)(E) \subseteq \mathcal{Y}$, for all $t \geq 0$;*

(iii) *For all $\tau > 0$ there exists a constant $k(\tau) > 0$ such that*

$$\|Z(t)c - Z(s)c\|_{\mathcal{B}} \leq k(\tau)|t-s||c| \quad \text{for all } t,s \in [0,\tau] \text{ and } c \in E.$$

(iv) *For any continuous function $f : [0,+\infty) \to E$, the functions*

$$t \mapsto \int_0^t Z(t-s)f(s)\,ds \quad \text{and} \quad t \mapsto \int_0^t S(t-s)f(s)\,ds$$

are continuously differentiable for all $t \geq 0$ and satisfy

$$\frac{d}{dt}(\int_0^t Z(t-s)f(s)\,ds) = \lim_{h\to 0^+}\frac{1}{h}\int_0^t \mathcal{T}(t-s)Z(h)f(s)ds \quad \text{for all } t \geq 0.$$

$$\begin{aligned} \mathcal{D}(\frac{d}{dt}\int_0^t Z(t-s)f(s)\,ds) &= \lim_{h\to 0^+}\frac{1}{h}\int_0^t S'(t-s)S(h)f(s)ds \\ &= \frac{d}{dt}\int_0^t S(t-s)f(s)ds. \end{aligned}$$

Sketch of Proof. Recalling that $\|S(t)\| \leq \bar{M}e^{\bar{\omega}t}$ for all $t \geq 0$, the proof of existence, uniqueness and (i) is only a particular case of [2, Lemma 3] where $h(t) = S(t)c$ and $v_0 = 0$. To prove (ii), it suffices to remark that for any $c \in E$, $S(t)c \in \overline{D(A)}$ for all $t \geq 0$ and $\mathcal{D}(Z(t)c) = S(t)c$. Then $Z(t)c \in \mathcal{Y}$ for all $t \geq 0$. We infer (iii) from the fact that $S(.)$ is locally Lipschitz continuous. Finally, the proof of (iv) is exactly the same as in [7]. Note that (iv) also ensures that $\int_0^t Z(t-s)f(s)ds$ is differentiable with respect to t. $\square$

For convenience of the reader about the main equation (1.1), we recall the following definitions.

Definition 2.3. Let $T > 0$ and $\varphi \in \mathcal{B}$. We consider the following definitions. We say that a function $u := u(.,\varphi) : (-\infty, T) \to E$, $0 < T \leq +\infty$, is an integral solution of (1.1) if:

(1) u is continuous on $[0,T)$,
(ii) $\int_0^t \mathcal{D}u_s\,ds \in D(A)$ for $t \in [0,T)$,
(iii) $\mathcal{D}u_t = \mathcal{D}\varphi + A\int_0^t \mathcal{D}u_s\,ds + \int_0^t F(s,u_s)\,ds$ for $t \in [0,T)$, (**iv**) $u(t) = \varphi(t)$, for all $t \in (-\infty, 0]$.

We deduce from [1] and [36] that integral solutions of (1.1) are given for $\varphi \in \mathcal{B}$ such that $\mathcal{D}\varphi \in \overline{D(A)}$ by the system

$$\begin{gathered} \mathcal{D}u_t = S'(t)\mathcal{D}\varphi + \frac{d}{dt}\int_0^t S(t-s)F(s,u_s)ds, \quad t \in [0,T), \\ u(t) = \varphi(t), \quad t \in (-\infty, 0], \end{gathered} \tag{2.4}$$

Definition 2.4. *Let $\varphi \in \mathcal{B}$. We say that a function $u := u(.,\varphi) : (-\infty,T) \to E$, $0 < T \leq +\infty$, is a strict solution of Eq.* (1.1) *if the following conditions hold:*

(i) $t \to \mathcal{D}u_t \in \mathcal{C}^1([0,T),E) \cap \mathcal{C}([0,T),D(A))$,
(ii) *u satisfies* (1.1) *on* $(-\infty,T)$.

Remark 2.5. It was proved in [1] that if $u := u(.,\varphi) : (-\infty,T) \to E$, $0 < T \leq +\infty$, is an integral solution of (1.1) such that $t \to \mathcal{D}u_t$ belongs to $\mathcal{C}^1([0,T),E)$, then $t \to \mathcal{D}u_t$ belongs to $\mathcal{C}([0,T),D(A))$.

Since our method of proof needs computing integrals in $\mathcal{B}$ from integrals in E, we suppose that $\mathcal{B}$ is normed and satisfies one of the following two extra axioms.

(C1) If $(\phi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{B}$ and if $(\phi_n)_{n\geq 0}$ converges compactly to ϕ on $(-\infty,0]$, then ϕ is in $\mathcal{B}$ and $\|\phi_n - \phi\|_{\mathcal{B}} \to 0$, as $n \to \infty$.
(D1) For a sequence $(\varphi_n)_{n\geq 0}$ in $\mathcal{B}$, if $\|\varphi_n\|_{\mathcal{B}} \to 0$, as $n \to \infty$, then $|\varphi_n(\theta)| \to 0$, as $n \to \infty$, for each $\theta \in (-\infty,0]$.

We remark that Axiom (D1) implies that the space $\mathcal{B}$ is normed.

Lemma 2.6 ([31]). *Let $\mathcal{B}$ be a normed space which satisfies Axiom (C1) and $f : [0,a] \to \mathcal{B}$, $a > 0$, be a continuous function such that $f(t)(\theta)$ is continuous for $(t,\theta) \in [0,a] \times (-\infty,0]$. Then,*

$$\Big[\int_0^a f(t)dt\Big](\theta) = \int_0^a f(t)(\theta)dt, \quad \theta \in (-\infty,0].$$

In [1], we have also obtained the following similar result using (D1).

Lemma 2.7 ([1, 9]). *Assume that $\mathcal{B}$ satisfies Axiom (D1) and $f : [0,a] \to \mathcal{B}$ is a continuous function. Then, for all $\theta \in (-\infty, 0]$, the function $f(.)(\theta)$ is continuous on $[0,a]$ and satisfies*

$$\Big[\int_0^a f(t)dt\Big](\theta) = \int_0^a f(t)(\theta)dt, \quad \theta \in (-\infty, 0].$$

Proposition 2.8. *Let $\mathcal{B}$ be a normed space which satisfies Axiom (C1) or Axiom (D1) with $K(0)\|\mathcal{D}_0\| < 1$. If there exists an integral solution $u := u(.,\varphi) : (-\infty, T) \to E$, $0 < T \leq +\infty$, of (1.1), then the function $[0,T) \ni t \mapsto u_t \in \mathcal{B}$ satisfies*

$$\begin{aligned} u_t &= \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s,u_s)ds \\ &= \mathcal{T}(t)\varphi + \lim_{h\to 0^+}\frac{1}{h}\int_0^t \mathcal{T}(t-s)Z(h)F(s,u_s)ds. \end{aligned} \tag{2.5}$$

Conversely, if there exists a function $v \in \mathcal{C}([0,T),\mathcal{B})$ such that

$$v(t) = \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s,v(s))ds, \quad t \in [0,T) \tag{2.6}$$

then $v(t) = u_t$ for all $t \in [0,T)$, where

$$u(t) = \begin{cases} v(t)(0) & t \in [0,T) \\ \varphi(t) & t \in (-\infty, 0] \end{cases}$$

and $u(.)$ is an integral solution of (1.1).

Proof. First, by Proposition 2.2, it is immediate that for any continuous function $f : [0,T) \to E$,

$$W(t) := \int_0^t Z(t-s)f(s)ds$$

is continuously differentiable and $W'(0) = 0$. Set

$$w(t) = \begin{cases} W(t)(0) & \text{if } t \geq 0 \\ 0 & \text{if } t \in (-\infty, 0]. \end{cases}$$

By Axiom (A1)(ii'), $w(t)$ is continuously differentiable. Lemma 2.6 or Lemma 2.7 implies that for all $t \in [0,T)$,

$$\begin{aligned} w(t) &= \Big(\int_0^t Z(t-s)f(s)ds\Big)(0) \\ &= \int_0^t (Z(t-s)f(s))(0)ds \\ &= \int_0^t z(t-s)f(s)ds. \end{aligned}$$

In general, for all $t \in [0, T)$ and $\theta \in (-\infty, 0]$,

$$\begin{aligned}(W(t))(\theta) &= (\int_0^t Z(t-s)f(s)ds)(\theta)\\ &= \int_0^t (Z(t-s)f(s))(\theta)ds\\ &= \int_0^t z(t+\theta-s)f(s)ds.\end{aligned}$$

Moreover, since $z(s) = 0$ for all $s \in (-\infty, 0]$,

$$\int_0^t z(t+\theta-s)f(s)ds = \int_0^{t+\theta} z(t+\theta-s)f(s)ds$$

and $(W(t))(\theta) = w(t+\theta)$. Which is equivalent to $W(t) = w_t$. On the other hand, we can see that for all $t \in [0, T)$ and $\theta \in (-\infty, 0]$,

$$(W'(t))(\theta) = w'(t+\theta).$$

Hence $W'(t) = (w')_t$ for all $t \in [0, T)$.

Now, suppose that $v(., \varphi)$ is a solution of (2.6). The function $\mathcal{T}(t)\varphi = x_t$ with $x : (-\infty, T) \to E$ is the integral solution of $\mathcal{D}x_t = S'(t)\mathcal{D}\varphi$ such that $x_0 = \varphi$. Set

$$w(t) = \int_0^t z(t-s)F(s, v(s))ds.$$

Then

$$v(t) = x_t + (w')_t = (x + w')_t\,.$$

If we set $u(t) = x(t) + w'(t)$, we obtain $v(t) = u_t$ and

$$\begin{aligned}u_t &= \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s, v(s))ds\\ &= \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds.\end{aligned}$$

Since $\mathcal{D}(\mathcal{T}(t)\varphi) = S'(t)\mathcal{D}\varphi$ and by Proposition 2.2,

$$\mathcal{D}\Big(\frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds\Big) = \frac{d}{dt}\int_0^t S(t-s)F(s, u_s)ds,$$

so that $u(t)$ is an integral solution of (1.1). Conversely, let $u(., \varphi)$ be an integral solution of (1.1) on $(-\infty, T)$. Then

$$\mathcal{D}u_t = S'(t)\mathcal{D}\varphi + \frac{d}{dt}\int_0^t S(t-s)F(s, u_s)ds.$$

By the definition of $\mathcal{T}(t)$,

$$\begin{aligned}\mathcal{D}u_t &= \mathcal{D}(\mathcal{T}(t)\varphi + \mathcal{D}(\frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds\\ &= \mathcal{D}\Big(\mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds\Big)\\ &= \mathcal{D}(x_t + (w')_t),\end{aligned}$$

where $x : (-\infty, T) \to E$ is the integral solution of $\mathcal{D}x_t = S'(t)\mathcal{D}\varphi$, and $w(t)$ is defined by

$$w(t) = \int_0^t z(t-s)F(s, v(s))ds.$$

We deduce that, $\mathcal{D}[(u-(x+w'))_t] = 0$, and hence $u-(x+w') = 0$. Consequently,

$$u_t = x_t + (w')_t = \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds.$$

Which completes the proof. $\square$

3. Existence and regularity of local solutions

To obtain our results on existence, uniqueness and regularity of solutions to (1.1), we add an extra condition

(H3) $F : [0, +\infty[\times\mathcal{B}$ is Lipschitz continuous with respect to φ on the balls of $\mathcal{B}$; i.e., for each $r > 0$ there exists a constant $c_0(r) > 0$ such that if $t \geq 0$, $\varphi_1, \varphi_2 \in \mathcal{B}$ and $\|\varphi_1\|_{\mathcal{B}}, \|\varphi_2\|_{\mathcal{B}} \leq r$ then

$$|F(t,\varphi_1) - F(t,\varphi_2)| \leq c_0(r)\|\varphi_1 - \varphi_2\|_{\mathcal{B}}.$$

Theorem 3.1. *Let $\mathcal{B}$ be a normed space which satisfies Axiom (C1) or Axiom (D1) with $K(0)\|\mathcal{D}_0\| < 1$. Assume that (H1) (H2) and (H3) hold. Let $\varphi \in \mathcal{B}$ such that $\mathcal{D}\varphi \in \overline{D(A)}$. Then, there exists a maximal interval of existence $(-\infty, b_\varphi)$, $b_\varphi > 0$, and a unique mild solution $u(.,\varphi)$ of (1.1), defined on $(-\infty, b_\varphi)$ and either $b_\varphi = +\infty$ or*

$$\limsup_{t\to b_\varphi^-} |u(t,\varphi)| = +\infty.$$

Moreover, $u(t,\varphi)$ is a continuous function of φ, in the sense that if $\varphi \in \mathcal{B}$, $\mathcal{D}\varphi \in \overline{D(A)}$ and $t \in [0, b_\varphi)$, then there exist positive constants β and α such that, for $\psi \in \mathcal{B}$, $\mathcal{D}\psi \in \overline{D(A)}$ and $\|\varphi-\psi\|_{\mathcal{B}} < \alpha$, we have $t \in [0, b_\psi)$ and

$$|u(s,\varphi) - u(s,\psi)| \leq \beta\|\varphi-\psi\|_{\mathcal{B}} \quad \text{for all } s \in [0,t].$$

Proof. The first part of the proof is contained in [10]. We prove that the solution depends continuously on the initial data. Let $\varphi \in \mathcal{B}$ such that $\mathcal{D}\varphi \in \overline{D(A)}$ and $t \in [0, b_\varphi)$ be fixed. Set

$$r = 1 + \sup_{0\leq s\leq t} \|u_s(.,\varphi)\|_{\mathcal{B}},$$
$$c(t) = \overline{M}e^{\omega t}\exp(\overline{M}e^{\omega t}c_0(r)kt).$$

Let $\alpha \in (0,1)$ be such that $c(t)\alpha < 1$ and $\psi \in \mathcal{B}$, $\mathcal{D}\psi \in \overline{D(A)}$ such that $\|\varphi-\psi\|_{\mathcal{B}} < \alpha$. We have

$$\|\psi\|_{\mathcal{B}} \leq \|\varphi\|_{\mathcal{B}} + \alpha < r.$$

Let

$$b_0 = \sup\{s \in (0, b_\psi) : \|u_\sigma(.,\psi)\|_{\mathcal{B}} \leq r \text{ for all } \sigma \in [0,s]\}.$$

Suppose that $b_0 < t$. We can see similarly as in [10] that for $s \in [0, b_0]$,

$$\|u_s(.,\varphi) - u_s(.,\psi)\|_{\mathcal{B}} \leq \overline{M}e^{\omega t}(\|\varphi-\psi\|_{\mathcal{B}} + c_0(r)k\int_0^s \|u_\sigma(.,\varphi) - u_\sigma(.,\psi)\|_{\mathcal{B}}\, d\sigma).$$

By Gronwall's lemma, we deduce that

$$\|u_s(.,\varphi) - u_s(.,\psi)\|_{\mathcal{B}} \leq c(t)\|\varphi-\psi\|_{\mathcal{B}}. \tag{3.1}$$

This implies

$$\|u_s(.,\psi)\|_{\mathcal{B}} \le c(t)\alpha + r - 1 < r \quad \text{for all } s \in [0, b_0].$$

By continuity, there exists $\delta > 0$ such that

$$\|u_s(.,\psi)\|_{\mathcal{B}} \le c(t)\alpha + r - 1 < r \quad \text{for all } s \in [0, b_0 + \delta].$$

It follows that b_0 cannot be the largest number $s > 0$ such that $\|u_\sigma(.,\psi)\|_{\mathcal{B}} \le r$, for all $\sigma \in [0, s]$. Thus, $b_0 \ge t$ and $t < b_\psi$. Furthermore, $\|u_s(.,\psi)\|_{\mathcal{B}} \le r$, for $s \in [0, t]$. Then, using inequality (3.1), we deduce the continuous dependence on the initial data. This completes the proof of Theorem 3.1. □

As in [3], we can obtain the strictness of the integral solution to (1.1) under similar restrictive conditions on φ and F; namely, (3.4) below, (H3) and

(H4) $F : [0, +\infty) \times \mathcal{B} \to E$ is continuously differentiable and the derivatives $D_t F$ and $D_\varphi F$ satisfy the locally Lipschitz condition (H3), i.e., for each $r > 0$ there exist constants $C_1(r), C_2(r) > 0$ such that if $t \ge 0$, $\varphi, \psi \in \mathcal{B}$ and $\|\varphi\|_{\mathcal{B}}, \|\psi\|_{\mathcal{B}} \le r$ then

$$|D_t F(t,\varphi) - D_t F(t,\psi)| \le C_1(r)\|\varphi - \psi\|_{\mathcal{B}}, \tag{3.2}$$

$$\|D_\varphi F(t,\varphi) - D_\varphi F(t,\psi)\| \le C_2(r)\|\varphi - \psi\|_{\mathcal{B}}\,. \tag{3.3}$$

Theorem 3.2. *Suppose that (H4) and the assumptions of Theorem 3.1 are satisfied. In addition, let an element φ of $\mathcal{B}$ be continuously differentiable such that*

$$\varphi' \in \mathcal{B}, \quad \mathcal{D}\varphi \in D(A), \quad \mathcal{D}\varphi' \in \overline{D(A)}, \quad \mathcal{D}\varphi' = A\mathcal{D}\varphi + F(0,\varphi). \tag{3.4}$$

Then, the integral solution asserted by Theorem 3.1 is a strict solution of (1.1).

Proof. Let $\varphi \in \mathcal{B}$ such that $\varphi' \in \mathcal{B}$, $\mathcal{D}\varphi \in D(A)$, $\mathcal{D}\varphi' \in \overline{D(A)}$ and $\mathcal{D}\varphi' = A\mathcal{D}\varphi + F(0,\varphi)$. Let $u := u(.,\varphi)$ be the unique integral solution of (1.1) on $(-\infty, b_\varphi)$. To prove that u is also a strict solution, by Remark 2.5, it suffices to show that $t \mapsto Du_t$ is continuously differentiable on $[0, b_\varphi)$. For that purpose, consider the linear equation

$$\begin{gathered} \frac{\partial}{\partial t}\mathcal{D}v_t = A\mathcal{D}v_t + D_t F(t,u_t) + D_\varphi F(t,u_t)\, v_t, \quad t \ge 0, \\ v_0 = \varphi'. \end{gathered} \tag{3.5}$$

Using Axiom (A2), we can set $r := \sup_{0\le s\le T}\|u_s\|_{\mathcal{B}}$ for each $0 \le T < b_\varphi$. Then the fact that F is continuously differentiable and (3.3) imply that there exists $\beta_0 > 0$ such that $\|D_\varphi F(t,u_t)\| \le \beta_0$ for all $t \in [0,T]$ where $0 \le T < b_\varphi$. Hence for all $0 \le T < b_\varphi$, the function $G : [0,T] \times \mathcal{B} \to E$ defined by $G(t,\psi) := D_t F(t,u_t) + D_\varphi F(t,u_t)\psi$ is uniformly Lipschitzian with respect to ψ. Then, using the same reasoning as in the proof in [1, Theorem 7], one can show that (3.5) has a unique integral solution v on $(-\infty, b_\varphi)$ given by

$$\begin{gathered} \mathcal{D}v_t = S'(t)\mathcal{D}\varphi' + \frac{d}{dt}\int_0^t S(t-s)(D_t F(s,u_s) + D_\varphi F(s,u_s)v_s)ds, \quad t \in [0, b_\varphi) \\ v_0 = \varphi'. \end{gathered}$$

Let $w : (-\infty, b_\varphi) \to E$ be the function defined by

$$w(t) = \begin{cases} \varphi(t) & \text{if } t \in (-\infty, 0], \\ \varphi(0) + \int_0^t v(s)\,ds & \text{if } t \in [0, b_\varphi). \end{cases}$$

Then, using Lemma 2.6 or Lemma 2.7,

$$w_t = \varphi + \int_0^t v_s \, ds \text{ for } t \in [0, b_\varphi).$$

Integrating the equation of v_t, we get

$$\int_0^t \mathcal{D} v_s ds = S(t)\mathcal{D}\varphi' + \int_0^t S(t-s)(D_t F(s, u_s) + D_\varphi F(s, u_s) v_s) ds. \tag{3.6}$$

Since

$$\int_0^t \mathcal{D} v_s ds = \mathcal{D}(\int_0^t v_s ds) = \mathcal{D} w_t - \mathcal{D}\varphi,$$

equality (3.6) becomes

$$\mathcal{D} w_t = \mathcal{D}\varphi + S(t)\mathcal{D}\varphi' + \int_0^t S(t-s)(D_t F(s, u_s) + D_\varphi F(s, u_s) v_s) ds.$$

On the other hand, from the assumption, $\mathcal{D}\varphi' = A\mathcal{D}\varphi + F(0, \varphi)$. Then

$$S(t)\mathcal{D}\varphi' = S(t)A\mathcal{D}\varphi + S(t)F(0, \varphi).$$

Since $\mathcal{D}\varphi \in D(A)$, we have $S(t)A\mathcal{D}\varphi = S'(t)\mathcal{D}\varphi - \mathcal{D}\varphi$. Hence

$$S(t)\mathcal{D}\varphi' = S'(t)\mathcal{D}\varphi - \mathcal{D}\varphi + S(t)F(0, \varphi).$$

Thus w_t satisfies

$$\mathcal{D} w_t = S'(t)\mathcal{D}\varphi + S(t)F(0, \varphi) + \int_0^t S(t-s)(D_t F(s, u_s) + D_\varphi F(s, u_s) v_s) ds. \tag{3.7}$$

Note that

$$\int_0^t S(t-s) F(s, w_s) ds = \int_0^t S(s) F(t-s, w_{t-s}) \, ds.$$

Since $t \mapsto w_t$ is continuously differentiable and $F(t-s, \varphi)$ is also continuously differentiable, it follows that $F(t-s, w_{t-s})$ is continuously differentiable with respect to t. Thus

$$\begin{aligned}
&\frac{d}{dt} \int_0^t S(t-s) F(s, w_s) ds \\
&= S(t)F(0, \varphi) + \int_0^t S(s)(D_t F(t-s, w_{t-s}) + D_\varphi F(t-s, w_{t-s}) \frac{d}{dt} w_{t-s}) ds \\
&= S(t)F(0, \varphi) + \int_0^t S(t-s)\Big(D_t F(s, w_s) + D_\varphi F(s, w_s) v_s\Big) ds.
\end{aligned}$$

We deduce that

$$S(t)F(0, \varphi) = \frac{d}{dt} \int_0^t S(t-s) F(s, w_s) ds - \int_0^t S(t-s)(D_t F(s, w_s) + D_\varphi F(s, w_s) v_s) ds.$$

Therefore, (3.7) becomes

$$\begin{aligned}
\mathcal{D} w_t = S'(t)\mathcal{D}\varphi &+ \frac{d}{dt} \int_0^t S(t-s) F(s, w_s) ds \\
&- \int_0^t S(t-s)(D_t F(s, w_s) + D_\varphi F(s, w_s) v_s) ds \\
&+ \int_0^t S(t-s)(D_t F(s, u_s) + D_\varphi F(s, u_s) v_s) ds.
\end{aligned}$$

Since the integral solution u satisfies

$$\mathcal{D}u_t = S'(t)\mathcal{D}\varphi + \frac{d}{dt}\int_0^t S(t-s)F(s,u_s)ds,$$

we get

$$\begin{aligned}\mathcal{D}(u_t - w_t) &= \frac{d}{dt}\int_0^t S(t-s)(F(s,u_s)-F(s,w_s))ds \\ &\quad - \int_0^t S(t-s)(D_tF(s,u_s)-D_tF(s,w_s))ds \\ &\quad - \int_0^t S(t-s)(D_\varphi F(s,u_s)-D_\varphi F(s,w_s))v_s ds.\end{aligned}$$

Let $0 \le T < b_\varphi$ and choose $T_1 := \min\{\varepsilon, T - T/2\}$ with $\varepsilon \in (0,T]$, we obtain for $t \in [0,T_1]$ and $\theta \in (-\infty,0]$

$$-\infty < t+\theta-\varepsilon \le t-\varepsilon \le 0.$$

Since $u(\theta) = w(\theta) = \varphi(\theta)$ for all $\theta \le 0$, it follows that

$$\begin{aligned}\tau_\varepsilon(u_t)(\theta) = u_t(\theta-\varepsilon) = u(t+\theta-\varepsilon) = \varphi(t+\theta-\varepsilon), \\ \tau_\varepsilon(w_t)(\theta) = w_t(\theta-\varepsilon) = w(t+\theta-\varepsilon) = \varphi(t+\theta-\varepsilon).\end{aligned}$$

Since W_ε is linear,

$$\mathcal{D}_\varepsilon(u_t - w_t) = W_\varepsilon \circ \tau_\varepsilon(u_t - w_t) = 0,$$

and

$$\begin{aligned}\mathcal{D}(u_t - w_t) &= u(t) - w(t) - \mathcal{D}_0(u_t - w_t) \\ &= u(t) - w(t) - (\mathcal{D}_0(u_t - w_t) - \mathcal{D}_\varepsilon(u_t - w_t)).\end{aligned}$$

Consequently,

$$\begin{aligned}u(t) - w(t) =& \mathcal{D}_0(u_t - w_t) - \mathcal{D}_\varepsilon(u_t - w_t) \\ &+ \frac{d}{dt}\int_0^t S(t-s)(F(s,u_s)-F(s,w_s))ds \\ &- \int_0^t S(t-s)(D_tF(s,u_s)-D_tF(s,w_s))ds \\ &- \int_0^t S(t-s)(D_\varphi F(s,u_s)-D_\varphi F(s,w_s))v_s ds.\end{aligned} \tag{3.8}$$

Recall that by Proposition 2.2,

$$\begin{aligned}&\frac{d}{dt}\int_0^t S(t-s)(F(s,u_s)-F(s,w_s))\,ds \\ &= \lim_{h\to 0^+}\frac{1}{h}\int_0^t S'(t-s)S(h)(F(s,u_s)-F(s,w_s))\,ds.\end{aligned}$$

Since

$$\limsup_{h\to 0^+}\frac{1}{h}\|S(h)\| < +\infty.$$

Hence, for suitable constants $\overline{M}, \overline{\omega} > 0$ and for all $t \in [0, T_1]$,

$$\begin{aligned}
&\big|\frac{d}{dt}\int_0^t S(t-s)(F(s,u_s)-F(s,w_s))\,ds\big| \\
&\leq \overline{M}e^{\overline{\omega}T_1}\int_0^t |F(s,u_s)-F(s,w_s)|\,ds.
\end{aligned}$$

Since $S(t)$ is assumed to be exponentially bounded, we have also for suitable positive constants $\bar{M}$ and $\overline{\omega}$,

$$\begin{aligned}
&\big|\int_0^t S(t-s)(D_tF(s,u_s)-D_tF(s,w_s))\,ds\big| \\
&\leq \bar{M}e^{\overline{\omega}T_1}\int_0^t |D_tF(s,u_s)-D_tF(s,w_s)|\,ds,
\end{aligned}$$

and

$$\begin{aligned}
&\big|\int_0^t S(t-s)(D_\varphi F(s,u_s)-D_\varphi F(s,w_s))v_s\,ds\big| \\
&\leq \bar{M}e^{\overline{\omega}T_1}\int_0^t \|D_\varphi F(s,u_s)-D_\varphi F(s,w_s)\|\|v_s\|_{\mathcal{B}}\,ds.
\end{aligned}$$

Set $K_T := \max_{0\leq t\leq T} K(t)$. Since $u_0 = w_0 = \varphi$, by Axiom (A1)(ii), for all $0 \leq t \leq T_1$,

$$\|u_t - w_t\|_{\mathcal{B}} \leq K_T \sup_{0\leq s\leq t}|u(s)-w(s)|.$$

From (H2) and inequality (3.8), we infer that

$$\begin{aligned}
|u(t)-w(t)| \leq& K_T\delta(\varepsilon)\sup_{0\leq s\leq t}|u(s)-w(s)| \\
&+\overline{M}e^{\overline{\omega}T}k\int_0^t |F(s,u_s)-F(s,w_s)|ds \\
&+\bar{M}e^{\overline{\omega}T}\int_0^t |D_tF(s,u_s)-D_tF(s,w_s)|ds \\
&+\bar{M}e^{\overline{\omega}T}\int_0^t \|D_\varphi F(s,u_s)-D_\varphi F(s,w_s)\|\|v_s\|_{\mathcal{B}}\,ds.
\end{aligned}$$

Choose ε small enough such that $K_T\delta(\varepsilon) < 1$. Thus for all $t \in [0, T_1]$,

$$\begin{aligned}
&\|u_t - w_t\|_{\mathcal{B}} \leq K_T \sup_{0\leq s\leq T_1}|u(s)-w(s)| \\
&\quad\leq K_T(1-K_T\delta(\varepsilon))^{-1}\overline{M}e^{\overline{\omega}T_1}k\int_0^t |F(s,u_s)-F(s,w_s)|ds \\
&\quad\quad+K_T(1-K_T\delta(\varepsilon))^{-1}\bar{M}e^{\overline{\omega}T_1}\int_0^t |D_tF(s,u_s)-D_tF(s,w_s)|ds \\
&\quad\quad+K_T(1-K_T\delta(\varepsilon))^{-1}\bar{M}e^{\overline{\omega}T_1}\int_0^t \|D_\varphi F(s,u_s)-D_\varphi F(s,w_s)\|\|v_s\|_{\mathcal{B}}\,ds.
\end{aligned}$$

Set

$$r := \max\big(\sup_{0\leq s\leq T_1}\|u_s\|_B\,,\ \sup_{0\leq s\leq T_1}\|v_s\|_{\mathcal{B}},\ \sup_{0\leq s\leq T_1}\|w_s\|_{\mathcal{B}}\big).$$

There exist $C_0(r), C_1(r), C_2(r) > 0$ such that, for $s \in [0, T_1]$,

$$\begin{aligned}|F(s,u_s) - F(s,w_s)| &\le C_0(r)\|u_s - w_s\|_{\mathcal{B}},\\ |D_tF(t,u_s) - D_tF(t,w_s)| &\le C_1(r)\|u_s - w_s\|_{\mathcal{B}},\\ \|D_\varphi F(t,u_s) - D_\varphi F(t,w_s)\| &\le C_2(r)\|u_s - w_s\|_{\mathcal{B}}.\end{aligned}$$

This implies that for suitable positive constants $\overline{M}$ and $\overline{\omega}$, for all $t \in [0, T_1]$,

$$\|u_t - w_t\|_{\mathcal{B}} \le \frac{K_T\overline{M}e^{\overline{\omega}T_1}}{1 - K_T\delta(\varepsilon)}(kC_0(r) + C_1(r) + rC_2(r))\int_0^t \|u_s - w_s\|_{\mathcal{B}}ds.$$

By the Gronwall lemma, $\|u_t - w_t\|_{\mathcal{B}}$ for any $t \in [0, T_1]$. Using Axiom (A1)(ii), we deduce that $u(t) = w(t)$ for all $t \in [0, T_1]$. We can repeat the previous argument on $[T_1, T_2]$, where $T_2 := \min\{2\varepsilon, T - T/2^2\}$ and $\varepsilon \in (0, T]$, $K_T\delta(\varepsilon) < 1$, with the initial condition u_{T_1}. We obtain for $t \in [T_1, T_2]$ and $\theta \in (-\infty, 0]$,

$$-\infty < t + \theta - \varepsilon \le t - \varepsilon \le T_2 - \varepsilon \le \varepsilon \le T_1.$$

Since $u_{T_1}(\theta) = w_{T_1}(\theta)$ for all $\theta \le 0$, it follows that for $t \in [T_1, T_2]$,

$$\tau_\varepsilon(u_t)(\theta) = u_t(\theta - \varepsilon) = u(t + \theta - \varepsilon) = w(t + \theta - \varepsilon) = w_t(\theta - \varepsilon) = \tau_\varepsilon(w_t)(\theta)$$

Since W_ε is linear, $\mathcal{D}_\varepsilon(u_t - w_t) = W_\varepsilon \circ \tau_\varepsilon(u_t - w_t) = 0$ and

$$\begin{aligned}\mathcal{D}(u_t - w_t) &= u(t) - w(t) - \mathcal{D}_0(u_t - w_t)\\ &= u(t) - w(t) - (\mathcal{D}_0(u_t - w_t) - \mathcal{D}_\varepsilon(u_t - w_t)).\end{aligned}$$

Consequently,

$$\begin{aligned}u(t) - w(t) =& \mathcal{D}_0(u_t - w_t) - \mathcal{D}_\varepsilon(u_t - w_t)\\ &+ \frac{d}{dt}\int_{T_1}^t S(t-s)(F(s,u_s) - F(s,w_s))ds\\ &- \int_{T_1}^t S(t-s)(D_tF(s,u_s) - D_tF(s,w_s))ds\\ &- \int_{T_1}^t S(t-s)(D_\varphi F(s,u_s) - D_\varphi F(s,w_s))v_s ds.\end{aligned}$$

Recall that by Proposition 2.2,

$$\begin{aligned}&\frac{d}{dt}\int_{T_1}^t S(t-s)(F(s,u_s) - F(s,w_s))\,ds\\ &= \lim_{h\to 0^+}\frac{1}{h}\int_{T_1}^t S'(t-s)S(h)(F(s,u_s) - F(s,w_s))\,ds,\end{aligned}$$

since $\limsup_{h\to 0^+}\frac{1}{h}\|S(h)\| < +\infty$. Hence, for suitable constants $\overline{M}, \overline{\omega} > 0$ and for all $t \in [T_1, T_2]$,

$$|\frac{d}{dt}\int_{T_1}^t S(t-s)(F(s,u_s) - F(s,w_s))\,ds| \le \overline{M}e^{\overline{\omega}T_2}\int_{T_1}^t |F(s,u_s) - F(s,w_s)|\,ds.$$

Since $S(t)$ is assumed to be exponentially bounded, we have also for suitable positive constants $\bar{M}$ and $\overline{\omega}$,

$$\begin{aligned}&\Big|\int_{T_1}^t S(t-s)(D_tF(s,u_s)-D_tF(s,w_s))\,ds\Big|\\&\le \bar{M}e^{\overline{\omega}T_2}\int_{T_1}^t |D_tF(s,u_s)-D_tF(s,w_s)|\,ds,\end{aligned}$$

and

$$\begin{aligned}&\Big|\int_{T_1}^t S(t-s)(D_\varphi F(s,u_s)-D_\varphi F(s,w_s))v_s\,ds\Big|\\&\le \bar{M}e^{\overline{\omega}T_2}\int_{T_1}^t \|D_\varphi F(s,u_s)-D_\varphi F(s,w_s)\|\|v_s\|_{\mathcal{B}}\,ds.\end{aligned}$$

Note that $\max_{T_1\le t\le T} K(t-T_1)\le K_T$. Since $u_{T_1}=w_{T_1}$, by Axiom (A1)(iii), for all $T_1\le t\le T_2$,

$$\|u_t-w_t\|_{\mathcal{B}}\le K_T\sup_{T_1\le s\le t}|u(s)-w(s)|,$$

and

$$\begin{aligned}|u(t)-w(t)|\le{}&K_T\delta(\varepsilon)\sup_{T_1\le s\le t}|u(s)-w(s)|\\&+\overline{M}e^{\overline{\omega}T}k\int_{T_1}^t|F(s,u_s)-F(s,w_s)|ds\\&+\bar{M}e^{\overline{\omega}T}\int_{T_1}^t|D_tF(s,u_s)-D_tF(s,w_s)|ds\\&+\bar{M}e^{\overline{\omega}T}\int_{T_1}^t\|D_\varphi F(s,u_s)-D_\varphi F(s,w_s)\|\|v_s\|_{\mathcal{B}}\,ds.\end{aligned}$$

Recall that $K_T\delta(\varepsilon)<1$. Thus for all $t\in[0,T_1)$,

$$\begin{aligned}\|u_t-w_t\|_{\mathcal{B}}&\le K_T\sup_{T_1\le s\le T_2}|u(s)-w(s)|\\&\le K_T(1-K_T\delta(\varepsilon))^{-1}\overline{M}e^{\overline{\omega}T_2}k\int_0^t|F(s,u_s)-F(s,w_s)|ds\\&\quad+K_T(1-K_T\delta(\varepsilon))^{-1}\bar{M}e^{\overline{\omega}T_2}\int_0^t|D_tF(s,u_s)-D_tF(s,w_s)|ds\\&\quad+K_T(1-K_T\delta(\varepsilon))^{-1}\bar{M}e^{\overline{\omega}T_2}\int_0^t\|D_\varphi F(s,u_s)-D_\varphi F(s,w_s)\|\|v_s\|_{\mathcal{B}}\,ds.\end{aligned}$$

Set

$$r:=\max\Big(\sup_{T_1\le s\le T_2}\|u_s\|_B,\sup_{T_1\le s\le T_2}\|v_s\|_{\mathcal{B}},\sup_{T_1\le s\le T_2}\|w_s\|_{\mathcal{B}}\Big),$$

There exist $C_0(r)$, $C_1(r),C_2(r)>0$ such that, for $s\in[T_1,T_2]$,

$$\begin{gathered}|F(s,u_s)-F(s,w_s)|\le C_0(r)\|u_s-w_s\|_{\mathcal{B}},\\|D_tF(t,u_s)-D_tF(t,w_s)|\le C_1(r)\|u_s-w_s\|_{\mathcal{B}},\\\|D_\varphi F(t,u_s)-D_\varphi F(t,w_s)\|\le C_2(r)\|u_s-w_s\|_{\mathcal{B}}.\end{gathered}$$

This implies that for suitable positive constants $\overline{M}$ and $\overline{\omega}$, and all $t \in [T_1, T_2]$,

$$\|u_t - w_t\|_{\mathcal{B}} \le \frac{K_T \overline{M} e^{\overline{\omega} T_2}}{1 - K_T \delta(\varepsilon)} (kC_0(r) + C_1(r) + rC_2(r)) \int_{T_1}^t \|u_s - w_s\|_{\mathcal{B}} ds.$$

By the Gronwall lemma, $\|u_t - w_t\|_{\mathcal{B}} = 0$ for any $t \in [T_1, T_2]$. Using Axiom (A1)(ii), we deduce that $u(t) = w(t)$ for all $t \in [T_1, T_2]$. Proceeding inductively we obtain $u(t) = w(t)$ for all $t \in [0, T]$ for any T in $[0, b_\varphi)$. Finally, since

$$t \mapsto \mathcal{D} w_t = \mathcal{D}\varphi + \mathcal{D}\Big(\int_0^t v_s ds \Big) = \mathcal{D}\varphi + \int_0^t \mathcal{D} v_s ds$$

is continuously differentiable, the function $t \mapsto \mathcal{D} u_t$ is continuously differentiable. This completes the proof of Theorem 3.2. $\square$

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