Electronic Journal of Differential Equations, Vol. 2006(2006), No. 94, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A BLOW UP CONDITION FOR A NONAUTONOMOUS SEMILINEAR SYSTEM

AROLDO PÉREZ-PÉREZ

ABSTRACT. We give a sufficient condition for finite time blow up of the nonnegative mild solution to a nonautonomous weakly coupled system with fractal diffusion having a time dependent factor which is continuous and nonnegative.

1. INTRODUCTION

This paper deals with the blow up of nonnegative solutions of the nonautonomous initial value problem for a weakly coupled system with a fractal diffusion

$$
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= k(t)\Delta_\alpha u(t,x) + v^{\beta_1}(t,x), \quad t>0, \quad x\in\mathbb{R}^d \\
\frac{\partial v(t,x)}{\partial t} &= k(t)\Delta_\alpha v(t,x) + u^{\beta_2}(t,x), \quad t>0, \quad x\in\mathbb{R}^d \\
u(0,x) &= \varphi_1(x), \quad x\in\mathbb{R}^d \\
v(0,x) &= \varphi_2(x), \quad x\in\mathbb{R}^d,
\end{aligned}
\tag{1.1}
$$

where $\Delta_\alpha := -(-\Delta)^{\alpha/2}$, $0 < \alpha \le 2$ denotes the α-Laplacian, $\beta_1,\ \beta_2 > 1$ are constants, $0 \le \varphi_1,\ \varphi_2 \in B(\mathbb{R}^d)$ (where $B(\mathbb{R}^d)$ is the space of bounded measurable functions on $\mathbb{R}^d$) do not vanish identically, $k : [0,\infty) \to [0,\infty)$ is continuous and satisfies

$$
\varepsilon_1 t^\rho \le \int_0^t k(r)dr \le \varepsilon_2 t^\rho, \quad \varepsilon_1,\ \varepsilon_2,\ \rho > 0, \tag{1.2}
$$

for all t large enough.

The associated integral system to (1.1) is given by

$$
u(t,x) = U(t,0)\varphi_1(x) + \int_0^t U(t,r)v^{\beta_1}(r,x)dr, \quad t>0,\ x\in\mathbb{R}^d, \tag{1.3}
$$

$$
v(t,x) = U(t,0)\varphi_2(x) + \int_0^t U(t,r)u^{\beta_2}(r,x)dr, \quad t>0,\ x\in\mathbb{R}^d, \tag{1.4}
$$

2000 *Mathematics Subject Classification.* 35B40, 35K45, 35K55, 35K57.
Key words and phrases. Finite time blow up; mild solution; weakly coupled system; nonautonomous initial value problem; fractal diffusion.

Submitted July 14, 2006. Published August 18, 2006.

where $\{U(t,s)\}_{t\geq s\geq 0}$ is the evolution family on $B(\mathbb{R}^d)$ that solves the homogeneous Cauchy problem for the family of generators $\{k(t)\Delta_\alpha\}_{t\geq 0}$. Clearly

$$U(t,s) = S(K(t,s)), \quad t \geq s \geq 0,$$

where $\{S(t)\}_{t\geq 0}$ is the semigroup with infinitesimal generator Δ_α, and $K(t,s) = \int_s^t k(r)dr$, $t \geq s \geq 0$.

A solution of (1.3)-(1.4) is called a mild solution of (1.1). If there exist a solution (u,v) of (1.1) in $[0,\infty)\times\mathbb{R}^d$ such that $\|u(t,\cdot)\|_\infty + \|v(t,\cdot)\|_\infty < \infty$ for any $t \geq 0$, we say that (u,v) is a global solution, and when there exist a number $T_{\varphi_1,\varphi_2} < \infty$ such that (1.1) has a bounded solution (u,v) in $[0,T]\times\mathbb{R}^d$ for all $T < T_{\varphi_1,\varphi_2}$ with $\lim_{t\uparrow T_{\varphi_1,\varphi_2}} \|u(t,\cdot)\|_\infty = \infty$ or $\lim_{t\uparrow T_{\varphi_1,\varphi_2}} \|v(t,\cdot)\|_\infty = \infty$ we say that (u,v) blows up in finite time.

The finite time blow up of (1.1) for $\alpha = 2$ and $k \equiv 1$ was initially considered by Escobedo and Herrero [4]. They proved that when $\beta_1\beta_2 > 1$ and $(\gamma+1)/(\beta_1\beta_2-1) \geq d/2$ with $\gamma = \max\{\beta_1,\beta_2\}$, any nontrivial positive solution to (1.1) blows up in finite time. Related results and more general cases for the Laplacian can be found for instance in [1, 3, 5, 6, 10, 12, 13, 15, 16]. The case for fractional powers of the Laplacian when $k \equiv 1$ for equations with different diffusion operators was considered in [8, 9]; see also [2, 11] for a probabilistic approach. Sugitani [14] has considered a scalar version of (1.1) with $k \equiv 1$ when the nonlinear term is given by an increasing nonnegative continuous and convex function $F(u)$, defined on $[0,\infty)$, and Guedda and Kirane [7] have considered a scalar version of (1.1) with $k \equiv 1$ when the nonlinear term is $h(t)u^\beta$, $\beta > 1$ with h being a nonnegative continuous function on $[0,\infty)$ satisfying $c_0 t^\sigma \leq h(t) \leq c_1 t^\sigma$ for sufficiently large t, where $c_0, c_1 > 0$ and $\sigma > -1$ are constants. They proved that in this scalar case, solutions blow up in finite time if $0 < d(\beta-1)/\alpha \leq 1+\sigma$ for any nontrivial nonnegative and continuous initial function on $\mathbb{R}^d$. Here we prove that if k satisfies (1.2) and $0 < d\rho(\beta_i - 1)/\alpha < 1$, $i = 1,2$, then any nontrivial positive solution of (1.1) blows up in finite time. Here solutions will be understood in the mild sense, that is, that solve (1.3)-(1.4).

2. Blow up condition

Let $(u(\cdot,\cdot), v(\cdot,\cdot))$ be a nonnegative solution of (1.1) and define

$$u(t) = \int_{\mathbb{R}^d} p(K(t,0),x)u(t,x)dx, \quad v(t) = \int_{\mathbb{R}^d} p(K(t,0),x)v(t,x)dx, \quad t > 0,$$

where $p(t,x)$, $t > 0$, $x \in \mathbb{R}^d$ denotes the density of the semigroup $S(t)$, $t \geq 0$.

Lemma 2.1. *For any $s,t > 0$, and $x,y \in \mathbb{R}^d$, $p(t,x)$ satisfies*

i) $p(ts,x) = t^{-\frac{d}{\alpha}} p(s, t^{-\frac{1}{\alpha}}x)$,
ii) $p(t,x) \leq p(t,y)$ *when* $|x| \geq |y|$,
iii) $p(t,x) \geq (\frac{s}{t})^{\frac{d}{\alpha}} p(s,x)$ *for* $t \geq s$,
iv) $p(t, \frac{1}{\tau}(x-y)) \geq p(t,x)p(t,y)$ *if* $p(t,0) \leq 1$ *and* $\tau \geq 2$.

Proof. See Guedda and Kirane [7] or Sugitani [14]. □

Lemma 2.2. *If there exist $T_0 > 0$ such that $u(t) = \infty$ or $v(t) = \infty$ for $t \geq T_0$, then the nonnegative solution of* (1.1) *blows up in finite time.*

Proof. Due to (1.2) and Lemma 2.1 i), we can assume that

$$p(K(t,0),0) \le 1 \quad \text{for all } t \ge T_0.$$

If $T_0 \le \varepsilon_1^{1/\rho} t$ and $\varepsilon_1^{1/\rho} t \le r \le (2\varepsilon_1)^{1/\rho} t$, we have from the conditions of $k(t)$,

$$\begin{aligned}
\tau \equiv \Big[\frac{K((10\varepsilon_2)^{1/\rho}t, r)}{K(r,0)}\Big]^{1/\alpha} &= \Big[\frac{K((10\varepsilon_2)^{1/\rho}t, 0) - K(r,0)}{K(r,0)}\Big]^{1/\alpha} \\
&\ge \Big[\frac{K((10\varepsilon_2)^{1/\rho}t, 0)}{K((2\varepsilon_1)^{1/\rho}t), 0)} - 1\Big]^{1/\alpha} \ge \Big[\frac{\varepsilon_1(10\varepsilon_2)t^\rho}{\varepsilon_2(2\varepsilon_1)t^\rho} - 1\Big]^{1/\alpha} \ge 2.
\end{aligned}$$

Hence, using Lemma 2.1 i), iv) with $\tau = \big[\frac{K((10\varepsilon_2)^{1/\rho}t, r)}{K(r,0)}\big]^{1/\alpha}$,

$$\begin{aligned}
&p(K((10\varepsilon_2)^{1/\rho}t, r), x-y) \\
&= p(K(r,0)\Big[\frac{K((10\varepsilon_2)^{1/\rho}t, r)}{K(r,0)}\Big], x-y) \\
&= \Big[\frac{K(r,0)}{K((10\varepsilon_2)^{1/\rho}t, r)}\Big]^{d/\alpha} p(K(r,0), \Big[\frac{K(r,0)}{K((10\varepsilon_2)^{1/\rho}t, r)}\Big]^{1/\alpha}(x-y)) \\
&\ge \Big[\frac{K(r,0)}{K((10\varepsilon_2)^{1/\rho}t), r)}\Big]^{d/\alpha} p(K(r,0),x)p(K(r,0),y), \quad x,y \in \mathbb{R}^d.
\end{aligned}$$

Hence, assuming that $u(t) = \infty$ for all $t \ge T_0$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} p(K((10\varepsilon_2)^{1/\rho}t, r), x-y)u(r,y)dy \\
&\ge \Big[\frac{K(r,0)}{K((10\varepsilon_2)^{1/\rho}t, r)}\Big]^{d/\alpha} p(K(r,0),x)u(r) = \infty, \quad x \in \mathbb{R}^d.
\end{aligned} \tag{2.1}$$

We know by (1.4) that

$$\begin{aligned}
v(t,x) &= \int_{\mathbb{R}^d} p(K(t,0), x-y)\varphi_2(y)dy \\
&\quad + \int_0^t \Big(\int_{\mathbb{R}^d} p(K(t,r), x-y)u^{\beta_2}(r,y)dy\Big)dr \\
&\ge \int_0^t \Big(\int_{\mathbb{R}^d} p(K(t,r), x-y)u^{\beta_2}(r,y)dy\Big)dr.
\end{aligned}$$

Thus,

$$v((10\varepsilon_2)^{1/\rho}t, x) \ge \int_0^{(10\varepsilon_2)^{1/\rho}t} \Big(\int_{\mathbb{R}^d} p(K((10\varepsilon_2)^{1/\rho}t, r), x-y)u^{\beta_2}(r,y)dy\Big)dr$$

and by Jensen's inequality and (2.1), we get

$$v((10\varepsilon_2)^{1/\rho}t, x) \ge \int_{\varepsilon_1^{1/\rho}t}^{(2\varepsilon_1)^{1/\rho}t} \Big(\int_{\mathbb{R}^d} p(K((10\varepsilon_2)^{1/\rho}t, r), x-y)u(r,y)dy\Big)^{\beta_2} dr = \infty,$$

so that $v(t,x) = \infty$ for any $t \ge (10\frac{\varepsilon_2}{\varepsilon_1})^{1/\rho}T_0$ and $x \in \mathbb{R}^d$. Similarly, when $v(t) = \infty$ for all $t \ge T_0$, it can be shown that $u(t,x) = \infty$ for all $t \ge (10\frac{\varepsilon_2}{\varepsilon_1})^{1/\rho}T_0$ and $x \in \mathbb{R}^d$. □

Theorem 2.3. *If $0 < d\rho(\beta_i - 1)/\alpha < 1$, $i = 1, 2$, then the nonnegative solution of system* (1.1) *blows up in finite time.*

Proof. Let $t_0 \geq 1$ be such that (1.2) holds for all $t \geq t_0$ and such that $p(K(t_0,0),0) \leq 1$. Using Lemma 2.1 i), iv), we have

$$\begin{aligned} p(K(t_0,0),x-y) =& p(K(t_0,0),\frac{1}{2}(2x-2y)) \geq p(K(t_0,0),2x)p(K(t_0,0),2y) \\ =& 2^{-d}p(2^{-\alpha}K(t_0,0),x)p(K(t_0,0),2y), \quad x,y \in \mathbb{R}^d. \end{aligned}$$

Therefore (see (1.3))

$$\begin{aligned} u(t_0,x) &\geq \int_{\mathbb{R}^d} p(K(t_0,0),x-y)\varphi_1(y)dy \\ &\geq 2^{-d}p(2^{-\alpha}K(t_0,0),x)\int_{\mathbb{R}^d} p(K(t_0,0),2y)\varphi_1(y)dy \\ &= N_1 p(2^{-\alpha}K(t_0,0),x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{2.2}$$

where $N_1 = 2^{-d}\int_{\mathbb{R}^d} p(K(t_0,0),2y)\varphi_1(y)dy$. Notice that

$$\begin{aligned} u(t+t_0,x) &= \int_{\mathbb{R}^d} p(K(t+t_0,0),x-y)\varphi_1(y)dy \\ &\quad + \int_0^{t+t_0}\Big(\int_{\mathbb{R}^d} p(K(t+t_0,r),x-y)v^{\beta_1}(r,y)dy\Big)dr \\ &= \int_{\mathbb{R}^d} p(K(t+t_0,t_0)+K(t_0,0),x-y)\varphi_1(y)dy \\ &\quad + \int_0^{t_0}\Big(\int_{\mathbb{R}^d} p(K(t+t_0,t_0)+K(t_0,r),x-y)v^{\beta_1}(r,y)dy\Big)dr \\ &\quad + \int_{t_0}^{t+t_0}\Big(\int_{\mathbb{R}^d} p(K(t+t_0,r),x-y)v^{\beta_1}(r,y)dy\Big)dr \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} p(K(t+t_0,t_0),x-z)p(K(t_0,0),z-y)dz\Big)\varphi_1(y)dy \\ &\quad + \int_0^{t_0}\Big[\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} p(K(t+t_0,t_0),x-z)p(K(t_0,r),z-y)dz\Big) \\ &\quad \times v^{\beta_1}(r,y)dy\Big]dr \\ &\quad + \int_0^{t}\Big(\int_{\mathbb{R}^d} p(K(t+t_0,r+t_0),x-y)v^{\beta_1}(r+t_0,y)dy\Big)dr, \end{aligned}$$

$t \geq 0$, $x \in \mathbb{R}^d$. From here, by Fubini's theorem and (1.3) we have

$$
\begin{aligned}
u(t+t_0,x) &= \int_{\mathbb{R}^d} p(K(t+t_0,t_0),x-z)\Big(\int_{\mathbb{R}^d} p(K(t_0,0),z-y)\varphi_1(y)dy\Big)dz \\
&\quad + \int_{\mathbb{R}^d} p(K(t+t_0,t_0),x-z)\Big[\int_0^{t_0}\Big(\int_{\mathbb{R}^d} p(K(t_0,r),z-y) \\
&\quad \times v^{\beta_1}(r,y)dy\Big)dr\Big]dz \\
&\quad + \int_0^t \Big(\int_{\mathbb{R}^d} p(K(t+t_0,r+t_0),x-y)v^{\beta_1}(r+t_0,y)dy\Big)dr \\
&= \int_{\mathbb{R}^d} p(K(t+t_0,t_0),x-y)u(t_0,y)dy \\
&\quad + \int_0^t \Big(\int_{\mathbb{R}^d} p(K(t+t_0,r+t_0),x-y)v^{\beta_1}(r+t_0,y)dy\Big)dr,
\end{aligned}
$$

$t \geq 0$, $x \in \mathbb{R}^d$. Thus, using (2.2) gives

$$
\begin{aligned}
u(t+t_0,x) &\geq N_1 \int_{\mathbb{R}^d} p(K(t+t_0,t_0),x-y)p(2^{-\alpha}K(t_0,0),y)dy \\
&\quad + \int_0^t \Big(\int_{\mathbb{R}^d} p(K(t+t_0,r+t_0),x-y)v^{\beta_1}(r+t_0,y)dy\Big)dr \\
&= N_1 p(K(t+t_0,t_0)+2^{-\alpha}K(t_0,0),x) \\
&\quad + \int_0^t \Big(\int_{\mathbb{R}^d} p(K(t+t_0,r+t_0),x-y)v^{\beta_1}(r+t_0,y)dy\Big)dr,
\end{aligned} \tag{2.3}
$$

$t \geq 0$, $x \in \mathbb{R}^d$. Multiplying both sides of (2.3) by $p(K(t+t_0,0),x)$ and integrating, we have

$$
\begin{aligned}
u(t+t_0) &= N_1 p(2K(t+t_0,t_0)+(2^{-\alpha}+1)K(t_0,0),0) \\
&\quad + \int_0^t \Big(\int_{\mathbb{R}^d} p(2K(t+t_0,0)-K(r+t_0,0),y)v^{\beta_1}(r+t_0,y)dy\Big)dr,
\end{aligned}
$$

$t \geq 0$. Applying Lemma 2.1 i), iii), we get

$$
\begin{aligned}
u(t+t_0) &\geq N_1[2K(t+t_0,t_0)+(2^{-\alpha}+1)K(t_0,0)]^{-d/\alpha}p(1,0) \\
&\quad + \int_0^t \Big(\frac{K(r+t_0,0)}{2K(t+t_0,0)}\Big)^{d/\alpha} v^{\beta_1}(r+t_0)dr.
\end{aligned}
$$

For a suitable choice of $\theta > 0$ given below, we define $f_1(t) = K^{d/\alpha}(t+t_0,0)u(t+t_0)$, $g_1(t) = K^{d/\alpha}(t+t_0,0)v(t+t_0)$, $t \geq \theta$. Then

$$
f_1(t) \geq \overline{N}_1 + 2^{-d/\alpha}\int_\theta^t K^{-d(\beta_1-1)/\alpha}(r+t_0,0)g_1^{\beta_1}(r)dr, \quad t \geq \theta,
$$

where $\overline{N}_1 = p(1,0)N_1\Big[\frac{K(\theta,0)}{2K(\theta+t_0,0)+(2^{-\alpha}+1)K(t_0,0)}\Big]^{d/\alpha}$. Similarly, it can be shown that

$$
g_1(t) \geq \overline{N}_2 + 2^{-d/\alpha}\int_\theta^t K^{-d(\beta_2-1)/\alpha}(r+t_0,0)f_1^{\beta_2}(r)dr, \quad t \geq \theta,
$$

where $\overline{N}_2 = p(1,0)N_2\Big[\frac{K(\theta,0)}{2K(\theta+t_0,0)+(2^{-\alpha}+1)K(t_0,0)}\Big]^{d/\alpha}$ with $N_2 = 2^{-d}\int_{\mathbb{R}^d} p(K(t_0,0), 2y)\varphi_2(y)dy$.

Letting $N=\min\{\overline{N}_1,\overline{N}_2\}$, we get

$$f_1(t)\geq N+2^{-d/\alpha}\int_\theta^t \min_{i\in\{1,2\}} K^{-d(\beta_i-1)/\alpha}(r+t_0,0)g_1^{\beta_1}(r)dr,\quad t\geq\theta,$$
$$g_1(t)\geq N+2^{-d/\alpha}\int_\theta^t \min_{i\in\{1,2\}} K^{-d(\beta_i-1)/\alpha}(r+t_0,0)f_1^{\beta_2}(r)dr,\quad t\geq\theta.$$

Let $(f_2(t),g_2(t))$ be the solution of the system integral equations

$$f_2(t)=N+2^{-d/\alpha}\int_\theta^t \min_{i\in\{1,2\}} K^{-d(\beta_i-1)/\alpha}(r+t_0,0)g_2^{\beta_1}(r)dr,\quad t\geq\theta,$$
$$g_2(t)=N+2^{-d/\alpha}\int_\theta^t \min_{i\in\{1,2\}} K^{-d(\beta_i-1)/\alpha}(r+t_0,0)f_2^{\beta_2}(r)dr,\quad t\geq\theta,$$

whose differential expression is

$$\begin{gathered}f_2'(t)=2^{-d/\alpha}\min_{i\in\{1,2\}} K^{-d(\beta_i-1)/\alpha}(t+t_0,0)g_2^{\beta_1}(t),\quad t>\theta,\\ g_2'(t)=2^{-d/\alpha}\min_{i\in\{1,2\}} K^{-d(\beta_i-1)/\alpha}(t+t_0,0)f_2^{\beta_2}(t),\quad t>\theta,\\ f_2(\theta)=N,\quad g_2(\theta)=N.\end{gathered}\tag{2.4}$$

From (2.4) it follows that

$$\int_\theta^t f_2^{\beta_2}(r)f_2'(r)dr=\int_\theta^t g_2^{\beta_1}(r)g_2'(r)dr,$$

that is,

$$\frac{1}{\beta_2+1}[f_2^{\beta_2+1}(t)-N^{\beta_2+1}]=\frac{1}{\beta_1+1}[g_2^{\beta_1+1}(t)-N^{\beta_1+1}].$$

Fix $\theta>0$ such that $0<N\leq 1$. This is possible due to $\overline{N}_1,\ \overline{N}_2\to 0$ when $\theta\to 0$. We assume without loss of generality that $\beta_2\geq\beta_1$. Then

$$\frac{f_2^{\beta_2+1}(t)}{\beta_2+1}\leq\frac{g_2^{\beta_1+1}(t)}{\beta_1+1}$$

or, equivalently

$$g_2(t)\geq\Big(\frac{\beta_1+1}{\beta_2+1}\Big)^{\frac{1}{\beta_1+1}} f_2^{\frac{\beta_2+1}{\beta_1+1}}(t),\quad t\geq\theta.$$

Substituting this in the first equation of (2.4), we have

$$f_2'(t)\geq 2^{-d/\alpha}\min_{i\in\{1,2\}} K^{-\frac{d(\beta_i-1)}{\alpha}}(t+t_0,0)\Big(\frac{\beta_1+1}{\beta_2+1}\Big)^{\frac{\beta_1}{\beta_1+1}} f_2^{\frac{\beta_1(\beta_2+1)}{\beta_1+1}}(t),\quad t\geq\theta,$$

that is,

$$f_2^{\frac{-\beta_1(\beta_2+1)}{\beta_1+1}}(t)f_2'(t)\geq 2^{-d/\alpha}\Big(\frac{\beta_1+1}{\beta_2+1}\Big)^{\frac{\beta_1}{\beta_1+1}}\min_{i\in\{1,2\}} K^{-\frac{d(\beta_i-1)}{\alpha}}(t+t_0,0),\quad t\geq\theta.$$

Integrating from θ to t yields

$$\begin{aligned}&\frac{\beta_1+1}{1-\beta_1\beta_2}\Big[f_2^{\frac{1-\beta_1\beta_2}{\beta_1+1}}(t)-N^{\frac{1-\beta_1\beta_2}{\beta_1+1}}\Big]\\ &\geq 2^{-d/\alpha}\Big(\frac{\beta_1+1}{\beta_2+1}\Big)^{\frac{\beta_1}{\beta_1+1}}\int_\theta^t\min_{i\in\{1,2\}} K^{-\frac{d(\beta_i-1)}{\alpha}}(r+t_0,0)dr.\end{aligned}$$

Thus (remember that $\beta_1,\, \beta_2 > 1$)

$$f_2(t) \geq \Big[N^{\frac{1-\beta_1\beta_2}{\beta_1+1}} - 2^{-d/\alpha}\Big(\frac{1-\beta_1\beta_2}{\beta_1+1}\Big)\Big(\frac{\beta_1+1}{\beta_2+1}\Big)^{\frac{\beta_1}{\beta_1+1}} H(t)\Big]^{\frac{\beta_1+1}{1-\beta_1\beta_2}},$$

where

$$H(t) \equiv \int_\theta^t \min_{i\in\{1,2\}} K^{-\frac{d(\beta_i-1)}{\alpha}}(r+t_0,0)dr, \quad t \geq \theta.$$

From (1.2) we have

$$H(t) \geq \int_\theta^t \min_{i\in\{1,2\}} (\varepsilon_2(r+t_0)^\rho)^{-\frac{d(\beta_i-1)}{\alpha}} dr.$$

Using the fact that $0 < d\rho\,(\beta_2 - 1)\,/\alpha < 1$ we get

$$\begin{aligned} H(t) &\geq \min_{i\in\{1,2\}} \varepsilon_2^{-\frac{d(\beta_i-1)}{\alpha}} \int_\theta^t (r+t_0)^{-\frac{d\rho(\beta_2-1)}{\alpha}} dr \\ &= \frac{\alpha}{\alpha - d\rho(\beta_2-1)} \min_{i\in\{1,2\}} \varepsilon_2^{-\frac{d(\beta_i-1)}{\alpha}} \Big[(t+t_0)^{\frac{\alpha-d\rho(\beta_2-1)}{\alpha}} - (\theta+t_0)^{\frac{\alpha-d\rho(\beta_2-1)}{\alpha}}\Big]. \end{aligned}$$

Thus $H(t) \to \infty$ when $t \to \infty$. So, we have that there exists $T_0 \geq \theta$ such that $f_2(t) = \infty$ for $t = T_0$. By comparison we have

$$K^{d/\alpha}(t+t_0,0)u(t+t_0) = f_1(t) \geq f_2(t) = \infty \quad \text{for } t = T_0,$$

which implies by Lemma 2.2 that $v(t,x) = \infty$ for all $t \geq (10\frac{\varepsilon_2}{\varepsilon_1})^{1/\rho}(T_0+t_0)$ and $x \in \mathbb{R}^d$. □

References

[1] D. Andreucci, M. A. Herrero, and J. J. L. Velázquez. Liouville theorems and blow up behaviour in semilinear reaction diffusion systems. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **14** (1997), No. 1, 1-53.

[2] M. Birkner, J. A. López-Mimbela, and A. Wakolbinger. Blow-up of semilinear PDE's at the critical demension. A probabilistic approach. *Proc. Amer. Math. Soc.* **130** (2002), 2431-2442.

[3] W. Deng. Global existence and finite time blow up for a degenerate reaction-diffusion system. *Nonlinear Anal. Theory Methods Appl.* **60** (2005), No. 5(A), 977-991.

[4] M. Escobedo and M. A. Herrero. Boundedness and blow up for a semilinear reaction-diffusion system. *J. Diff. Equations* **89** (1991), 176-202.

[5] M. Escobedo and H. A. Levine. Critical blow up and global existence numbers for a weakly coupled system of reaction-diffusion equations. *Arch. Ration. Mech. Anal.* **129** (1995), No. 1, 47-100.

[6] M. Fila, H. A. Levine, and Y. Uda. A Fujita-type global existence-global non-existence theorem for a system of reaction diffusion equations with differing diffusivities. *Math. Methods Appl. Sci.* **17** (1994), No. 10, 807-835.

[7] M. Guedda and M. Kirane. A note on nonexistence of global solutions to a nonlinear integral equation. *Bull. Belg. Math. Soc.* **6** (1999), 491-497.

[8] M. Guedda and M. Kirane. Critically for some evolution equations. *Differ. Equ.* **37** (2001), No. 4, 540-550.

[9] M. Kirane and M. Qafsaoui. Global nonexistence for the Cauchy problem of some nonlinear reaction-diffusion systems. *J. Math. Anal. Appl.* **268** (2002), 217-243.

[10] Y. Kobayashi. The life span of blow-up solutions for a weakly coupled system of reaction-diffusion equations. *Tokyo J. Math.* **24** (2001), No. 2, 487-498.

[11] J. A. López-Mimbela and A. Wakolbinger. Lenght of Galton-Watson trees and blow-up of semilinear systems. *J. Appl. Prob.* **35** (1998), 802-811.

[12] K. Mochizuki and Q. Huang. Existence and behaviour of solutions for a weakly coupled system of reaction-diffusion equations. *Methods Appl. Anal.* **5** (1998), No. 2, 109-124.

[13] P. Souplet and S. Tayachi. Optimal condition for non-simultaneous blow-up in a reaction-diffusion system, *J. Math. Soc. Japan* **56**, (2004), No. 2, 571-584.
[14] S. Sugitani. On nonexistence of global solutions for some nonlinear integral equations. *Osaka J. Math.* **12** (1975), 45-51.
[15] M. Wang. Blow-up rate estimates for semilinear parabolic systems. *J. Diff. Equations* **170** (2001), 317-324.
[16] S. Zheng. Nonexistence of positive solutions to a semilinear elliptic system and blow-up estimates for a reaction-diffusion system. *J. Math. Anal. Appl.* **232** (1999), 293-311.

Aroldo Pérez-Pérez
División Académica de Ciencias Básicas, Universidad Juárez Autónoma de Tabasco, Km. 1 Carretera Cunduacán-Jalpa de Méndez, C.P. 86690 A.P. 24, Cunduacán, Tabasco, México
E-mail address: aroldo.perez@dacb.ujat.mx