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POSITIVE SOLUTIONS OF A THREE-POINT BOUNDARY-VALUE PROBLEM FOR DIFFERENTIAL EQUATIONS WITH DAMPING AND ACTIVELY BOUNDED DELAYED FORCING TERM

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Abstract. We provide sufficient conditions for the existence of positive solutions of a three-point boundary value problem concerning a second order delay differential equation with damping and forcing term whose the delayed part is an actively bounded function, a meaning introduced in [19]. By writing the damping term as a difference of two factors one can extract more information on the solutions. (For instance, in an application, given in the last section, we can give the exact value of the norm of the solution).

1. Introduction

This paper is motivated by the work of Henderson [15] where the existence of two positive solutions of the differential equation $x''+f(x)=0$ satisfying the conditions $x(0)=0$ and $x(\eta)=x(1)$ is investigated.

To say exactly what we shall do in this paper we need some notation. For any interval Y of the real line $\mathbf{R}$ we shall denote by $C(Y)$ the Banach space of all continuous functions $x:Y\to\mathbf{R}$ furnished with the usual sup-norm $\|\cdot\|_Y$. If, in addition, the set Y contains the origin, we shall write $C_0(Y)$ for the set of all $\psi\in C(Y)$ with $\psi(0)=0$. In this paper we shall work, mainly, on sets of the form

$$C_0^+(Y):=\{x\in C_0(Y):x(t)\geq 0,\ t\in Y\}.$$

Consider the sets $I:=[0,1]$ and $J:=[-r,0]$ for a fixed $r\geq 0$.

Our intention is to provide sufficient conditions for the existence of positive solutions of a three-point boundary value problem concerning the second order delay differential equation

$$x''(t)+p(t)x'(t)+Q(t,x(t))+f(t,x_t)=0,\quad t\in I:=[0,1], \tag{1.1}$$

$$x_0=\phi,\quad x(\eta)=x(1), \tag{1.2}$$

where $\phi\in C_0^+(J)$, $0<\eta<1$ and the delayed part $f(t,x_t)$ of the forcing term is actively bounded function, in a sense introduced in [19]. Our technique is based

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on the fact that the coefficient $p(t)$ of the damping term can be written as the difference of two (suitable) functions:

$$p(t) = p_1(t) - p_2(t).$$

The advantage of such an approach is that we can vary the functions p_1, p_2. Then the conditions imposed as well as the existence range of the solutions also vary appropriately. A sight of what we mean is seen in the last section, where an application is presented. On the other hand such a decomposition of the damping term affects the Green function of the problem. Thus, our first intention is to construct such a kernel of the integral operator, which plays the most crucial role in our discussion.

As it is noticed elsewhere (see, e.g. [8, 14]), boundary-value problems associated with delay differential equations are generated from physics and control theory and other topics of applied mathematics. In the literature one can find a relatively great number of works dealing with the existence of solutions of boundary value problems which are associated not necessary with ordinary differential equations. For instance, in [1] one can find such problems for difference and integral equations, in [5] for equations whose the solutions depend on the past and on the future, in [11] for equations with deviating arguments, etc. Moreover a great deal can be met in the literature for the case of delay differential equations. We refer, for instance to [2, 3, 6, 7, 10, 12, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27] and to the references therein.

Most of the works mentioned above do use of the following important Fixed Point Theorem of Krasnoselskii.

Theorem 1.1 ([23])**.** *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$$

be a completely continuous operator such that either

$$\|Au\| \leq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \geq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_2,$$

or

$$\|Au\| \geq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \leq \|u\|, \quad u \in \mathcal{K} \cap \partial\Omega_2.$$

Then A has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We recall that an operator $A : X \to Y$ is called completely continuous if it is continuous and it maps bounded sets into precompact sets. We notice that when Theorem 1.1 is applied to boundary-value problems for functional differential equations, usually the most crucial point is to provide suitable conditions on the forcing delayed term which guarantee the fact that the corresponding integral operator satisfies the two alternatives of Krasnoselskii's fixed point theorem. As in [19], in this article, in order to cover the autonomous and nonautonomous cases, the continuous and discrete delay, as well as the atomic and the nonatomic response, we assume that the function f is a so called *actively bounded* function. To be more precise we shall repeat its definition here.

Definition 1.2 ([19])**.** We call a function $f(\cdot,\cdot) : I \times C_0^+(J) \to [\mathbf{0}, +\infty)$ *actively bounded*, if for each $t \in I$ there exist a nonempty closed set $\Theta_t \subseteq J$ and two real nonnegative functions $L_0(t;\cdot,\cdot)$ and $\omega(t;\cdot,\cdot)$, such that

$$\omega(t; m, M) \leq f(t, \psi) \leq L_0(t; m, M),$$

for all $t \in I$ and $\psi \in P(t; m, M)$, where

$$P(t; m, M) := \{\psi \in C_0^+(J) : m \leq \inf_{s \in \Theta_t} \psi(s),\ \|\psi\|_J \leq M\}.$$

Let $\Theta_t(f)$ be the smallest set of the form Θ_t. In [19] it was shown that the class of the actively bounded functions is closed under summation and multiplication. Also, several examples of such functions were given.

2. Formulation of the BVP

The basic theory of delay differential equations is exhibited in several places of the literature. Especially we refer to the classical books [7, 13]. For any continuous function y defined on the interval $[-r, 1]$ and any $t \in [0, 1] =: I$, the symbol y_t (appeared, also, in (1.2)) is used to denote the element of C_r defined by

$$y_t(s) = y(t+s), \quad s \in J.$$

Our purpose is to establish sufficient conditions for the existence of positive solutions of the boundary value problem (1.1)-(1.2). Here we want to make clear what makes the difference between the ordinary and the delay case and in particular what is going to be proved for the delay boundary value problem.

(We find it convenient to repeat some comments made, also, in [19].) It is well known that in the ordinary case, namely, when $r = 0$, (thus (1.1) is an ordinary differential equation), we look for conditions which guarantee the truth of the following fact: *There is a solution x of the (ordinary differential equation)* (1.1) *with* $x(0) = 0$ *and satisfying condition* (1.2). It follows that uniqueness of such a solution means that there is exactly one function with these properties.

But in the (nontrivial) delay case the problem is quite different. Indeed, here we are invited to give our response to the following challenge: *Determine a class S of initial functions with the property that for each $\phi \in S$ there is a solution x of* (1.1) *satisfying condition* (1.2). (Notice that some authors use to extend the situation from the ordinary case by simply assuming that $\phi(s) = 0$, for all $s \in J$, see, e.g. [4].) Therefore uniqueness of solutions of the BVP (1.1)-(1.2) presupposes that there is only one solution with initial value the fixed initial function ϕ. Any new initial function from the class S implies new solution of the boundary value problem (1.1)-(1.2). As we shall see later, in this paper the set S will be a closed ball in $C_0^+(J)$.

We shall reformulate the problem (1.1)-(1.2) by transforming it into a fixed point problem. Then the existence of a solution of the latter is guaranteed by Theorem 1.1.

To proceed, fix a $\phi \in C_0^+(J)$. For each function $x \in C_0(I)$ we shall denote by $T(\cdot, x; \phi)$ the function defined on $[-r, 1]$ by

$$T(s, x; \phi) := \begin{cases} \phi(s), & s \in J, \\ x(s), & s \in I, \end{cases}$$

It is easy to see that

$$\|T_t(\cdot, x_1; \phi) - T_t(\cdot, x_2; \phi)\|_J \leq \|x_1 - x_2\|_I, \tag{2.1}$$

for all $t \in I$ and $x_1, x_2 \in C_0(I)$. (Recall that for each $t \in I$ the symbol $T_t(\cdot, x; \phi)$ denotes the element of $C(J)$ defined by $T_t(s, x; \phi) := T(t+s, x; \phi)$, $s \in J$.) Thus the function

$$x \to T_t(\cdot, x; \phi) : C_0(I) \to C(J)$$

is continuous (uniformly with respect to t).

By a solution of the boundary-value problem (1.1)-(1.2) we mean a function $x \in C_0(I)$ satisfying (1.2) and its second derivative $x''(t)$ exists for all $t \in I$ satisfying the relation

$$x''(t) + p(t)x'(t) + q(t)x(t) + f(t, T_t(\cdot, x; \phi)) = 0. \tag{2.2}$$

Our first basic condition of the problem states as follows:

(H) The functions $p, q : I \to \mathbf{R}$ are continuous and such that p can be written in the form

$$p = p_1 - p_2,$$

where p_1 is continuous, p_2 is positive and differentiable and, moreover, they satisfy the inequality

$$Q(t, \xi) + (p_2'(t) + p_1(t)p_2(t))\xi \geq 0,$$

for all $t \in I$ and $\xi \geq 0$.

To simplify our presentation we set

$$V(u, s, t) := e^{\int_u^s p_1(\theta)d\theta + \int_u^t p_2(\theta)d\theta},$$

$$Y(t) := \int_0^t e^{\int_\theta^1 (p_1(u)+p_2(u))du} d\theta,$$

$$\upsilon_i(s) := e^{-\int_s^1 p_i(u)du}, \quad i = 1, 2.$$

Specially, we shall denote by υ the value $\upsilon_2(\eta)$. Clearly it holds

$$\int_0^\theta V(u, s, t)du = \upsilon_1(s)\upsilon_2(t)Y(\theta),$$

for all $\theta, s, t \in I$.

Remark. We observe that for all $s \geq \eta$ it holds

$$Y(s) - \upsilon Y(\eta) = \int_\eta^s e^{\int_v^1 (p_1(u)+p_2(u))du} dv + (1 - \upsilon)Y(\eta) > 0. \tag{2.3}$$

To proceed, we set $y(t) := x'(t)$ and write equation (2.2) in the form

$$y'(t) + p_1(t)y(t) - p_2(t)x'(t) + Q(t, x(t)) + f(t, T_t(\cdot, x; \phi)) = 0.$$

Integrate from t (≥ 0) to 1 and get

$$\begin{aligned} y(t) &= y(1)e^{\int_t^1 p_1(s)ds} \\ &\quad + \int_t^1 [-p_2(u)x'(u) + Q(u, x(u)) + f(u, T_u(\cdot, x; \phi))]e^{\int_t^u p_1(s)ds} du, \end{aligned}$$

which leads to

$$x'(t) - p_2(t)x(t) = [x'(1) - p_2(1)x(1)]e^{\int_t^1 p_1(s)ds} + \int_t^1 z(u)e^{\int_t^u p_1(s)ds} du,$$

where, for simplicity, we have put

$$z(u) := f(u, T_u(\cdot, x; \phi)) + Q(u, x(u)) + [p_1(u)p_2(u) + p_2'(u)]x(u), \quad u \in I.$$

Thus the solution x satisfies

$$x(t) = [x'(1) - p_2(1)x(1)]\upsilon_2(t)Y(t) + \int_0^t \int_u^1 V(u,s,t)z(s)\,ds\,du, \quad t \in I. \tag{2.4}$$

In (2.4) we set $t = 1$ and find

$$x'(1) = \frac{1}{Y(1)}\Big[x(1)[1 + p_2(1)Y(1)] - \int_0^1 \int_u^1 V(u,s,1)z(s)dsdu\Big].$$

Substitute this value in (2.4) and obtain

$$\begin{aligned} x(t) = \frac{\upsilon_2(t)Y(t)}{Y(1)}\Big[x(1) - \int_0^1 \int_u^1 V(u,s,1)z(s)\,ds\,du\Big] \\ + \int_0^t \int_u^1 V(u,s,t)z(s)\,ds\,du, \quad t \in I. \end{aligned} \tag{2.5}$$

Now take into account that $x(\eta) = x(1)$. From (2.5) it follows that

$$x(1) = \gamma\Big[Y(1)\int_0^\eta \int_u^1 V(u,s,\eta)z(s)ds\,du - \upsilon Y(\eta)\int_0^1 \int_u^1 V(u,s,1)z(s)ds\,du,$$

where

$$\gamma := \Big(Y(1) - \upsilon Y(\eta)\Big)^{-1}.$$

Because of (2.3) the constant γ is positive. Substituting this value to (2.5), after some manipulation, we derive

$$\begin{aligned} x(t) = \gamma\upsilon_2(t)Y(t)\Big[\int_0^\eta \int_u^1 V(u,s,\eta)z(s)\,ds\,du - \int_0^1 \int_u^1 V(u,s,1)z(s)\,ds\,du\Big] \\ + \int_0^t \int_u^1 V(u,s,t)z(s)\,ds\,du. \end{aligned} \tag{2.6}$$

Lemma 2.1. *A function x is a solution of the boundary-value problem* (1.1)-(1.2) *if and only if it satisfies the operator equation*

$$x = A_\phi x, \tag{2.7}$$

where A_ϕ is the operator

$$(A_\phi x)(t) := \int_0^1 G(t,s)F(s, T_s(\cdot, x; \phi))ds, \quad x \in C_0^+(I). \tag{2.8}$$

Here we have set

$$F(s, T_s(\cdot, x; \phi)) := f(s, T_s(\cdot, x; \phi)) + Q(s, x(s)) + [p_1(s)p_2(s) + p_2'(s)]x(s), \quad u \in I$$

and the kernel $G(t,s)$ is defined by

$$\begin{aligned} G(t,s) : = \gamma\upsilon_1(s)\upsilon_2(t)\Big[\upsilon Y(s \wedge \eta)Y(t) - Y(t)Y(s) \\ + Y(1)Y(s \wedge t) - \upsilon Y(s \wedge t)Y(\eta)\Big], \end{aligned}$$

where, as usually, $\alpha \wedge \beta := \min\{\alpha, \beta\}$.

Proof. Assume that x is a solution. Then it satisfies (2.6) and, so, we have

$$x(t)=\int_0^1\int_u^1 U(u,s,t)z(s)\,ds\,du, \tag{2.9}$$

where

$$U(u,s,t):=\gamma\upsilon_2(t)Y(t)\Big[V(u,s,\eta)\chi_{[0,\eta]}(u)-V(u,s,1)\Big]+V(u,s,t)\chi_{[0,t]}(u),$$

where $\chi_{[0,t]}(\cdot)$ stands for the characteristic function of the interval $[0,t]$. We apply Fubini's Theorem in the right side of (2.9) and get

$$x(t)=\int_0^1 G(t,s)z(s)ds,$$

where

$$G(t,s):=\int_0^s U(u,s,t)du.$$

The inverse is proved by the inverse way. The proof is complete. □

Next we simplify the form of the kernel G by examining the following cases:
Case 1.1: $s\le t\le\eta$. Then we have

$$\begin{aligned}
G(t,s)&=\gamma\upsilon_1(s)\upsilon_2(t)\Big[\upsilon Y(s)Y(t)-Y(t)Y(s)+Y(1)Y(s)-\upsilon Y(s)Y(\eta)\Big]\\
&=\gamma\upsilon_1(s)\upsilon_2(t)Y(s)\Big[[Y(1)-Y(t)]+\upsilon[Y(t)-Y(\eta)]\Big]\\
&=\gamma\upsilon_1(s)\upsilon_2(t)Y(s)\Big[\int_t^1 V(u,1,1)du-\upsilon\int_t^\eta V(u,1,1)du\Big]\\
&=\gamma\upsilon_1(s)\upsilon_2(t)Y(s)\Big[(1-\upsilon)\int_t^\eta V(u,1,1)du+\int_\eta^1 V(u,1,1)du\Big].
\end{aligned}$$

Case 1.2: $t<s\le\eta$. Then we have

$$\begin{aligned}
G(t,s)&=\gamma\upsilon_1(s)\upsilon_2(t)\Big[\upsilon Y(s)Y(t)-Y(t)Y(s)+Y(1)Y(t)-\upsilon Y(t)Y(\eta)\Big]\\
&=\gamma\upsilon_1(s)\upsilon_2(t)Y(t)\Big[[Y(1)-Y(s)]+\upsilon[Y(s)-Y(\eta)]\Big]\\
&=\gamma\upsilon_1(s)\upsilon_2(t)Y(t)\Big[\int_s^1 V(u,1,1)du-\upsilon\int_s^\eta V(u,1,1)du\Big]\\
&=\gamma\upsilon_1(s)\upsilon_2(t)Y(t)\Big[(1-\upsilon)\int_s^\eta V(u,1,1)du+\int_\eta^1 V(u,1,1)du\Big].
\end{aligned}$$

Case 1.3: $t\le\eta<s$. Then we have

$$\begin{aligned}
G(t,s)&=\gamma\upsilon_1(s)\upsilon_2(t)\Big[\upsilon Y(\eta)Y(t)-Y(t)Y(s)+Y(1)Y(t)-\upsilon Y(t)Y(\eta)\Big]\\
&=\gamma\upsilon_1(s)\upsilon_2(t)Y(t)\Big[Y(1)-Y(s)\Big]\\
&=\gamma\upsilon_1(s)\upsilon_2(t)Y(t)\Big[\int_s^1 V(u,1,1)du\Big].
\end{aligned}$$

Case 2.1: $s \leq \eta < t$. Then we have

$$
\begin{aligned}
G(t,s) &= \gamma \upsilon_1(s)\upsilon_2(t)\Big[\upsilon Y(s)Y(t) - Y(t)Y(s) + Y(1)Y(s) - \upsilon Y(s)Y(\eta)\Big] \\
&= \gamma \upsilon_1(s)\upsilon_2(t)Y(s)\Big[[Y(1) - Y(t)] + \upsilon[Y(t) - Y(\eta)]\Big] \\
&= \gamma \upsilon_1(s)\upsilon_2(t)Y(s)\Big[\int_t^1 V(u,1,1)du + \upsilon \int_\eta^t V(u,1,1)du\Big].
\end{aligned}
$$

Case 2.2: $\eta < s \leq t$. Then we have

$$
\begin{aligned}
G(t,s) &= \gamma \upsilon_1(s)\upsilon_2(t)\Big[\upsilon Y(\eta)Y(t) - Y(t)Y(s) + Y(1)Y(s) - \upsilon Y(s)Y(\eta)\Big] \\
&= \gamma \upsilon_1(s)\upsilon_2(t)\Big[Y(s)[Y(1) - Y(t)] + \upsilon Y(\eta)[Y(t) - Y(s)]\Big] \\
&= \gamma \upsilon_1(s)\upsilon_2(t)\Big[Y(s)\int_t^1 V(u,1,1)du + \upsilon Y(\eta)\int_s^t V(u,1,1)du\Big].
\end{aligned}
$$

Case 2.3: $\eta < t < s$. Then we have

$$
\begin{aligned}
G(t,s) &= \gamma \upsilon_1(s)\upsilon_2(t)\Big[\upsilon Y(\eta)Y(t) - Y(t)Y(s) + Y(1)Y(t) - \upsilon Y(t)Y(\eta)\Big] \\
&= \gamma \upsilon_1(s)\upsilon_2(t)Y(t)\Big[Y(1) - Y(s)\Big] \\
&= \gamma \upsilon_1(s)\upsilon_2(t)Y(t)\int_s^1 V(u,1,1)du.
\end{aligned}
$$

3. Main Result

Now we are ready to present our main result of this article.

Theorem 3.1. *Suppose that assumption (H) is satisfied and $f(t,\phi)$ is an actively bounded continuous function with $\Theta_t(f), t \in I$ being the set-valued function defined in Definition 1.2. Assume, also, that the functions $L_0(\cdot;m,M)$ and $\omega(\cdot;m,M)$ are measurable for all $m < M$. Finally, assume that there is a $\delta \in [0,1]$ and two (distinct) real numbers ρ_1, ρ_2 such that*

$$
\frac{1}{\rho_1}\int_0^1 G(s,s)L(s,\frac{\mu}{\Lambda}\rho_1,\rho_1)ds \leq \frac{1}{\Lambda}, \tag{3.1}
$$

$$
\frac{1}{\rho_2}\sup_{t\in I}\int_S G(t,s)\omega(s,\frac{\mu}{\Lambda}\rho_2,\rho_1 \vee \rho_2)ds \geq 1, \tag{3.2}
$$

where

$$
L(t,m,M) := \sup_{0\leq\xi\leq M}\Big(Q(t,\xi) + [p_2'(t) + p_1(t)p_2(t)]\xi\Big) + L_0(t,m,M).
$$

$$
S := \{s \in [0,1] : s + \theta \in [\delta,1],\ \theta \in \Theta_s(f)\},
$$

$$
\Lambda := \max\{e^{\int_0^\eta p_2(v)dv}, e^{\int_\eta^1 p_2(v)dv}\}. \tag{3.3}
$$

$$
\mu := \min\Big\{\frac{Y(1)-Y(\eta)}{Y(1)-\upsilon Y(\eta)},\quad \frac{Y(\delta)}{Y(1)}\upsilon_2(0),\quad \frac{Y(\eta)}{Y(1)}\upsilon\Big\}. \tag{3.4}
$$

Then, for any $\phi \in C_r^+(0)$ with $\|\phi\| \leq \rho_1$, there is a positive solution of the boundary-value problem (1.1)-(1.2) having norm in the interval with ends the numbers ρ_1, ρ_2.

Proof. First we shall obtain some properties of the kernel G of the operator A_ϕ. In Case 1.1 we have

$$\begin{aligned}\frac{G(t,s)}{G(s,s)}&=\frac{\upsilon_2(t)\Big[(1-\upsilon)\int_t^\eta V(u,1,1)du+\int_\eta^1 V(u,1,1)du\Big]}{\upsilon_2(s)\Big[(1-\upsilon)\int_s^\eta V(u,1,1)du+\int_\eta^1 V(u,1,1)du\Big]}\\&\le e^{\int_s^t p_2(v)dv}\le e^{\int_0^\eta p_2(v)dv}\le\Lambda,\end{aligned}\tag{3.5}$$

where Λ is given by (3.3). Also in this case we get

$$\begin{aligned}\frac{G(t,s)}{G(s,s)}&\ge\frac{\int_\eta^1 V(u,1,1)du}{(1-\upsilon)\int_0^\eta V(u,1,1)du+\int_\eta^1 V(u,1,1)du}\\&=\frac{Y(1)-Y(\eta)}{Y(1)-\upsilon Y(\eta)}\ge\mu.\end{aligned}\tag{3.6}$$

In Cases 1.2, 1.3 and 2.3 we get

$$\frac{G(t,s)}{G(s,s)}=\frac{\upsilon_2(t)Y(t)}{\upsilon_2(s)Y(s)}\le e^{-\int_t^s p_2(v)dv}\le 1\le\Lambda.\tag{3.7}$$

If, in addition, we have $\delta\le t$, then it follows that

$$\frac{G(t,s)}{G(s,s)}=\frac{\upsilon_2(t)Y(t)}{\upsilon_2(s)Y(s)}=e^{-\int_t^s p_2(v)dv}\frac{Y(t)}{Y(s)}\ge e^{-\int_t^s p_2(v)dv}\frac{Y(\delta)}{Y(1)}\ge\mu.\tag{3.8}$$

In Case 2.1 it holds

$$\begin{aligned}\frac{G(t,s)}{G(s,s)}&=\frac{\upsilon_2(t)\Big[\int_t^1 V(u,1,1)du+\upsilon\int_\eta^t V(u,1,1)du\Big]}{\upsilon_2(s)\Big[\int_s^1 V(u,1,1)du+\upsilon\int_\eta^s V(u,1,1)du\Big]}\\&\le\upsilon^{-1}e^{\int_s^t p_2(v)dv}\le e^{\int_0^\eta p_2(v)dv}\le\Lambda.\end{aligned}\tag{3.9}$$

Also, we get

$$\frac{G(t,s)}{G(s,s)}\ge\upsilon e^{\int_s^t p_2(v)dv}\ge\upsilon\ge\mu.\tag{3.10}$$

Finally, in Case 2.2 we have

$$\begin{aligned}\frac{G(t,s)}{G(s,s)}&=\frac{\upsilon_2(t)\Big[Y(s)\int_t^1 V(u,1,1)du+\upsilon Y(\eta)\int_s^t V(u,1,1)du\Big]}{\upsilon_2(s)\Big[Y(s)\int_s^1 V(u,1,1)du\Big]}\\&\le e^{\int_s^t p_2(v)dv}\frac{Y(s)\int_t^1 V(u,1,1)du+\upsilon Y(s)\int_s^t V(u,1,1)du}{Y(s)\int_s^1 V(u,1,1)du}\\&\le e^{\int_\eta^1 p_2(v)dv}\le\Lambda,\end{aligned}\tag{3.11}$$

and moreover

$$\begin{aligned}\frac{G(t,s)}{G(s,s)} &= \frac{\upsilon_2(t)\Big[Y(s)\int_t^1 V(u,1,1)du + \upsilon Y(\eta)\int_s^t V(u,1,1)du\Big]}{\upsilon_2(s)\Big[Y(s)\int_s^1 V(u,1,1)du\Big]}\\ &\geq e^{\int_s^t p_2(v)dv}\frac{\upsilon Y(\eta)\int_s^1 V(u,1,1)du}{Y(s)\int_s^1 V(u,1,1)du}\\ &\geq \upsilon\frac{Y(\eta)}{Y(1)} \geq \mu.\end{aligned} \tag{3.12}$$

From (3.5), (3.7), (3.9) and (3.11) we see that for all $s,t \in I$,

$$G(t,s) \leq \Lambda G(s,s), \tag{3.13}$$

where (recall that) Λ is the constant defined in (3.3). Also from (3.6), (3.8), (3.10) and (3.12) we see that for all $t \in [\delta,1]$ and $s \in [0,1]$ it holds

$$G(t,s) \geq \mu G(s,s), \tag{3.14}$$

where μ is defined in (3.4).

Now define the set

$$\mathcal{K} := \{x \in C_0^+(I): \quad x(t) \geq \frac{\mu}{\Lambda}\|x\|, \quad t \in [\delta,1]\}$$

and observe that it is a cone in the space $C_0(I)$.

Consider a initial function $\phi \in C_r^+(0)$ with $\|\phi\|_J \leq \rho_1$, where ρ_1 satisfies (3.1) and (3.2).

Let A_ϕ be the corresponding operator defined by (2.8). Because of Lemma 2.1 it is enough to show that the operator A_ϕ has a fixed point. To this end we let $x \in \mathcal{K}$. Then we have $(A_\phi x)(0) = 0$ and from (3.13) we get

$$\begin{aligned}\|A_\phi x\|_I &= \sup_{t\in I}\int_0^1 G(t,s)F(s,T_s(\cdot,x;\phi))ds\\ &\leq \Lambda\int_0^1 G(s,s)F(s,T_s(\cdot,x;\phi))ds.\end{aligned} \tag{3.15}$$

From (H) and the definition of f we have $F(s,T_s(\cdot,x;\phi)) \geq 0$, for all $s \in I$ and therefore $(A_\phi x)(t) \geq 0$ for all $t \in I$. Let $t \in [\delta,1]$; from (3.14) and (3.15) we get

$$\begin{aligned}(A_\phi x)(t) &= \int_0^1 G(t,s)F(s,T_s(\cdot,x;\phi))ds\\ &\geq \mu\int_0^1 G(s,s)F(s,T_s(\cdot,x;\phi))ds \geq \frac{\mu}{\Lambda}\|A_\phi x\|_I.\end{aligned} \tag{3.16}$$

Relation (3.16) guarantees that the operator A_ϕ maps the cone $\mathcal{K}$ into itself. Furthermore from (2.1) and the first argument in Definition 1.2 we conclude that the function $y \to F(\cdot,T_\cdot(\cdot,y;\phi))$ is continuous and it maps bounded sets into bounded sets; thus the operator A_ϕ is completely continuous.

Next take any $x \in \mathcal{K}$. By definition, for any $s \in S$ we have $s+\theta \in [\delta,1] \subseteq I$, for all $\theta \in \Theta_s(f)$. Thus it holds

$$T_s(\theta,x;\phi) = x(s+\theta) \geq \frac{\mu}{\Lambda}\|x\|_I. \tag{3.17}$$

Let $x \in \mathcal{K}$ with $\|x\|_I = \rho_1$. Taking it into account together with the choice of $\|\phi\|_J$, we have $\|T_s(\cdot, x; \phi)\|_J \le \rho_1$. Thus, because of (3.13), Definition 1.2 and (3.1) for all $t \in I$ we have

$$\begin{aligned} \|A_\phi x\|_I &= \sup_{t\in I} \int_0^1 G(t,s)F(s, T_s(\cdot, x; \phi))ds \\ &\le \Lambda \int_0^1 G(s,s)F(s, T_s(\cdot, x; \phi))ds \\ &\le \Lambda \int_0^1 G(s,s)L(s, \frac{\mu}{\Lambda}\rho_1, \rho_1)ds \le \rho_1 = \|x\|_I. \end{aligned} \tag{3.18}$$

Also, let $x \in \mathcal{K}$, with $\|x\|_I = \rho_2$. Then we derive

$$\|T_s(\cdot; x, \phi)\|_J \le \rho,$$

where, recall that, $\rho := \rho_1 \vee \rho_2$. Consequently, because of (H), Definition 1.2 and (3.2), we get

$$\begin{aligned} \|A_\phi x\|_I &= \sup_{t\in I} \int_0^1 G(t,s)F(s, T_s(\cdot, x; \phi))ds \\ &\ge \sup_{t\in I} \int_S G(t,s)F(s, T_s(\cdot, x; \phi))ds \\ &\ge \sup_{t\in I} \int_S G(t,s)\omega(s; \frac{\mu}{\eta}\|x\|_I, \rho)ds \ge \|x\|_I. \end{aligned} \tag{3.19}$$

Finally, define Ω_1 and Ω_2 to be the open balls with radius $\rho_1 \wedge \rho_2$ and $\rho_1 \vee \rho_2$ respectively. The previous arguments together with (3.18) and (3.19) permit us to apply Theorem 1.1 to get the result. □

4. An Application

In this section we show that our technique (namely, to write the damping term as the difference of two factors), helps a lot to obtain more information on the existence of the solutions. We show that given any $\rho > 0$ there exist solutions having norm equal to ρ. Consider the delay differential equation

$$x''(t) + te^{-x(t-\frac{1}{2})} \exp^{1000x(\frac{t}{2})} = 0, \quad t \in [0,1], \tag{4.1}$$

associated with the conditions

$$x_0 = \phi, \quad x(\frac{1}{2}) = x(1). \tag{4.2}$$

Here we have $\eta = \frac{1}{2}$, $r = \frac{1}{2}$ (thus $J = [-\frac{1}{2}, 0]$), $Q(t, \xi) := 0$ and $p(t) := 0$. Let c be a positive parameter and write $p(t) = c - c$. First observe that the function

$$f(t, \psi) := te^{-\psi(-\frac{1}{2})} \exp^{1000\psi(-\frac{t}{2})}$$

is actively bounded with

$$\Theta_t(f) := \{-\frac{1}{2}, -\frac{t}{2}\},$$
$$\omega(t; m, M) := te^{-M} \exp^{1000m},$$
$$L_0(t, m, M) := te^{-m} \exp^{1000M}.$$

Hence

$$L(t, m, M) := c^2 M + te^{-m} \exp^{1000M}.$$

Choose $\delta = \frac{1}{10}$, thus we obtain $S = [\frac{3}{5}, 1]$.

In the sequel all constants involved in our conditions are given as expressions of the parameter c. So, first we obtain

$$\begin{gathered}
\upsilon_1(t;c) = \upsilon_2(t;c) = e^{-c(1-t)}, \quad \upsilon(c) = e^{-\frac{c}{2}}, \\
Y(t;c) = \frac{e^{c(2-t)}}{c}\sinh(ct), \\
\gamma(c) = \frac{ce^{-c}}{\sinh(c) - \sinh(\frac{c}{2})}.
\end{gathered}$$

Also we obtain

$$\begin{gathered}
\Lambda(c) = e^{\frac{c}{2}}, \\
\mu(c) = \min\Big\{\frac{e^{\frac{-c}{2}}\sinh(\frac{c}{2})}{\sinh(c) - \sinh(\frac{c}{2})}, \quad e^{-(\frac{11c}{10})}\frac{\sinh(\frac{c}{10})}{\sinh(c)}, \quad \frac{\sinh(\frac{c}{2})}{\sinh(c)}\Big\}.
\end{gathered}$$

We can see that for all $c > 0$,

$$\mu(c) = \frac{e^{-\frac{c}{4}}\sinh(\frac{c}{4})}{\sinh(c)}.$$

Next we compute $G(s,s)$ for $s \in I$: If $s \le \frac{1}{2}$, then

$$G(s,s) = \frac{(\sinh(cs))^2}{c(\sinh(c) - \sinh(\frac{c}{2}))}[1 + e^{c(\frac{3}{4}-s)}\sinh(\frac{c}{4})] + \frac{\sinh(cs)}{c},$$

while, if $s \ge \frac{1}{2}$, then

$$G(s,s) = \frac{\cosh(c) - \cosh(2cs - c)}{2c(\sinh(c) - \sinh(\frac{c}{2}))}.$$

Also we obtain

$$G(\frac{1}{2}, s) = \frac{\sinh(\frac{c}{2})\sinh(c - cs)}{c(\sinh(c) - \sinh(\frac{c}{2}))}.$$

Now we seek for the existence of positive reals ρ_1, ρ_2 satisfying (3.1), namely,

$$\begin{aligned}
&\frac{\rho_1}{\sinh(c) - \sinh(\frac{c}{2})}\Big[\frac{1}{2}\sinh(\frac{c}{4})[c\sinh(\frac{3c}{4}) - 2\sinh(\frac{c}{2})\sinh(\frac{c}{4})] - 2e^{-c}\Big] \\
&+ \frac{\exp^{1000\rho_1}}{8c^2(\sinh(c) - \sinh(\frac{c}{2}))}\Big[c\sinh(\frac{c}{4})\sinh(\frac{3c}{4}) + 2\sinh(c) \\
&+ \sinh(\frac{c}{2}) - \frac{3(1+c^2)}{2c}\cosh(c) \\
&+ \frac{1}{2c}\Big]\exp\big(\frac{-e^{-(\frac{8c}{5})}\sinh(\frac{c}{10})}{\sinh(c)}\rho_1\big) \le \rho_1 e^{-\frac{c}{2}}
\end{aligned} \tag{4.3}$$

and (3.2). The latter becomes

$$\int_S G(\frac{1}{2}, s)\omega(s, \frac{\mu(c)}{\Lambda(c)}\rho_2, \rho_1 \vee \rho_2)ds \ge \rho_2,$$

which takes the form

$$\begin{aligned}\frac{\sinh(\frac{c}{2})}{c(\sinh(c)-\sinh(\frac{c}{2})}\Big(\frac{3}{5c}\cosh(\frac{2c}{5})+\frac{1}{c^2}\sinh(\frac{2c}{5})-\frac{1}{c}\Big)\\ \exp(-1100c\rho_2\frac{\sinh(c/10)}{\sinh(c)})\geq\rho_2 e^{\rho_1\vee\rho_2}.\end{aligned}\tag{4.4}$$

Let us restrict our discussion to the case

$$\rho_1<\rho_2. \tag{4.5}$$

By the use of a graphing calculator, we can take a view of the set of pairs (ρ_1,ρ_2) satisfying the implicit algebraic inequalities (4.3), (4.4) and (4.5). We find out that there are two points c_1 and c_2 (approximately equal to 0.1 and 1.66527, respectively) such that for all $c\in[c_1,c_2]$ inequalities (4.3), (4.4) are satisfied by all $\rho_1,\rho_2>0$.

We shall show the following result.

Theorem 4.1. *Let $\rho_2>0$ and any (initial) function $\phi\in C_0^+(J)$ with $\|\phi\|_J\leq\rho_2$. Then there is a solution x of the problem* (4.1)- (4.2) *such that $\|x\|_I=\rho_2$.*

Proof. Consider a $c\in(c_1,c_2]$ and a (strictly) increasing sequence of positive reals R_n converging to ρ_2. By the previous arguments it follows that $\rho_2>R_n$ satisfies (4.4) and R_n satisfies (4.3). By Theorem 3.1 there is a solution x_n of (4.1) such that $x_n(s)=\phi(s)$, for all $s\in[-\frac{1}{2},0]$,

$$x_n(\frac{1}{2})=x(1) \tag{4.6}$$

and

$$R_n\leq\|x_n\|_I\leq\rho_2, \tag{4.7}$$

for all n. From (4.6) it follows that there is a $t_n\in[\frac{1}{2},1]$ such that $x_n'(t_n)=0$ and hence from (4.1) by integration we get

$$x_n'(t)=-\int_{t_n}^t se^{-x_n(s-\frac{1}{2})}e^{1000x_n(\frac{s}{2})}ds. \tag{4.8}$$

This shows that (x_n') is bounded. Also, from (4.1) we see that (x_n'') is bounded. Applying Arzela-Ascoli theorem twice it follows that there is a subsequence (x_{k_n}) converging (in the C^1 sense) to some differentiable function x satisfying the integral equation (4.8). It is easy to see that x is a solution of the original problem, and because of (4.7), it satisfies $\|x\|_I=\rho_2$. The proof is complete. $\square$

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