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STABLE TIME PERIODIC SOLUTIONS FOR DAMPED SINE-GORDON EQUATIONS

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Abstract. We investigate existence and stability of time periodic solutions for damped Sine-Gordon equations with delay under reasonable assumptions. The key-step is constructing suitable Lyapunov functionals and establishing the priori bound for all possible periodic solutions. We also use the Schaefer fixed-point theorem.

1. Introduction

In this paper, we are interested in obtaining existence and stability of time periodic solutions for the following damped Sine-Gordon equations with delay

$$\begin{gathered}\frac{\partial^2}{\partial t^2}u(t,x)=\frac{\partial^2}{\partial x^2}u(t,x)-\delta\frac{\partial}{\partial t}u(t,x)-\int_{t-\tau}^{t}e^{-\alpha(t-s)}\frac{\partial}{\partial s}u(s,x)ds\\ +\beta\sin(u(t,x))+g(t,x),\quad 0<x<1,\ t>0,\\ u(t,0)=u(t,1)=0,\quad t>0,\end{gathered}$$

where $0<\tau<+\infty$, β and δ are two constants. The Sine-Gordon equation appears in a number of physical applications, including the propagation of junction between two superconductors, the motion of rigid pendula attached to a stretched wire, and the dislocations in crystals.

The existence of periodic solutions of nonlinear partial differential equations with delay has been considered in several works, see for example [1, 2, 3, 4, 7, 8, 9, 10, 14] and references listed therein. Most of these results are established by applying semi-group theory [4, 7], Leray-Schauder continuation theorem [9], coincidence degree theory [10] and so on. Hino, Naito et al. [7] investigated the existence of (almost) periodic solutions for damped wave equations by using the key assumption that there exists a bounded solution. Burton and Zhang [1] obtained the time periodic solution to some evolution equations with infinity delay by means of Granas's fixed point theorem [13]. The theory of partial differential equations with delay(s) has seen considerable development, see the monographs of Wu [8] and Hale [3, 5], where numerous properties of their solutions are studied. For the more special work, we can read the references therein [14, 2, 11, 6].

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The paper is organized as follows. In section 2 we will establish existence of time periodic solution for abstract differential equation in a certain Banach space. Some existence results for time periodic solutions are given in section 3. At last section, the uniformly asymptotic stable time periodic solution will be shown.

For the remainder of the introduction, we state the following lemma which will be used in the sequel.

Lemma 1.1 (Schaefer [12]). *Let $(X, \|\cdot\|)$ be a normed linear space, H a continuous mapping of X into X which is compact on each bounded subset D of X. Then either*

(i) *$x = \lambda Hx$ has a solution in X for $\lambda = 1$, or*
(ii) *the set of all such solutions, $0 < \lambda < 1$, is unbounded.*

2. Time periodic solutions for abstract equations

In this section, we obtain the existence results of periodic solutions for the wave equation

$$\begin{gathered}\frac{\partial^2}{\partial t^2}u(t,x) + \delta\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \lambda C(t,u), \quad 0 < x < 1,\ t > 0, \\ u(t,0) = u(t,1) = 0, \quad t > 0,\end{gathered} \tag{2.1}$$

where $C(t+T,u) = C(t,u)$ for some $T > 0$, $\lambda \in (0,1)$. Let $v(t,x) = \frac{\partial}{\partial t}u(t,x)$, we define a linear (unbounded) operator A by

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{\partial^2 u}{\partial x^2} - \delta v \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in D(A) = (H^2 \cap H_0^1) \times H_0^1,$$

where $H_0^1 = W_0^{1,2}(0,1)$, $H^2 = W^{2,2}(0,1)$. Set

$$\tilde{C}(t,w) = \begin{pmatrix} 0 \\ C(t,u) \end{pmatrix} \quad \text{for } w = \begin{pmatrix} u \\ v \end{pmatrix} \in L^2(0,1;R) \times L^2(0,1;R).$$

We rewrite (2.1) as an abstract equation in $L^2(0,1;R) \times L^2(0,1;R)$. That is,

$$w'(t) = Aw(t) + \lambda\tilde{C}(t,w). \tag{2.2}$$

Let

$$\begin{gathered}X = \{w \in C(R, H_0^1 \times H^0) | w(t+T) = w(t)\}, \\ \|w\|_X = \sup\{(\int_0^1 (u_x^2 + v^2)dx)^{1/2} | 0 \le t \le T\}.\end{gathered} \tag{2.3}$$

Then $(X, \|\cdot\|_X)$ is a Banach space.

Theorem 2.1. *Suppose the following conditions hold:*

(1) *There exists $D > 0$ such that if $w(t)$ is a T-periodic solution of (2.2) for some $\lambda \in (0,1)$, then $\|w\|_X < D$;*
(2) *$C : [0,T] \times X \to H^0$ is continuous and takes bounded sets into bounded sets. Then (2.1) has a T-periodic solution for $\lambda = 1$.*

Proof. By the definition of the operator A, we see that A generates the strongly continuous semigroup $\{T(t)\}_{t\ge0}$ and there exist $c > 0$, $\alpha > 0$ such that $\alpha \in \rho(A)$ and $\|T(t)\| \le ce^{\alpha t}$ for $t \ge 0$. Take $S(t) = T(t)e^{-\alpha t}$ for $t \ge 0$, then $\|S(t)\| \le c$ for $t \ge 0$. Therefore, $\{S(t)\}_{t\ge0}$ is bounded strongly continuous semigroup with the generator $A - \alpha I$. For the constant $T > 0$, from the equality $\sigma(S(T)) = \overline{e^{T\sigma(A-\alpha I)}}$ [7, Theorem 1.8], we conclude that $1 \notin \sigma(S(T))$. Thus there exists a positive number N such that $\|(I-S(T))^{-1}\| \le N$. We will complete the proof with using the following two lemmas.

Lemma 2.2. *If $L : X \to X$ is defined by*

$$Lw(t) = \int_{t-T}^{t} S(t-s)[I-S(T)]^{-1}[\lambda\tilde{C}(s,w(s)) + \alpha w(s)]ds$$

and if w is a fixed point of L then w satisfies (2.2).

Proof. For any $w \in X$, since

$$\begin{aligned} Lw(t+T) &= \int_{t}^{t+T} S(t+T-s)[I-S(T)]^{-1}[\lambda\tilde{C}(s,w(s)) + \alpha w(s)]ds \\ &= \int_{t-T}^{t} S(t-r)[I-S(T)]^{-1}[\lambda\tilde{C}(r+T,w(r+T)) + \alpha w(r+T)]dr \\ &= \int_{t-T}^{t} S(t-s)[I-S(T)]^{-1}[\lambda\tilde{C}(s,w(s)) + \alpha w(s)]ds = Lw(t), \end{aligned}$$

we see that L is well-defined. On the other hand, if $w(t)$ is a fixed point of L, by the well-known equality $\frac{d}{dt}S(t)w = (A-\alpha I)S(t)w$ for any $w \in X$, we have

$$\begin{aligned} \frac{d}{dt}w(t) &= \frac{d}{dt}Lw(t) = [I-S(T)]^{-1}[\lambda\tilde{C}(t,w(t)) + \alpha w(t)] \\ &\quad - S(T)[I-S(T)]^{-1}[\lambda\tilde{C}(t-T,w(t-T)) + \alpha w(t-T)] \\ &\quad + (A-\alpha I)\int_{t-T}^{t} S(t-s)[I-S(T)]^{-1}[\lambda\tilde{C}(s,w(s)) + \alpha w(s)]ds \\ &= Aw(t) + \lambda\tilde{C}(t,w(t)). \end{aligned}$$

This completes the proof. □

Lemma 2.3. *The operator L is a compact operator in X.*

Proof. Let $E(w(s)) = \lambda\tilde{C}(s,w(s)) + \alpha w(s)$, by assumption (2) of Theorem 2.1, $E(\cdot)$ is continuous. Then, for any $w_1, w_2 \in X$, we obtain

$$Lw_1(t) - Lw_2(t) = \int_{t-T}^{t} S(t-s)[I-S(T)]^{-1}[E(w_1(s)) - E(w_2(s))]ds.$$

Thus

$$\|Lw_1(t) - Lw_2(t)\|_{H_0^1\times H^0} \le cTN \sup_{s\in[0,T]} \|E(w_1(s)) - E(w_2(s))\|_{H_0^1\times H^0}.$$

Then the continuity of $E(\cdot)$ implies that L is continuous.

Next, we show L maps the bounded sets into compact sets. Let B be any bounded set in X and $M = \sup_{(t,w)\in[0,T]\times B} |C(t,u(t))|_{H^0}$. Since C takes $[0,T]\times B$ into a bounded set, we see that

$$\|Lw\|_X \le cTN[\sup_{(t,w)\in[0,T]\times B} |C(t,u(t))|_{H^0} + \alpha K] \le cTN[M + \alpha K],$$

where K is the boundedness of B. This implies that $L(B)$ is uniformly bounded.

On the other hand, for any $w \in B$, and $0 \le t_1 < t_2 \le T$, we have

$$Lw(t_2) - Lw(t_1)$$

$$
\begin{aligned}
&= \int_{t_2-T}^{t_2} S(t_2-s)[I-S(T)]^{-1}[\lambda\tilde{C}(s,w(s))+\alpha w(s)]ds \\
&\quad - \int_{t_1-T}^{t_1} S(t_1-s)[I-S(T)]^{-1}[\lambda\tilde{C}(s,w(s))+\alpha w(s)]ds \\
&= \int_{t_1}^{t_2} S(t_2-s)[I-S(T)]^{-1}[\lambda\tilde{C}(s,w(s))+\alpha w(s)]ds \\
&\quad - \int_{t_1-T}^{t_2-T} S(t_1-s)[I-S(T)]^{-1}[\lambda\tilde{C}(s,w(s))+\alpha w(s)]ds \\
&\quad + [S(t_2-t_1)-I]\int_{t_2-T}^{t_1} S(t_1-s)[I-S(T)]^{-1}[\lambda\tilde{C}(s,w(s))+\alpha w(s)]ds.
\end{aligned}
$$

Since $\lim_{t_2\to t_1} S(t_2-t_1)w = w$ for $w\in X$, we conclude that the right-hand side of above equality tends to zero as $t_2\to t_1$. This implies $L(B)$ is equicontinuous. By Ascoli's lemma, $L(B)$ is a compact set. This completes the proof. □

From Assumption (1) in Theorem 2.1, we see that the set $\{x\in X : x=\lambda Lx,$ for some $\lambda\in(0,1)\}$ is bounded. Then from Lemma 2.2, Lemma 2.3 and Lemma 1.1, we see that equation (2.2) has a T-periodic solution $w(t,x)$ for $\lambda=1$. That is, (2.1) has a T-periodic solution $u(t,x)$ for $\lambda=1$. Then the proof of Theorem 2.1 is complete. □

3. Time periodic solutions for Sine-Gordon equations

In this section, we consider the existence of periodic solutions for the Sine-Gordon equations with delay:

$$
\begin{gathered}
\frac{\partial^2}{\partial t^2}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - \delta\frac{\partial}{\partial t}u(t,x) - \int_{t-\tau}^{t} e^{-\alpha(t-s)}\frac{\partial}{\partial s}u(s,x)ds \\
+\beta\sin(u(t,x)) + g(t,x), \quad 0<x<1,\ t>0, \\
u(t,0)=u(t,1)=0, \quad t>0,
\end{gathered}
\tag{3.1}
$$

where g is continuous. We assume that there exist positive constants K and T satisfying

- (A1) $a=\int_{-\tau}^{0} e^{\alpha s}ds<\delta$;
- (A2) $\delta<\frac{\pi^2}{3}$ and $0<|\beta|<\sqrt{(2\delta-\frac{(2+\delta)\delta}{\pi^2}-a)(2\delta-\frac{\delta(5\delta+a)}{\pi^2})}$;
- (A3) $g\neq 0, g(t+T,x)=g(t,x)$ and $|g(t,x)|\le K$.

First, we consider the homotopy equation of (3.1),

$$
u_{tt}+\delta u_t = u_{xx}+\lambda[\beta\sin(u)+g(t,x)-\int_{t-\tau}^{t} e^{-\alpha(t-s)}u_s(s,x)ds], \quad 0<\lambda<1. \tag{3.2}
$$

Lemma 3.1. *Suppose that (A1), (A2) and (A3) hold. Then there is a constant $C_1>0$ such that all T-periodic solution $u(t)$ of (3.2) satisfy*

$$
\sup_{0\le t\le T}\int_0^1 (u_x^2+u_t^2)dx \le C_1.
$$

Proof. Let $u(t,x)$ be a T-periodic solution of (3.2) and define

$$
V(t)=\int_0^1 [v^2+2kuv+u_x^2]dx,
$$

where $v = u_t$, $k = \frac{\delta}{\pi^2}$. Since $\int_0^1 \pi^2 u^2 dx \le \int_0^1 u_x^2 dx$ for $u \in H_0^1$, we obtain

$$(1-k)\int_0^1 [u_x^2 + v^2]dx \le V(t) \le (1+k)\int_0^1 [u_x^2 + v^2]dx. \tag{3.3}$$

Noting the assumptions (A2) and (A3), we see that $|\beta \sin(u) + g(t,x)| \le |\beta||u| + K$. On the other hand, since

$$0 < |\beta| < \sqrt{(2\delta - \frac{(2+\delta)\delta}{\pi^2} - a)(2\delta - \frac{\delta(5\delta + a)}{\pi^2})},$$

then we choose positive constant ε_1 such that

$$\frac{|\beta|}{2\delta - \frac{(2+\delta)\delta}{\pi^2} - a} < \varepsilon_1 < \frac{2\delta - \frac{\delta(5\delta+a)}{\pi^2}}{|\beta|}.$$

Thus we have

$$\frac{(2+\delta)\delta}{\pi^2} + a + |\beta|\varepsilon_1^{-1} < 2\delta$$

and (since $|\beta| < 2\delta$)

$$\frac{\delta}{\pi^2}(\delta + 2|\beta| + a) + |\beta|\varepsilon_1 < \frac{\delta(5\delta + a)}{\pi^2} + |\beta|\varepsilon_1 < 2\delta.$$

Therefore, there exist constants ε_2 and ε_3 such that

$$\frac{(2+\delta)\delta}{\pi^2} + a + |\beta|\varepsilon_1^{-1} + \varepsilon_2 < 2\delta$$

and

$$\frac{\delta}{\pi^2}(\delta + 2|\beta| + a + \varepsilon_3) + |\beta|\varepsilon_1 < 2\delta.$$

Hence, by choosing $k = \delta/\pi^2$, we have

$$(2\delta - 2k) - (k\delta + |\beta|\varepsilon_1^{-1} + \varepsilon_2 + a) > 0, \tag{3.4}$$

$$2k - \frac{1}{\pi^2}[k(\delta + 2|\beta| + \varepsilon_3 + a) + |\beta|\varepsilon_1] > 0. \tag{3.5}$$

So that

$$\begin{aligned}
&V'(t) \\
&= 2\int_0^1 [vv_t + u_x u_{xt} + k(u_t v + uv_t)]dx \quad (\text{since } \int_0^1 u_x u_{xt} dx = -\int_0^1 v u_{xx} dx) \\
&= 2\int_0^1 [vu_{tt} - vu_{xx} + k(u_{tt}u + v^2)]dx \\
&= 2\int_0^1 [-\delta v^2 + \lambda v[\beta \sin(u) + g(t,x) - \int_{t-\tau}^t e^{-\alpha(t-s)} v(s,x)ds]]dx \\
&\quad + 2k\int_0^1 [v^2 + u_{xx}u - \delta vu + \lambda u[\beta \sin(u) + g(t,x) - \int_{t-\tau}^t e^{-\alpha(t-s)} v(s,x)ds]]dx \\
&\le 2\int_0^1 [-(\delta - k)v^2 - ku_x^2 - \delta kuv]dx \\
&\quad + 2\int_0^1 (ku + v)(|\beta||u| + K + \int_{t-\tau}^t e^{-\alpha(t-s)}|v(s,x)|ds)dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 [-(2\delta-2k)v^2 - 2ku_x^2 + k\delta(u^2+v^2)]dx \\
&\quad + \int_0^1 (2|\beta|ku^2 + \varepsilon_3 ku^2 + k\frac{K^2}{\varepsilon_3} + aku^2 + k\int_{t-\tau}^t e^{-\alpha(t-s)}v^2(s,x)ds)dx \\
&\quad + \int_0^1 (|\beta|\varepsilon_1 u^2 + \frac{|\beta|}{\varepsilon_1}v^2 + \varepsilon_2 v^2 + \frac{K^2}{\varepsilon_2} + av^2 + \int_{t-\tau}^t e^{-\alpha(t-s)}v^2(s,x)ds)dx \\
&= \int_0^1 [-(2\delta-2k) + k\delta + |\beta|\varepsilon_1^{-1} + \varepsilon_2 + a]v^2 dx \\
&\quad + \int_0^1 -2ku_x^2 + [k(\delta + 2|\beta| + \varepsilon_3 + a) + |\beta|\varepsilon_1]u^2 dx \\
&\quad + \frac{K^2}{\varepsilon_2} + k\frac{K^2}{\varepsilon_3} + (1+k)\int_0^1\int_{t-\tau}^t e^{-\alpha(t-s)}v^2(s,x)dsdx \\
&\quad \Big(\text{since } \int_0^1 \pi^2 u^2 dx \le \int_0^1 u_x^2 dx\Big) \\
&\le -c_1\int_0^1 v^2 + u_x^2 dx + c_2\int_0^1\int_{t-\tau}^t e^{-\alpha(t-s)}v^2(s,x)dsdx + c_3,
\end{aligned}
$$

where $c_2 = 1+k$, $c_3 = \frac{K^2}{\varepsilon_2} + k\frac{K^2}{\varepsilon_3}$ and (by (3.4) and (3.5))

$$
\begin{aligned}
c_1 = \min\big\{&(2\delta-2k) - (k\delta + |\beta|\varepsilon_1^{-1} + \varepsilon_2 + a), \\
&2k - \frac{1}{\pi^2}[k(\delta+2|\beta|+\varepsilon_3+a) + |\beta|\varepsilon_1]\big\} > 0.
\end{aligned}
$$

By (3.3), we see that there exist three positive constants r_1, r_2 and r_3 such that

$$
V'(t) \le -r_1 V(t) + r_2\int_0^1\int_{t-\tau}^t e^{-\alpha(t-s)}v^2(s,x)ds\,dx + r_3. \tag{3.6}
$$

Since $V(t)$ and v are T-periodic, we have

$$
\begin{aligned}
\int_0^T V(t)dt &\le \frac{r_2}{r_1}\int_0^T\int_0^1\int_{t-\tau}^t e^{-\alpha(t-s)}v^2(s,x)dsdxdt + r_3T/r_1 \\
&= \frac{r_2 a}{r_1}\int_0^T\int_0^1 v^2(t,x)dx\,dt + r_3T/r_1.
\end{aligned}
\tag{3.7}
$$

To obtain the boundedness of $\int_0^T\int_0^1 v^2 dxdt$, we introduce the function

$$
V_1(t) = \int_0^1 [v^2 + u_x^2 + 2\lambda\cos(u)]dx.
$$

Then

$$
\begin{aligned}
V_1'(t) &= 2\int_0^1 \Big[-\delta v^2 + \lambda v[\beta\sin(u) + g(t,x) \\
&\qquad - \int_{t-\tau}^t e^{-\alpha(t-s)}v(s,x)ds] - \lambda\beta\sin(u)v\Big]dx \\
&\le \int_0^1 \Big[-(2\delta - a - \varepsilon_4)v^2 + \int_{t-\tau}^t e^{-\alpha(t-s)}v^2(s,x)ds\Big]dx + \frac{K}{\varepsilon_4},
\end{aligned}
\tag{3.8}
$$

where $0 < \varepsilon_4 < 2(\delta - a)$. Therefore,
$$\int_0^T \int_0^1 v^2(t,x)dxdt \le \frac{KT}{\varepsilon_4(2\delta - 2a - \varepsilon_4)} := K_1.$$
By (3.7), we obtain
$$\int_0^T V(t)dt \le \frac{r_2 a}{r_1} K_1 + r_3 T/r_1.$$
Since v and u_x are continuous, there is a $t_0 \in [0,T]$ with
$$V(t_0) \le \frac{\frac{r_2 a}{r_1} K_1 + r_3 T/r_1}{T} := K_2.$$
Hence, if $t_0 \le t \le t_0 + T$, then by (3.6), we obtain
$$\begin{aligned} V(t) &= V(t_0) + \int_{t_0}^t V'(s)ds \\ &\le K_2 + r_2 \int_{t_0}^{t_0+T} \int_0^1 \int_{t-\tau}^t e^{-\alpha(t-s)} v^2(s,x)dsdxdt + r_3 T \\ &\le K_2 + r_2 a K_1 + r_3 T. \end{aligned}$$
Note that (3.3) yields
$$\int_0^1 (u_x^2 + v^2)dx \le \frac{K_2 + r_2 a K_1 + r_3 T}{1 - \frac{\delta}{\pi^2}} := C_1.$$
Thus
$$\sup_{0 \le t \le T} \int_0^1 (u_x^2 + v^2)dx \le C_1.$$
This completes the proof. □

Theorem 3.2. *Suppose that (A1)–(A3) hold. Then* (3.1) *admits a nontrivial T-periodic solution.*

Proof. Let $C(t,u) = \beta \sin(u) + g(t,x) - \int_{t-\tau}^t e^{-\alpha(t-s)} u_s(s)ds$. Since
$$\begin{aligned} \int_{t-\tau}^t e^{-\alpha(t-s)} u_s(s)ds &= \int_{-\tau}^0 e^{\alpha s} u_s(t+s)ds \\ &= u(t) - u(t-\tau)e^{-\alpha\tau} - \alpha \int_{-\tau}^0 e^{\alpha s} u(t+s)ds, \end{aligned}$$
from the assumptions (A1)–(A3), we see that C is continuous and takes bounded sets into bounded sets. By Lemma 3.1 and Theorem 2.1, we see that (3.1) has a T-periodic solution. Since $g \ne 0$, we see that the T-periodic solution is nontrivial. This completes the proof. □

4. Stable periodic solutions for Sine-Gordon equations

In this section, we investigate the uniformly asymptotic stability of time periodic solutions for Sine-Gordon equation
$$\begin{gathered} \frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(t,x)}{\partial x^2} - \delta \frac{\partial u(t,x)}{\partial t} + \beta \sin(u(t,x)) + g(t,x), \quad 0 < x < 1,\ t > 0, \\ u(t,0) = u(t,1) = 0, \quad t > 0, \end{gathered} \tag{4.1}$$

where g is continuous. We assume that

(H1) $0 < \delta < \frac{2\pi^2}{5}$ and $0 < |\beta| < \delta\sqrt{(2-\frac{(2+\delta)}{\pi^2})(2-\frac{5\delta}{\pi^2})}$.

Theorem 4.1. *Assume that (H1) and (A3) hold. Then* (4.1) *admits a nontrivial uniformly asymptotic stable T-periodic solution.*

Proof. By applying Theorem 3.2 and assumptions (H1) and (A3), we conclude that (4.1) admits a nontrivial T-periodic solution $\bar{u}(t)$. Let $u(t)$ be a solution for (4.1), and define

$$V_2(t) = \int_0^1 [(u_t - \bar{u}_t)^2 + 2k(u-\bar{u})(u_t-\bar{u}_t) + (u-\bar{u})_x^2]dx,$$

where $k = \frac{\delta}{\pi^2}$. By the similar arguments as in Lemma 3.1, we have

$$\begin{aligned}
V_2'(t) &= 2\int_0^1 [(u_t-\bar{u}_t)(u_t-\bar{u}_t)_t + (u-\bar{u})_x(u-\bar{u})_{xt}]dx \\
&\quad + 2k\int_0^1 [(u_t-\bar{u}_t)^2 + (u-\bar{u})(u_t-\bar{u}_t)_t]dx \\
&= 2\int_0^1 [-\delta(u_t-\bar{u}_t)^2 + \beta[(u_t-\bar{u}_t) + k(u-\bar{u})](\sin(u)-\sin(\bar{u}))]dx \\
&\quad + 2k\int_0^1 [(u_t-\bar{u}_t)^2 + (u-\bar{u})_{xx}(u-\bar{u}) - \delta(u_t-\bar{u}_t)(u-\bar{u})]dx \\
&\le 2\int_0^1 [-(\delta-k)(u_t-\bar{u}_t)^2 - k(u-\bar{u})_x^2 - k\delta(u-\bar{u})(u_t-\bar{u}_t)]dx \\
&\quad + 2|\beta|\int_0^1 (k(u-\bar{u}) + (u_t-\bar{u}_t))|u-\bar{u}|dx \\
&\le \int_0^1 [-(2\delta-2k)(u_t-\bar{u}_t)^2 - 2k(u-\bar{u})_x^2 + k\delta((u-\bar{u})^2 + (u_t-\bar{u}_t)^2)]dx \\
&\quad + \int_0^1 (2|\beta|k(u-\bar{u})^2 + |\beta|\varepsilon(u-\bar{u})^2 + \frac{|\beta|}{\varepsilon}(u_t-\bar{u}_t)^2dx \\
&= \int_0^1 [-(2\delta-2k) + k\delta + |\beta|\varepsilon^{-1}](u_t-\bar{u}_t)^2dx \\
&\quad + \int_0^1 -2k(u-\bar{u})_x^2 + [k(\delta+2|\beta|) + |\beta|\varepsilon](u-\bar{u})^2dx \\
&\le -\tilde{c}\int_0^1 (u_t-\bar{u}_t)^2 + (u-\bar{u})_x^2dx,
\end{aligned}$$

where

$$\frac{|\beta|}{2\delta - \frac{2\delta+\delta^2}{\pi^2}} < \varepsilon < \frac{2\delta - \frac{5\delta^2}{\pi^2}}{|\beta|}$$

and

$$\tilde{c} = \min\{2\delta - (\frac{2+|\beta|}{\pi^2} + |\beta|\varepsilon^{-1}), \frac{1}{\pi^2}[2\delta - (\frac{\delta}{\pi^2}(\delta+2|\beta|) + |\beta|\varepsilon)]\}.$$

Since

$$\frac{1}{2}\int_0^1 (u_t-\bar{u}_t)^2 + (u-\bar{u})_x^2dx \le V_2(t) \le 2\int_0^1 (u_t-\bar{u}_t)^2 + (u-\bar{u})_x^2dx,$$

we obtain that

$$V_2'(t) \le -\frac{\tilde{c}}{2} V_2(t).$$

This implies that for any $t > t_0$,

$$\begin{aligned}
&\|u(t) - \bar{u}(t)\|_{H_0^1} + \|u_t(t) - \bar{u}_t(t)\|_{H^0} \\
&\le 2e^{-\frac{\tilde{c}}{2}(t-t_0)}[\|u(t_0) - \bar{u}(t_0)\|_{H_0^1} + \|u_t(t_0) - \bar{u}_t(t_0)\|_{H^0}].
\end{aligned}$$

Thus the T-periodic solution $\bar{u}(t)$ is uniformly asymptotic stable. Then the proof of Theorem 4.1 is complete. □

Especially, in above arguments, let $u(t)$ be any T-periodic solution of (4.1). Then, for any positive integer n, we have

$$\begin{aligned}
&\|u(t) - \bar{u}(t)\|_{H_0^1} + \|u_t(t) - \bar{u}_t(t)\|_{H^0} \\
&= \|u(t+nT) - \bar{u}(t+nT)\|_{H_0^1} + \|u_t(t+nT) - \bar{u}_t(t+nT)\|_{H^0} \\
&\le 2e^{-\frac{\tilde{c}}{2}(t+nT-t_0)}[\|u(t_0) - \bar{u}(t_0)\|_{H_0^1} + \|u_t(t_0) - \bar{u}_t(t_0)\|_{H^0}].
\end{aligned}$$

Let $n \to +\infty$, we obtain $\|u(t) - \bar{u}(t)\|_{H_0^1} = 0$ and $\|u_t(t) - \bar{u}_t(t)\|_{H^0} = 0$, which implies that $u(t) = \bar{u}(t)$ a.e. in [0,T]. Thus we have the following result.

Corollary 4.2. *Suppose that (H1) and (A3) hold. Then* (4.1) *admits a unique uniformly asymptotic stable T-periodic solution.*

Note that in Theorem 4.1, the assumption $0 < |\beta| < \delta\sqrt{(2 - \frac{(2+\delta)}{\pi^2})(2 - \frac{5\delta}{\pi^2})}$ is necessary. If this condition is omitted, the time periodic solution maybe not uniformly asymptotic stable. Now, we show an example to explain this case.

Example Consider the equation

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t^2} + 2\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + 4\pi^2 u(t,x), \quad 0 < x < 1,\ t > 0, \\
u(t,0) = u(t,1) = 0, \quad t > 0.
\end{gathered} \tag{4.2}$$

It is easy to see that $u(t,x) = \sin(2\pi x)$ is a time periodic solution of (4.2). On the other hand, when $p = \sqrt{3\pi^2 + 1} - 1$, $u(t,x) = \sin(2\pi x) + e^{pt}\sin(\pi x)$ is a solution of (4.2) too, but unbounded. Meanwhile $u(t,x) \equiv 0$ is time periodic solution of (4.2). Hence both $u(t,x) \equiv 0$ and $u(t,x) = \sin(2\pi x)$ are not uniformly asymptotic stable time periodic solutions.

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