Electronic Journal of Differential Equations, Vol. 2006(2006), No. 93, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ASYMPTOTIC BEHAVIOUR OF THE SOLUTION FOR THE SINGULAR LANE-EMDEN-FOWLER EQUATION WITH NONLINEAR CONVECTION TERMS

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Abstract. We show the exact asymptotic behaviour near the boundary for the classical solution to the Dirichler problem

$$-\Delta u = k(x)g(u) + \lambda|\nabla u|^q, \quad u > 0,\ x \in \Omega, \quad u\big|_{\partial\Omega} = 0,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$. We use the Karamata regular varying theory, a perturbed argument, and constructing comparison functions.

1. Introduction and statement of the main results

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 1)$. Consider the singular Dirichlet problem for the Lane-Emden-Fowler equation

$$-\Delta u = k(x)g(u) + \lambda|\nabla u|^q, \quad u > 0,\ x \in \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $q \in [0, 2]$, and the functions g, k satisfy the hypotheses:

(H1) $g \in C^1((0,\infty),(0,\infty))$, $g'(s) \leq 0$ for all $s > 0$, $\lim_{s\to 0^+} g(s) = \infty$
(H2) $k \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0, 1)$, is non-negative and non-trivial on Ω.

The problem above arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials [5, 8, 10, 18, 21].

The main feature of this paper is the presence of the three terms: the singularity term $g(u)$ which is regular varying at zero of index $-\gamma$ with $\gamma \in (0, 1)$, the weight $k(x)$ which may be vanishing at the boundary, the both of them include a large class of functions, and the nonlinear convection terms $\lambda|\nabla u|^q$.

This problem was discussed in a number of works; see, for instance, [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 27, 28, 29]. When $\lambda = 0$, i.e., problem (1.1) becomes

$$-\Delta u = k(x)g(u), \quad u > 0,\ x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35J65, 35B05, 35O75, 35R05.
Key words and phrases. Semilinear elliptic equations; Dirichlet problem; singularity; nonlinear convection terms; Karamata regular variation theory; unique solution; exact asymptotic behaviour.

Submitted December 23, 2005. Published August 18, 2006.
Supported by Grant 10071066 from the National Natural Science Foundation of China.

For $k \equiv 1$ on Ω. Fulks and Maybee [8], Stuart [21], Crandall, Rabinowitz and Tartar [5] showed that (1.2) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$. Moreover, Crandall, Rabinowitz and Tartar [5, Theorems 2.2 and 2.5] showed that if $p \in C[0,a] \cap C^2(0,a]$ is the local solution to the problem

$$-p''(s) = g(p(s)), \quad p(s) > 0,\ 0 < s < a, \quad p(0) = 0,$$

then there exist positive constants C_1 and C_2 such that

(i) $C_1 p(d(x)) \le u(x) \le C_2 p(d(x))$ near $\partial\Omega$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$
(ii) $|\nabla u(x)| \le C_2[d(x) g(C_1 p(d(x))) + p(d(x))/d(x)]$ near $\partial\Omega$.

In particular, u is Lipschitz continuous on $\bar{\Omega}$ if and only if $\int_0^1 g(s)ds < \infty$. Recently, Ghergu and Rădulescu [9] showed that if g satisfies (H1) and

(H3) There exist positive constants C_0, η_0 and $\gamma \in (0,1)$ such that $g(s) \le C_0 s^{-\gamma}$, for all $s \in (0, \eta_0)$
(H4) There exist $\theta > 0$ and $t_0 \ge 1$ such that $g(\xi t) \ge \xi^{-\theta} g(t)$ for all $\xi \in (0,1)$ and $0 < t \le t_0 \xi$
(H5) The mapping $\xi \in (0,\infty) \to T(\xi) = \lim_{t\to 0^+} \frac{g(\xi t)}{\xi g(t)}$ is a continuous function; and k satisfies (H2) and the following assumptions: there exist $\delta_0 > 0$ and a positive non-decreasing function $h \in C(0, \delta_0)$ such that
(H6) $\lim_{d(x)\to 0} \frac{k(x)}{h(d(x))} = c_0$
(H7) $\lim_{t\to 0^+} h(t) g(t) = +\infty$.

Then (1.2) has a unique solution $u \in C^{1,1-\alpha}(\bar{\Omega}) \cap C^2(\Omega)$ satisfying

$$\lim_{d(x)\to 0} \frac{u(x)}{p(d(x))} = \xi_0, \tag{1.3}$$

where $T(\xi_0) = c_0^{-1}$, and $p \in C^1[0,a] \cap C^2(0,a] (a \in (0, \delta_0))$ is the local solution to the problem

$$-p''(s) = h(s) g(p(s)), \quad p(s) > 0,\ 0 < s < a, \quad p(0) = 0. \tag{1.4}$$

The exact asymptotic behaviour of the unique solution to (1.2) with $\int_0^1 g(s)ds = \infty$ has been studied in [27].

For $\lambda \ne 0$, existence and uniqueness of solutions to problem (1.1), see [10, 11, 26, 28], and the exact asymptotic behaviour of the unique solution to (1.1) with $\int_0^1 g(s)ds = \infty$, see [11, 28, 29].

In this paper, we generalize the Ghergu and Rădulescu's results [9] to problem (1.1), and we showed that the asymptotic behaviour (1.3) of the unique solution u_λ to problem (1.1) is independent on $\lambda|\nabla u_\lambda|^q$.

First we recall a basic definition (see [17, 19, 20]).

Definition 1.1. A positive measurable function g defined on some neighborhood $(0,a)$ for some $a > 0$, is called regular varying at zero with index β, written $g \in RVZ_\beta$ if for each $\xi > 0$ and some $\beta \in \mathbb{R}$,

$$\lim_{t\to 0^+} \frac{g(\xi t)}{g(t)} = \xi^\beta.$$

Our main result is summarized in the following theorem.

Theorem 1.2. *Let g satisfy (H1) and $g \in RVZ_{-\gamma}$ with $\gamma \in (0,1)$ and k satisfy (H2), (H6) and $h \in RVZ_\beta$ with $\beta \in [0,1)$. If $\beta < \gamma$, then the unique solution*

$u_\lambda \in C^{1,1-\alpha}(\bar{\Omega}) \cap C^2(\Omega)$ to problem (1.1) *satisfies*

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{p(d(x))} = \xi_0,$$

where $\xi_0 = c_0^{1/(1+\gamma)}$, and $p \in C^1[0,a] \cap C^2(0,a]$ is the local solution to problem (1.4).

Remark 1.3. By (H1) and the proof of the maximum principle [12, Theorems 10.1 and 10.2] we see that (1.1) has at most one solution in $C^2(\Omega) \cap C(\bar{\Omega})$ for each fixed λ.

Remark 1.4. In section 2, we will see that $g \in RVZ_{-\gamma}$ with $\gamma > 0$ implies $\lim_{s\to 0^+} g(s) = \infty$ and $h \in RVZ_\beta$ with $\beta > 0$ implies $\lim_{t\to 0^+} h(t) = 0$.

Remark 1.5. For the existence of solutions to (1.4) with $a \in (0,1)$, see [1, Corollary 2.1].

The outline of this article is as follows. In section 2, we recall some basic definitions and the properties to Karamata regular varying theory. In section 3, we prove the asymptotic behaviour of the unique solution in Theorem 1.2.

2. Karamata regular varying theory

Let us recall some basic definitions and the properties to Karamata regular varying theory, which is a basic tool in probability theory (see [17, 19, 20]).

Definition 2.1. A positive measurable function f defined on $[a,\infty)$, for some $a > 0$, is called regular varying at infinity with index ρ, written $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{f(\xi t)}{f(t)} = \xi^\rho. \tag{2.1}$$

The real number ρ is called the index of regular variation.

Definition 2.2. When $\rho = 0$, a positive measurable function L defined on $[a,\infty)$, for some $a > 0$, is called slowly varying at infinity, if for each $\xi > 0$

$$\lim_{t\to\infty} \frac{L(\xi t)}{L(t)} = 1. \tag{2.2}$$

It follows by the definition that if $f \in RV_\rho$ it can be represented in the form

$$f(t) = t^\rho L(t).$$

Some basic examples of slowly varying functions are:

(i) $\lim_{t\to\infty} L(t) = c \in (0,\infty)$;
(ii) $L(t) = \prod_{m=1}^{m=n} (\log_m(t))^{\alpha_m}$, $\alpha_m \in \mathbb{R}$;
(iii) $L(t) = e^{(\prod_{m=1}^{m=n} (\log_m(t))^{\alpha_m})}$, $0 < \alpha_m < 1$;
(iv) $L(t) = \dfrac{1}{t}\displaystyle\int_a^t \frac{ds}{\ln s}$;
(v) $L(t) = e^{((\ln t)^{1/3} \cos((\ln t)^{1/3}))}$, where $\lim_{t\to\infty} \inf L(t) = 0$, $\lim_{t\to\infty} \sup L(t) = +\infty$.

Lemma 2.3 (Uniform convergence theorem)**.** *If $f \in RV_\rho$, then* (2.1) *(and so* (2.2)*) holds uniformly for $\xi \in [a,b]$ with $0 < a < b$.*

Lemma 2.4 (Representation theorem). *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(t) = c(t) \exp\big(\int_a^t \frac{y(s)}{s} ds\big), \quad t \geq a,$$

for some $a > 0$, where $c(t)$ and $y(t)$ are measurable and for $t \to \infty$, $y(t) \to 0$ and $c(t) \to c$, with $c > 0$.

Lemma 2.5. *If functions L, L_1 are slowly varying at infinity, then*

(i) *L^α for every $\alpha \in \mathbb{R}$, $L(t) + L_1(t)$, $L(L_1(t))$ (if $L_1(t) \to \infty$ as $t \to \infty$), are also slowly varying at infinity;*

(ii) *for every $\theta > 0$ and $t \to \infty$,*

$$t^\theta L(t) \to \infty, \quad t^{-\theta} L(t) \to 0;$$

(iii) *for $t \to \infty$, $\ln(L(t))/\ln t \to 0$.*

Definition 2.6. A positive measurable function H defined on some neighborhood $(0, a)$ for some $a > 0$, is called slowly varying at zero, if for each $\xi > 0$

$$\lim_{t \to 0^+} \frac{H(\xi t)}{H(t)} = 1.$$

It follows by Definitions 1.1 and 2.6 that if $g \in RVZ_{-\gamma}$ it can be represented in the form $g(t) = t^{-\gamma} H(t)$.

Lemma 2.7. *Definition 1.1 is equivalent to saying that $f^*(t) = g(1/t)$ is regular varying at infinity of index $-\beta$.*

Thus we transfer our attention from infinity to the origin.

Corollary 2.8 (Representation theorem). *A function H is slowly varying at zero if and only if it may be written in the form*

$$H(t) = c(t) \exp\big(\int_t^a \frac{y(s)}{s} ds\big), \quad 0 < t < a,$$

for some $a > 0$, where $c(t)$ and $y(t)$ are measurable and for $t \to 0^+$, $y(t) \to 0$ and $c(t) \to c$, with $c > 0$.

Corollary 2.9. *If a function H is slowly varying at zero, then for every $\theta > 0$ and $t \to 0^+$, $t^{-\theta} H(t) \to \infty$, $t^\theta H(t) \to 0$.*

Corollary 2.10. *If g satisfies (H1), $g \in RVZ_{-\gamma}$ with $\gamma \in (0, 1)$, and k satisfies (H2), (H6), $h \in RVZ_\beta$ with $\beta \in (0, 1)$, then*

$$g(t) = t^{-\gamma} c_1(t) \exp\big(\int_t^a \frac{y_1(s)}{s} ds\big), \quad h(t) = t^\beta c_2(t) \exp\big(\int_t^a \frac{y_2(s)}{s} ds\big),$$

where $y_1, y_2, c_1, c_2 \in C[0, a]$, $y_1(0) = y_2(0) = 0$, $c_1(0) > 0$, $c_2(0) > 0$.

3. Asymptotic behaviour

First we give some preliminary considerations.

Lemma 3.1. *If g satisfies (H1) and $g \in RVZ_{-\gamma}$ with $\gamma \in (0, 1)$, then*

$$\int_0^1 g(t) dt < \infty.$$

Proof. We see by corollaries 2.9 and 2.10 that there exists $\gamma_1 \in (\gamma, 1)$ such that

$$\lim_{t\to 0^+} t^{\gamma_1} g(t) = \lim_{t\to 0^+} t^{\gamma_1-\gamma} c_1(t) \exp\Big(\int_t^a \frac{y_1(s)}{s} ds\Big) = 0.$$

It follow that there exists $\delta \in (0,1)$ such that $g(t) < t^{-\gamma_1}$, for all $t \in (0,\delta)$ and so $g \in L^1(0,1)$. □

Lemma 3.2. *Under the assumptions in Theorem 1.2, the local solution* p *to problem* (1.4) *has the following properties*

(i) $p \in C^1[0,a]$;
(ii) $\lim_{s\to 0^+} p''(s) = -\infty$;
(iii) $\lim_{s\to 0^+} \frac{(p'(s))^q}{p''(s)} = 0$ *for* $q \in [0,2]$.

Proof. (i) Since $-((p'(s))^2)' = 2h(s)g(p(s))p'(s)$ for $s \in (0,a]$, and $p(s)$ is a positive concave on $(0,a]$, $p(0) = 0$, $p''(s) < 0$, we see that $p'(s)$ is decreasing and $p'(s) > 0$ on $(0,a]$, so $p(s)$ is increasing. Since h is non-decreasing, multiplying (1.4) by $p'(s)$ and integrating on $[t,a]$, $0 < t < a$, we get by Lemma 3.1 that

$$\begin{aligned}
(p'(a))^2 + 2h(t)\int_t^a g(p(s))p'(s)ds &= (p'(a))^2 + 2h(t)\int_{p(t)}^{p(a)} g(y)dy \\
&\le (p'(a))^2 + 2\int_t^a h(s)g(p(s))p'(s)ds \\
&= (p'(t))^2 \\
&\le (p'(a))^2 + 2h(a)\int_{p(t)}^{p(a)} g(y)dy, \\
&\le (p'(a))^2 + 2h(a)\int_0^{p(a)} g(y)dy < \infty.
\end{aligned}$$

Thus $p'(0) \in (0,\infty)$, i.e., $p \in C^1[0,a]$.

(ii) Let $b = p'(0)$. Since $p'(s)$ is decreasing on $[0,a]$, it follows by the Lagrange mean value theorem that there exists $\tau_s \in (0,s)$ such that

$$p(s)/s = (p(s) - p(0))/s = p'(\tau_s) < b, \quad \forall s \in (0,a].$$

Thus $p(s) < bs$ for all $s \in (0,a]$ and so $g(p(s)) \ge g(bs)$, for all $s \in (0,a]$. Since $\gamma > \beta$, we see by corollaries 2.9 and 2.10 that

$$\lim_{t\to 0^+} h(t)g(t) = \lim_{t\to 0^+} t^{-(\gamma-\beta)} c_1(t)c_2(t) \exp\Big(\int_t^a \frac{y_1(s)}{s} ds\Big) \exp\Big(\int_t^a \frac{y_2(s)}{s} ds\Big) = \infty,$$

and

$$\lim_{t\to 0^+} \frac{g(bt)}{g(t)} = b^{-\gamma}.$$

Thus

$$-p''(t) = h(t)g(p(t)) \ge h(t)g(t)\frac{g(bt)}{g(t)}, \quad \forall t \in (0,a],$$
$$\lim_{s\to 0^+} p''(s) = -\infty.$$

(iii) is follows by (i) and (ii). The proof is complete. □

Proof of the asymptotic behaviour in Theorem 1.2. Set $\xi_0 = c_0^{1/(1+\gamma)}$ and for a fix $\varepsilon \in (0,1/4)$ let

$$\xi_{1\varepsilon} = \big(\frac{c_0}{1-2\varepsilon}\big)^{1/(1+\gamma)}, \quad \xi_{2\varepsilon} = \big(\frac{c_0}{1+2\varepsilon}\big)^{1/(1+\gamma)},$$

we see that

$$\big(\frac{c_0}{2}\big)^{1/(1+\gamma)} < \xi_{2\varepsilon} < \xi_{1\varepsilon} < (2c_0)^{1/(1+\gamma)}.$$

For any $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega : d(x) \le \delta\}$. By the regularity of $\partial\Omega$ and lemma 3.2, we can choose δ sufficiently small such that

- (i) $d(x) \in C^2(\Omega_\delta)$;
- (ii) $|\frac{p'(s)}{p''(s)}\Delta d(x) + \lambda \xi_{i\varepsilon}^{q-1}\frac{(p'(s))^q}{p''(s)}| < \varepsilon$, for all $(x,s) \in \Omega_\delta \times (0,\delta)$, $i = 1,2$ and fixed λ;
- (iii) $\frac{\xi_{2\varepsilon}h(d(x))g(p(d(x)))}{g(p(d(x))\xi_{2\varepsilon})}(1+\varepsilon) < k(x) < \frac{\xi_{1\varepsilon}h(d(x))g(p(d(x)))}{g(p(d(x))\xi_{1\varepsilon})}(1-\varepsilon)$ in Ω_δ.

For any $x \in \Omega_\delta$, define $\bar u = \xi_{1\varepsilon}p(d(x))$, and $\underline{u} = \xi_{2\varepsilon}p(d(x))$. It follows from $|\nabla d(x)| = 1$ that

$$\begin{aligned}
&\Delta\overline{u}(x) + k(x)g(\overline{u}(x)) + \lambda|\nabla\overline{u}(x)|^q\\
&= k(x)g(\xi_{1\varepsilon}p(d(x))) + \xi_{1\varepsilon}p'(d(x))\Delta d(x) + \xi_{1\varepsilon}p''(d(x)) + \lambda\xi_{1\varepsilon}^q(p'(d(x)))^q\\
&= \xi_{1\varepsilon}h(d(x))g(p(d(x)))\Big[\frac{k(x)g(\xi_{1\varepsilon}p(d(x)))}{\xi_{1\varepsilon}h(d(x))g(p(d(x)))} - 1 - \frac{p'(d(x))}{p''(d(x))}\Delta d(x)\\
&\quad - \lambda\xi_{1\varepsilon}^{q-1}\frac{(p'(d(x))^q}{p''(d(x))}\Big]\\
&\le \xi_{1\varepsilon}h(d(x))g(p(d(x)))\Big[(1-2\varepsilon) - 1 - \frac{p'(d(x))}{p''(d(x))}\Delta d(x) - \lambda\xi_{1\varepsilon}^{q-1}\frac{(p'(d(x))^q}{p''(d(x))}\Big] \le 0;
\end{aligned}$$

and

$$\begin{aligned}
&\Delta\underline{u}(x) + k(x)g(\underline{u}(x)) + \lambda|\nabla\underline{u}(x)|^q\\
&= k(x)g(\xi_{2\varepsilon}p(d(x))) + \xi_{2\varepsilon}p'(d(x))\Delta d(x) + \xi_{2\varepsilon}p''(d(x)) + \lambda\xi_{2\varepsilon}^q(p'(d(x)))^q\\
&= \xi_{2\varepsilon}h(d(x))g(p(d(x)))\Big[\frac{k(x)g(\xi_{2\varepsilon}p(d(x)))}{\xi_{2\varepsilon}h(d(x))g(p(d(x)))} - 1 - \frac{p'(d(x))}{p''(d(x))}\Delta d(x)\\
&\quad - \lambda\xi_{2\varepsilon}^{q-1}\frac{(p'(d(x))^q}{p''(d(x))}\Big]\\
&\ge \xi_{2\varepsilon}h(d(x))g(p(d(x)))\Big[(1+2\varepsilon) - 1 - \frac{p'(d(x))}{p''(d(x))}\Delta d(x) - \lambda\xi_{2\varepsilon}^{q-1}\frac{(p'(d(x))^q}{p''(d(x))}\Big] \ge 0.
\end{aligned}$$

Let $u_\lambda \in C(\bar\Omega) \cap C^{2+\alpha}(\Omega)$ be the unique solution to problem (1.1). We assert

$$\xi_{2\varepsilon}p(d(x)) = \underline{u}(x) \le u_\lambda(x) \le \bar u(x) = \xi_{1\varepsilon}p(d(x)), \quad \forall x \in \Omega_\delta.$$

In fact, denote $\Omega_\delta = \Omega_{\delta+} \cup \Omega_{\delta-}$, where $\Omega_{\delta+} = \{x \in \Omega_\delta : u_\lambda(x) \ge \underline{u}(x)\}$ and $\Omega_{\delta-} = \{x \in \Omega_\delta : u_\lambda(x) < \underline{u}(x)\}$. We need to show $\Omega_{\delta-} = \emptyset$. Assume the contrary, we see that there exists $x_0 \in \Omega_{\delta-}$ such that

$$0 < \underline{u}(x_0) - u_\lambda(x_0) = \max_{x\in\bar\Omega_{\delta-}}(\underline{u}(x) - u_\lambda(x)),$$

and

$$\nabla\underline{u}(x_0) = \nabla u_\lambda(x_0), \quad \Delta(\underline{u}(x_0) - u_\lambda(x_0)) \le 0.$$

On the other hand, we see by (H1) that

$$-\Delta(u_\lambda - \underline{u})(x_0) = k(x_0)(g(\underline{u}(x_0)) - g(u_\lambda(x_0))) < 0,$$

which is a contradiction. Hence $\Omega_{\delta-} = \emptyset$, i.e., $u_\lambda(x) \geq \underline{u}(x)$ in Ω_δ. As the same way, we can see that $u_\lambda(x) \leq \bar{u}(x)$, for all $x \in \Omega_\delta$. Let $\varepsilon \to 0$, we see that $\lim_{d(x)\to 0} \frac{u_\lambda(x)}{p(d(x))} = \xi_0$. As the same proof as in [9, 10], we see that $u_\lambda \in C^{1,1-\alpha}(\bar{\Omega}) \cap C^2(\Omega)$. The proof is complete. $\square$

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